Canad. Math. Bull. Vol. **25** (1), 1982

REMARKS ON TWO WEAK FORMS OF CONTINUITY

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ABSTRACT. New characterizations of weakly-continuous and θ-continuous functions are presented, and θ-continuity is applied to characterize $H(i)$ spaces; a recent characterization of closed graph functions is utilized to characterize H-closed spaces. Noiri has shown that a function λ which is almost-continuous in the sense of Husain is weakly-continuous if $\text{cl}(\lambda^{-1}(W)) \subset \lambda^{-1}(\text{cl}(W))$ for all open W. It is established here that almost-continuity is superfluous in this statement.

All spaces will be topological spaces, $\text{cl}(A)$ and $\text{int}(A)$ will represent respectively the closure and interior of a subset A of a space, and $\Sigma(A)$ ($\Sigma(x)$ if $A = \{x\}$) will represent the collection of open sets which contain A. Levine [L] has called a function λ from a space X to a space Y *weakly-continuous* if for each $x \in X$ and $W \in \Sigma(\lambda(x))$ there is a $V \in \Sigma(x)$ satisfying $\lambda(V) \subset \text{cl}(W)$. Fomin [F] has defined a function $\lambda : X \to Y$ to be *θ-continuous* if for each $x \in X$ and $W \in \Sigma(\lambda(x))$ there is a $V \in \Sigma(x)$ satisfying $\lambda(\text{cl}(V)) \subset \text{cl}(W)$.

As the main results in this paper, we present new characterizations of weakly-continuous and θ-continuous functions, apply θ-continuity to characterize $H(i)$ spaces, and utilize a recent characterization of closed graph functions by Hamlett and Long [H–L] to obtain characterizations of H-closed spaces. These concepts have been considered by numerous authors (e.g. see [N_1], [N_2], [R]. [J_1], [He–L], and [H]). A function $\lambda : X \to Y$ is called *almost-continuous* by Husain [H] if for each $x \in X$ and each $W \in \Sigma(\lambda(x))$, $\text{cl}(\lambda^{-1}(W))$ is a neighborhood of x in X. Noiri [N_1] has shown that a weakly-continuous function λ satisfies $\text{cl}(\lambda^{-1}(W)) \subset \lambda^{-1}(\text{cl}(W))$ for all open W and that an almost-continuous function λ which satisfies $\text{cl}(\lambda^{-1}(W)) \subset \lambda^{-1}(\text{cl}(W))$ for all open W is weakly-continuous. One of our results establishes that any function λ which satisfies $\text{cl}(\lambda^{-1}(W)) \subset \lambda^{-1}(\text{cl}(W))$ for all open W is weakly-continuous.

Let X be a space, let $A \subset X$ and let $x \in X$. A is *regular-closed* (*regular-open*) if $A = \text{cl}(\text{int}(A))$ ($X - A$ is regular-closed); we say that $x \in X$ is in the *θ-closure* of $A(\text{cl}_\theta(A))$ if each $V \in \Sigma(x)$ satisfies $A \cap \text{cl}(V) \neq \emptyset$, and that A is *θ-closed* if $A = \text{cl}_\theta(A)$. We come now to our first main result.

Received by the editors, February 22, 1980 and, in revised form, August 4, 1980 and September 30, 1980

AMS (*MOS*) *Subject classifications* (1980). Primary 54C05. Secondary 54D25.

Key words and phrases. θ-continuity, weak-continuity, H-closed spaces.

THEOREM 1. *The following statements are equivalent for spaces X, Y and function* $\lambda : X \to Y$:

(a) *The function* λ *is weakly-continuous.*

(b) *Each* $A \subset Y$ *satisfies* $\text{cl}[\lambda^{-1}(\text{int}(\text{cl}_\theta(A)))] \subset \lambda^{-1}(\text{cl}_\theta(A))$.

(c) *Each open subset W of Y satisfies* $\text{cl}[\lambda^{-1}(\text{int}(\text{cl}(W)))] \subset \lambda^{-1}(\text{cl}(W))$.

(d) *Each regular-closed subset R of Y satisfies* $\text{cl}[\lambda^{-1}(\text{int}(R))] \subset \lambda^{-1}(R)$.

(e) *Each open subset W of Y satisfies* $\text{cl}(\lambda^{-1}(W)) \subset \lambda^{-1}(\text{cl}(W))$.

Proof. To see that (a) implies (b), assume that $A \subset Y$ and $x \in X - \lambda^{-1}(\text{cl}_\theta(A))$. Then $\lambda(x) \notin \text{cl}_\theta(A)$. So some $V \in \Sigma(x)$ in X satisfies $\lambda(V) \subset \text{cl}(Y\text{-cl}_\theta(A))$. This gives $\lambda(V) \cap \text{int}(\text{cl}_\theta(A)) = \emptyset$ and, consequently, $x \notin \text{cl}[\lambda^{-1}(\text{int}(\text{cl}_\theta(A)))]$. It is clear that (b) implies (c) since $\text{cl}_\theta(W) = \text{cl}(W)$ for open W. Now assume (c) and let $R \subset Y$ be regular-closed. Then $\text{cl}[\lambda^{-1}(\text{int}(R))] = \text{cl}[\lambda^{-1}(\text{int}(\text{cl}(\text{int}(R))))] \subset \lambda^{-1}(\text{cl}(\text{int}(R))) = \lambda^{-1}(R)$, so (d) holds. If (d) holds and W is open in Y then $\text{cl}(W)$ is regular-closed in Y and, therefore, $\text{cl}(\lambda^{-1}(W)) \subset \text{cl}[\lambda^{-1}(\text{int}(\text{cl}(W)))] \subset \lambda^{-1}(\text{cl}(W))$, so (e) follows. Finally, to verify that (a) is implied by (e), let $x \in X$ and $W \in \Sigma(\lambda(x))$. Then $\lambda(x) \notin \text{cl}(Y\text{-cl}(W))$ and $x \notin \lambda^{-1}(\text{cl}(Y - \text{cl}(W)))$. Since $Y - \text{cl}(W)$ is open in Y, we conclude from (e) that $x \notin \text{cl}(\lambda^{-1}(Y\text{-cl}(W)))$. Hence some $V \in \Sigma(x)$ satisfies $V \cap \lambda^{-1}(Y\text{-cl}(W)) = \emptyset$ and $\lambda(V) \subset \text{cl}(W)$. The proof is complete.

The proof of Theorem 2 is similar in nature to that of Theorem 1 and is omitted.

THEOREM 2. *The following statements are equivalent for spaces X, Y and function* $\lambda : X \to Y$:

(a) *The function* λ *is* θ*-continuous.*

(b) *Each* $A \subset Y$ *satisfies* $\text{cl}_\theta[\lambda^{-1}(\text{int}(\text{cl}_\theta(A)))] \subset \lambda^{-1}(\text{cl}_\theta(A))$.

(c) *Each open subset W of Y satisfies* $\text{cl}_\theta[\lambda^{-1}(\text{int}(\text{cl}(W)))] \subset \lambda^{-1}(\text{cl}(W))$.

(d) *Each regular-closed subset R of Y satisfies* $\text{cl}_\theta[\lambda^{-1}(\text{int}(R))] \subset \lambda^{-1}(R)$

(e) *Each open subset W of Y satisfies* $\text{cl}_\theta(\lambda^{-1}(W)) \subset \lambda^{-1}(\text{cl}(W))$.

In our next main results, we give several new characterizations of $H(i)$ spaces in terms of θ-continuous functions on the spaces and the θ-closures of values of the functions. We recall that the *θ-adherence* of a filterbase Ω on a space X is $\bigcap_\Omega \text{cl}_\theta(F)$. $A \subset X$ is *quasi H-closed* (*QHC*) *relative* to X if each filterbase Ω on A satisfies $A \cap ad_\theta\Omega \neq \emptyset$. X is $H(i)$ if X is *QHC* relative to X. Hausdorff $H(i)$ spaces are called *H-closed.* Alexandroff and Urysohn [A–U] originally defined a Hausdorff space to be H-closed if it is a closed subset of any Hausdorff space which contains it.

THEOREM 3. *The following statements are equivalent for a space* X:

(a) X is $H(i)$.

(b) *Each* θ*-continuous function* λ *on* X *satisfies* $\text{cl}(\lambda(\text{cl}_\theta(A))) \subset \bigcup_{\text{cl}_\theta(A)} \text{cl}_\theta(\lambda(x))$ *for each* $A \subset X$.

https://doi.org/10.4153/CMB-1982-008-8 Published online by Cambridge University Press

(c) *Each θ-continuous function λ on X satisfies* $\mathrm{cl}(\lambda(\mathrm{ad}_\theta\Omega)) \subset \bigcap_{A\in\Omega}(\bigcup_{\mathrm{cl}_\theta(A)} \mathrm{cl}_\theta(\lambda(x)))$ *for each filterbase Ω on X.*

(d) *Each θ-continuous function λ on X satisfies* $\mathrm{cl}(\lambda(\mathrm{cl}(A))) \subset \bigcup_{\mathrm{cl}(A)} \mathrm{cl}_\theta(\lambda(x))$ *for each open $A \subset X$.*

(e) *Each θ-continuous function λ on X satisfies* $\mathrm{cl}(\lambda(X)) \subset \bigcup_X \mathrm{cl}_\theta(\lambda(x))$.

(f) *Each continuous function λ on X satisfies* $\mathrm{cl}(\lambda(X)) \subset \bigcup_X \mathrm{cl}_\theta(\lambda(x))$.

(g) $Y = \bigcup_X \mathrm{cl}_\theta(x)$ *for any space Y in which X is embedded as a dense (dense open) subspace.*

Proof. It is immediate that (c) implies (d), that (d) implies (e), that (e) implies (f), and that (f) implies (g). If we assume (a) the inequality in (b) follows from the fact [J_2] that $\mathrm{cl}_\theta(A)$ is *QHC* relative to X, the known result that the θ-continuous image of a *QHC* relative subset is a *QHC* relative subset, and the inequality $\mathrm{cl}(P) \subset \bigcup_P \mathrm{cl}_\theta(x)$ for *QHC* relative subsets which may be established by appeal to the simple observation that $y \in \mathrm{cl}_\theta(x)$ when $x \in \mathrm{cl}_\theta(y)$. If (b) holds and Ω is a filterbase on X and λ is θ-continuous, we have that

$$\mathrm{cl}(\lambda(\mathrm{ad}_\theta\Omega)) \subset \bigcap_{\Omega} \mathrm{cl}(\lambda(\mathrm{cl}_\theta(A))) \subset \bigcap_{A\in\Omega} \Big(\bigcup_{\mathrm{cl}_\theta(A)} \mathrm{cl}_\theta(\lambda(x))\Big).$$

Finally, we show that (g) implies (a). It is well-known, and easily seen from the fact that any filterbase Ω on a space satisfies $\mathrm{ad}_\theta\Omega = \bigcap_\Omega \mathrm{cl}_\theta(F) = \bigcap_\Omega \mathrm{ad}\,\Sigma(F) = \mathrm{ad} \bigcup_\Omega \Sigma(F)$, that it is enough to show that each open filterbase on X has a nonempty adherence. Let Ω be an open filterbase on X and choose $\pi \notin X$. Let $Y = X \cup \{\pi\}$ with the topology having as base the topology on X along with all sets of the form $F \cup \{\pi\}$, where $F \in \Omega$. X is clearly embedded in Y as a dense open subspace and by (g) we have $Y = \bigcup_X \mathrm{cl}_\theta(x)$. Hence there is an $x \in X$ with $\pi \in \mathrm{cl}_\theta(x)$. For such an x we have $x \in \mathrm{cl}_\theta(\pi)$ and all $V \in \Sigma(x)$ in X and $F \in \Omega$ satisfy $V \cap F \neq \varnothing$. Thus Ω has a nonempty adherence in X.

The proof is complete.

From the characterizations in Theorem 3 and the known fact that a space X is Hausdorff if and only if all of its singleton sets are θ-closed, the following result may be readily obtained.

COROLLARY 4. *The following statements are equivalent for a Hausdorff space X:*

(a) *X is H-closed.*

(b) *Each θ-continuous function λ from X to a Hausdorff space maps θ-closed subsets onto closed subsets.*

(c) *Each θ-continuous function λ from X to a Hausdorff space maps regular-closed subsets onto closed subsets.*

It is well-known that a continuous function $\lambda : X \to Y$ into a Hausdorff space has a closed graph, i.e., $\{(x, \lambda(x)) : x \in X\}$ is a closed subset of $X \times Y$. Hamlett and Long [H–L] have proved that $\lambda : X \to Y$ has a closed graph if and only if

https://doi.org/10.4153/CMB-1982-008-8 Published online by Cambridge University Press

each $y \in Y$ and each (some) open set base Σ at y satisfy $\lambda^{-1}(y) = \bigcap_\Sigma \text{cl}(\lambda^{-1}(V))$. We apply this characterization along with continuous functions on the spaces and the θ-closure operator to offer several new characterizations of H-closed spaces. For economy in stating our next theorem, if X is a space, we let $\mathscr{C}(X, T_2)$ represent the class of all continuous functions from X to Hausdorff spaces.

THEOREM 5. *The following statements are equivalent for a Hausdorff space* X:

(a) X *is* H-*closed.*

(b) *All* $\lambda \in \mathscr{C}(X, T_2)$ *satisfy* $\text{ad}\,\lambda(\Omega) \subset \lambda(\text{ad}_\theta\,\Omega)$ *for all filterbases* Ω *on* X.

(c) *All* $\lambda \in \mathscr{C}(X, T_2)$ *satisfy* $\text{ad}\,\lambda(\Omega) \subset \lambda(\text{ad}\,\Omega)$ *for all open filterbases* Ω *on* X.

(d) *All* $\lambda \in \mathscr{C}(X, T_2)$ *satisfy* $\text{cl}(\lambda(A)) \subset \lambda(\text{cl}_\theta(A))$ *for all* $A \subset X$.

(e) *All* $\lambda \in \mathscr{C}(X, T_2)$ *satisfy* $\text{cl}(\lambda(A)) \subset \lambda(\text{cl}(A))$ *for all open* $A \subset X$.

(f) *All* $\lambda \in \mathscr{C}(X, T_2)$ *satisfy* $\text{cl}(\lambda(A)) \subset \lambda(\text{cl}(A))$ *for all regular-open* $A \subset X$.

Proof. The proofs that (b) implies (c), that (d) implies (e), and that (e) implies (f) are all clear. Moreover, assuming (f), X is obviously a closed subspace of any Hausdorff space which contains it. So, X is H-closed and (f) implies (a). Now, assume (a), let Y be Hausdorff; let $\lambda: X \to Y$ be continuous and let Ω be a filterbase on X with $y \in \text{ad}\,\lambda(\Omega)$. Then $\Omega^* = \{F \cap \lambda^{-1}(V) : V \in \Sigma(y),\ F \in \Omega\}$ is a filterbase on X and, since X is H-closed, $\text{ad}_\theta\,\Omega^* \neq \varnothing$. Moreover, $\text{ad}_\theta\,\Omega^* \subset \bigcap_{\Sigma(y)} \text{cl}_\theta(\lambda^{-1}(A)) \cap \text{ad}_\theta\,\Omega = \lambda^{-1}(y) \cap \text{ad}_\theta\,\Omega$. Hence $y \in \lambda(\text{ad}_\theta\,\Omega)$ and (b) is established. Finally let Y be Hausdorff, $\lambda: X \to Y$ be continuous, $\varnothing \neq A \subset X$ and suppose $y \in Y - \lambda(\text{cl}_\theta(A))$. Then $y \notin \lambda(\text{ad}\,\Sigma(A))$. Hence, from (c), $y \notin \text{ad}\,\lambda(\Sigma(A))$. This implies that $y \notin \text{cl}(\lambda(A))$ and (d) holds.

The proof is complete.

REMARK 6. It is fairly obvious that if the requirement "all functions" in Theorem 5 is replaced throughout by the requirement "all bijections", the resulting statement is valid.

A bijection $\lambda: X \to Y$ is a θ-*homeomorphism* if λ and λ^{-1} are both θ-continuous [V_1]. By the characterization of θ-continuity in [J_1], it is obvious that a bijection $\lambda: X \to Y$ is a θ-homeomorphism if and only if $\lambda(\text{cl}_\theta(A)) = \text{cl}_\theta(\lambda(A))$ for each $A \subset X$. Hence, since continuous functions are θ-continuous, the results in Theorem 2 in [T] and its attendant corollary may be extended to the class of θ-continuous functions, and θ-homeomorphisms, respectively.

THEOREM 7. *If* X *is* $H(i)$, $\lambda: X \to Y$ *is a* θ-*continuous bijection and QHC subsets relative to* Y *are* θ-*closed, then* λ *is a* θ-*homeomorphism.*

Proof. We need show that λ^{-1} is θ-continuous. If $A \subset X$, then $\text{cl}_\theta(A)$ is *QHC* relative to X. Thus $\lambda(\text{cl}_\theta(A))$ is *QHC* relative to Y. Hence $\lambda(\text{cl}_\theta(A))$ is

θ-closed by hypothesis. Since $\lambda(A) \subset \lambda(\text{cl}_\theta(A))$, it follows that $\text{cl}_\theta(\lambda(A)) \subset \lambda(\text{cl}_\theta(A))$. The proof is complete.

Our final result comes as a result of a decomposition theorem for θ-rigid subsets by the authors [E–J]. A subset A of a space X is *θ-rigid* if for each cover Ω of A by open subsets of X, some finite $\Omega^* \subset \Omega$ satisfies $A \subset \text{int}(\text{cl}(\bigcup_{\Omega^*} V))$. It has been shown [E–J] that any θ-rigid subset A of a space satisfies $\text{cl}_\theta(A) = \bigcup_A \text{cl}_\theta(x)$.

THEOREM 8. *If $A \subset X$ is θ-rigid, Y is T_2, and $\lambda : X \to Y$ is θ-continuous, then $\lambda(\text{cl}_\theta(A)) = \lambda(A)$.*

Proof. We see from the decomposition result provided above that $\lambda(A) \subset \lambda(\text{cl}_\theta(A)) = \lambda(\bigcup_A \text{cl}_\theta(x)) \subset \bigcup_A \text{cl}_\theta(\lambda(x))$. The last set in this inequality is $\lambda(A)$ because Y is T_2. The proof is complete.

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