BULL. AUSTRAL. MATH. SOC. 57M05, 20F34, 57N05
VOL. 67 (2003) [1–14]

IRREDUCIBLE DECOMPOSITION OF THE MAGNUS REPRESENTATION OF THE TORELLI GROUP

MASAAKI SUZUKI

In this paper we describe the irreducible decomposition of the Magnus representation of the Torelli group.

1. INTRODUCTION

Let Σ_g be a closed oriented surface of genus g and $\Sigma_{g,1}$ be an oriented surface obtained from Σ_g by removing an open disk. We denote by $\mathcal{M}_{g,1}$ the mapping class group of $\Sigma_{g,1}$ relative to the boundary, that is, the group of path components of the group of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ which restrict to the identity on the boundary. Let $\mathcal{I}_{g,1}$ be the Torelli group of $\Sigma_{g,1}$, namely the normal subgroup of $\mathcal{M}_{g,1}$ consisting of all the elements which act on the homology of $\Sigma_{g,1}$ trivially. Various results concerning the structure of the mapping class group and the Torelli group have been obtained (for example see [**5**, **7**]).

We investigate the Magnus representation of the Torelli group. The aim of this paper is to give the irreducible decomposition of this representation. The Magnus representations are defined for a wide class of subgroups of automorphism group of free groups (see [**2**] for details). For example, the classical Burau representation and the Gassner representation, for braid groups and pure braid groups respectively, belong to this class. The Magnus representation for the mapping class group $\mathcal{M}_{g,1}$

$$r : \mathcal{M}_{g,1} \to GL(2g; \mathbb{Z}[\Gamma_0])$$

was studied in [**6**], where $\Gamma_0 = \pi_1(\Sigma_{g,1})$. This mapping is not a group homomorphism. We restrict it to the Torelli group $\mathcal{I}_{g,1}$ and reduce the coefficients to $\mathbb{Z}[H]$ which is induced by the Abelianisation $\mathfrak{a} : \Gamma_0 \to H$, where $H = H_1(\Sigma_{g,1}; \mathbb{Z})$. Then we obtain a homomorphism

$$\bar{r} : \mathcal{I}_{g,1} \longrightarrow GL(2g; \mathbb{Z}[H]).$$

Received 23rd October, 2001
The author would like to express his gratitude to Professor Shigeyuki Morita for helpful suggestions and encouragements. He also would like to thank Professor Nariya Kawazumi, Professor Teruaki Kitano and Professor Takayuki Morifuji for valuable discussions and advices.

We call this mapping $\overline{r}$ the Magnus representation of the Torelli group. In Section 2, we recall the definition of the Magnus representation of the Torelli group more precisely from [**6**].

Each of the Burau and Gassner representations has a 1-dimensional trivial subrepresentation. In Section 3, we show that the Magnus representation of the Torelli group also has a 1-dimensional trivial subrepresentation. In contrast to the classical cases, however, this 1-dimensional trivial subrepresentation is not a direct summand. In Section 4, we show that the quotient representation is reducible in our case. In fact, it has a $(2g-2)$-dimensional irreducible subrepresentation. That is to say, we arrive at the following main result of this paper.

Main Theorem. *For $g \geqslant 2$ there exsists a non-singular matrix $P \in GL(2g; R)$ such that for any element $\varphi \in \mathcal{I}_{g,1}$*

$$P^{-1}\,\overline{r}(\varphi)\,P = \left(\begin{array}{c|ccc|c} 1 & & * & & * \\ \hline 0 & & & & \\ \vdots & & \rho_B(\varphi) & & * \\ & & & & \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right).$$

Moreover, ρ_B is a $(2g-2)$-dimensional irreducible representation of $\mathcal{I}_{g,1}$.

Here $R = \mathbb{Z}\left[{x_i}^{\pm 1}, {y_i}^{\pm 1}, 1/(1-y_i)\right]$ $\left(\supset \mathbb{Z}[H]\right)$ where x_i, y_i $(i = 1, \ldots, g)$ is a symplectic basis of H obtained by Abelianising a system of generators α_i, β_i of Γ_0 as shown in Figure 1. We show that the representation ρ_B is irreducible by making use of Formanek's technique used in [**4**] to determine whether the reduced Burau representation obtained by the complex specialisation is irreducible or not. His technique was also used in [**1**] to give a necessary and sufficient condition for the specialisation of the reduced Gassner representation to be irreducible.

In Section 6, we give some remarks and applications of the irreducible decomposition.

2. Definition of the Magnus representation

We denote by $\mathbb{Z}[\Gamma_0]$ the integral group ring of $\Gamma_0 = \pi_1(\Sigma_{g,1})$. We fix a system of generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of the free group Γ_0 as shown in Figure 1. Let us simply write $\gamma_1, \ldots, \gamma_{2g}$ for them.

Definition 2.1: We call the following mapping

$$\begin{array}{rccc} r: & \mathcal{M}_{g,1} & \longrightarrow & GL\left(2g; \mathbb{Z}[\Gamma_0]\right) \\ & \varphi & \longmapsto & \left(\overline{\dfrac{\partial \varphi(\gamma_j)}{\partial \gamma_i}}\right)_{i,j} \end{array}$$

https://doi.org/10.1017/S0004972700033475 Published online by Cambridge University Press

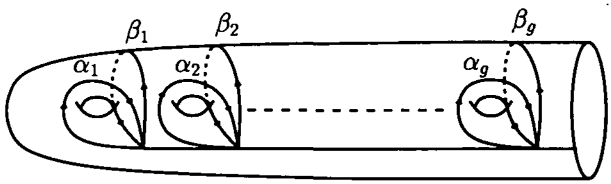

Figure 1: Generators of Γ_0

the Magnus representation for the mapping class group $\mathcal{M}_{g,1}$. Here $\dfrac{\partial}{\partial\gamma_i}$ is the Fox derivation and $\bar{\ }: \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ is the antiautomorphism induced by the mapping $\gamma \mapsto \gamma^{-1}$.

However, this mapping r is not a group homomorphism.

Proposition 2.2. ([**6**]) *For any two elements $\varphi, \psi \in \mathcal{M}_{g,1}$, we have*

$$r(\varphi\psi) = r(\varphi)\cdot {}^{\varphi}r(\psi)$$

where ${}^{\varphi}r(\psi)$ denotes the matrix obtained from $r(\psi)$ by applying the automorphism $\varphi : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ on each entry.

We restrict this mapping r to the Torelli group $\mathcal{I}_{g,1}$ and reduce the coefficients to $\mathbb{Z}[H]$. Since the Torelli group $\mathcal{I}_{g,1}$ acts trivially on H, we obtain a genuine homomorphism

$$\bar{r} : \mathcal{I}_{g,1} \to GL\big(2g; \mathbb{Z}[H]\big).$$

This is the definition of the Magnus representation of the Torelli group.

3. Reducibility of the Magnus representation of the Torelli group and a quotient representation

As is mentioned in [**2**], the classical Burau representation as well as the Gassner representation has a 1-dimensional trivial subrepresentation. The Magnus representation of the Torelli group has a similar property.

We define the vector v as follows. Let ζ be a simple closed curve on $\Sigma_{g,1}$ which is parallel to the boundary. We may regard ζ as an element of Γ_0.

$$\zeta = [\beta_g, \alpha_g] \cdots [\beta_1, \alpha_1] \in \Gamma_0$$

The vector v is obtained by applying Fox derivations to the boundary curve ζ. That is, we set

$$\begin{aligned} v &= {}^t\left(\mathfrak{a}\left(\overline{\frac{\partial\zeta}{\partial\alpha_1}}\right) \cdots \mathfrak{a}\left(\overline{\frac{\partial\zeta}{\partial\alpha_g}}\right) \mathfrak{a}\left(\overline{\frac{\partial\zeta}{\partial\beta_1}}\right) \cdots \mathfrak{a}\left(\overline{\frac{\partial\zeta}{\partial\beta_g}}\right)\right) \\ &= {}^t\,(-b_1 \ \cdots \ -b_g \ a_1 \ \cdots \ a_g) \end{aligned}$$

where a_i, b_i are $1-\bar{x}_i$, $1-\bar{y}_i$ respectively and $\bar{x}_i = {x_i}^{-1}$, $\bar{y}_i = {y_i}^{-1}$.

https://doi.org/10.1017/S0004972700033475 Published online by Cambridge University Press

PROPOSITION 3.1. *The Magnus representation of the Torelli group has a 1-dimensional trivial subrepresentation. In fact, a 1-dimensional subspace spanned by the vector v is invariant under the action of $\mathcal{I}_{g,1}$.*

PROOF: For a given element $\varphi \in \mathcal{I}_{g,1}$, we write simply

$$\overline{r}(\varphi) = \begin{pmatrix} m_{1,1} & \cdots & m_{1,2g} \\ \vdots & \ddots & \vdots \\ m_{2g,1} & \cdots & m_{2g,2g} \end{pmatrix}.$$

Let τ_ζ be the Dehn twist along a simple closed curve ζ. By straightforward calculation, we obtain the following matrix.

$$\overline{r}(\tau_\zeta) = I_{2g} + {}^t(-b_1 \ \cdots \ -b_g \ a_1 \ \cdots \ a_g) \cdot (a_1 \ \cdots \ a_g \ b_1 \ \cdots \ b_g)$$

Here I_{2g} denotes the $2g \times 2g$ unit matrix. Therefore

$$\begin{aligned} \overline{r}(\varphi\tau_\zeta) &= \overline{r}(\varphi)\overline{r}(\tau_\zeta) \\ &= \begin{pmatrix} m_{1,1} & \cdots & m_{1,2g} \\ \vdots & \ddots & \vdots \\ m_{2g,1} & \cdots & m_{2g,2g} \end{pmatrix} \\ &\quad + \begin{pmatrix} m_{1,1} & \cdots & m_{1,2g} \\ \vdots & \ddots & \vdots \\ m_{2g,1} & \cdots & m_{2g,2g} \end{pmatrix} \begin{pmatrix} -b_1 \\ \vdots \\ -b_g \\ a_1 \\ \vdots \\ a_g \end{pmatrix} (a_1 \ \cdots \ a_g \ b_1 \ \cdots \ b_g). \end{aligned} \tag{3.1}$$

On the other hand, we can easily compute the same matrix directly by the definition, because the action of $\varphi\tau_\zeta \in \mathcal{I}_{g,1}$ on $\pi_1(\Sigma_{g,1})$ is given by

$$\varphi\tau_\zeta(\gamma_j) = [\beta_g, \alpha_g] \cdots [\beta_1, \alpha_1]\varphi(\gamma_j)[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g].$$

For example, the Fox derivations of them are computed as

$$\begin{aligned} \frac{\partial(\varphi\tau_\zeta(\alpha_j))}{\partial\alpha_i} = {} & [\beta_g, \alpha_g] \cdots [\beta_{i+1}, \alpha_{i+1}]\beta_i \\ & - [\beta_g, \alpha_g] \cdots [\beta_i, \alpha_i] \\ & + [\beta_g, \alpha_g] \cdots [\beta_1, \alpha_1] \frac{\partial\varphi(\alpha_j)}{\partial\alpha_i} \\ & + [\beta_g, \alpha_g] \cdots [\beta_1, \alpha_1]\varphi(\alpha_j)[\alpha_1, \beta_1] \cdots [\alpha_{i-1}, \beta_{i-1}] \\ & - [\beta_g, \alpha_g] \cdots [\beta_1, \alpha_1]\varphi(\alpha_j)[\alpha_1, \beta_1] \cdots [\alpha_{i-1}, \beta_{i-1}]\alpha_i\beta_i\overline{\alpha}_i. \end{aligned}$$

Since $\mathfrak{a}\Big(\overline{\dfrac{\partial\varphi(\alpha_j)}{\partial\alpha_i}}\Big) = m_{i,j}$ and $\mathfrak{a}(\varphi(\alpha_j)) = x_j$, the (i,j)-component of the matrix $\overline{r}(\varphi\tau_\zeta)$ is expressed as

$$\mathfrak{a}\Big(\overline{\frac{\partial(\varphi\tau_\zeta(\alpha_j))}{\partial\alpha_i}}\Big) = m_{i,j} - (1-\overline{y}_i)(1-\overline{x}_j) \quad i,j = 1,\ldots,g\,.$$

Similarly we have the following $(i,j = 1,\ldots,g)$.

$$\mathfrak{a}\Big(\overline{\frac{\partial(\varphi\tau_\zeta(\beta_j))}{\partial\alpha_i}}\Big) = m_{i,g+j} - (1-\overline{y}_i)(1-\overline{y}_j)$$
$$\mathfrak{a}\Big(\overline{\frac{\partial(\varphi\tau_\zeta(\alpha_j))}{\partial\beta_i}}\Big) = m_{g+i,j} + (1-\overline{x}_i)(1-\overline{x}_j)$$
$$\mathfrak{a}\Big(\overline{\frac{\partial(\varphi\tau_\zeta(\beta_j))}{\partial\beta_i}}\Big) = m_{g+i,g+j} + (1-\overline{x}_i)(1-\overline{y}_j)$$

Thus we obtain the matrix $\overline{r}(\varphi\tau_\zeta)$

$$(3.2)\qquad \overline{r}(\varphi\tau_\zeta) = \begin{pmatrix} m_{1,1} & \cdots & m_{1,2g} \\ \vdots & \ddots & \vdots \\ m_{2g,1} & \cdots & m_{2g,2g} \end{pmatrix} + \begin{pmatrix} -b_1 \\ \vdots \\ -b_g \\ a_1 \\ \vdots \\ a_g \end{pmatrix} (a_1 \; \cdots \; a_g \; b_1 \; \cdots \; b_g)\,.$$

We note that $\mathbb{Z}[H]$ is an integral domain. By comparing (3.1) with (3.2), we arrive at the following equation.

$$\begin{pmatrix} m_{1,1} & \cdots & m_{1,2g} \\ \vdots & \ddots & \vdots \\ m_{2g,1} & \cdots & m_{2g,2g} \end{pmatrix} \begin{pmatrix} -b_1 \\ \vdots \\ -b_g \\ a_1 \\ \vdots \\ a_g \end{pmatrix} = \begin{pmatrix} -b_1 \\ \vdots \\ -b_g \\ a_1 \\ \vdots \\ a_g \end{pmatrix}$$

Hence we can conclude

$$\overline{r}(\varphi)v = v.$$

This completes the proof. ∎

We define the matrix P_1 as follows. First, we consider the following $2g$ elements

$$\gamma'_j = [\beta_g,\alpha_g]\cdots[\beta_j,\alpha_j]\,, \quad \gamma'_{g+j} = \beta_j \quad (j = 1,\ldots,g),$$

and set

$$P_1 = \mathfrak{a}\left(\overline{\frac{\partial\gamma'_j}{\partial\gamma_i}}\right)_{i,j} = \begin{pmatrix} P_{11} & 0 \\ P_{21} & I_g \end{pmatrix}$$

https://doi.org/10.1017/S0004972700033475 Published online by Cambridge University Press

where

$$P_{11} = -\begin{pmatrix} b_1 & & & 0 \\ b_2 & b_2 & & \\ \vdots & \vdots & \ddots & \\ b_g & b_g & \cdots & b_g \end{pmatrix}, \quad P_{21} = \begin{pmatrix} a_1 & & & 0 \\ a_2 & a_2 & & \\ \vdots & \vdots & \ddots & \\ a_g & a_g & \cdots & a_g \end{pmatrix}.$$

We can easily check that $P_1 e_1 = v$ where $e_1 = {}^t(1\ 0\ \cdots\ 0)$. However, we note that $\pi_1(\Sigma_{g,1})$ is not generated by γ'_j, and also that $P_1{}^{-1}$ is not contained in $GL(2g; \mathbb{Z}[x_i{}^{\pm 1}, y_i{}^{\pm 1}])$ but $GL\Big(2g; \mathbb{Z}\big[x_i{}^{\pm 1}, y_i{}^{\pm 1}, 1/(1-y_i)\big]\Big)$. Therefore we have the following.

Corollary 3.2. *For any element $\varphi \in \mathcal{I}_{g,1}$, the non-singular matrix P_1 satisfies the equation*

$$P_1{}^{-1}\,\overline{r}(\varphi)\,P_1 = \left(\begin{array}{c|c} 1 & \rho_a(\varphi) \\ \hline 0 & \\ \vdots & \rho_A(\varphi) \\ 0 & \end{array}\right).$$

The Burau representation is the direct sum of its 1-dimensional trivial subrepresentation and the quotient representation. The Gassner representation has the same property. However, The Magnus representation of the Torelli group does not have this property. In fact, we can deduce the following result.

Proposition 3.3. *The Magnus representation of the Torelli group is not completely reducible.*

Proof: To get a contradiction, suppose that $\overline{r}$ is completely reducible. Then there exists a non-singular matrix U such that

$$U^{-1}\left(\begin{array}{c|c} 1 & \rho_a(\varphi) \\ \hline 0 & \\ \vdots & \rho_A(\varphi) \\ 0 & \end{array}\right) U = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \rho_A{}'(\varphi) & \\ 0 & & & \end{array}\right). \tag{3.3}$$

The coefficients of U belong to a field including $\mathbb{Z}[H]$. By setting

$$U = \left(\begin{array}{c|c} U_1 & U_2 \\ \hline U_3 & U_4 \end{array}\right),$$

the equation (3.3) implies

$$\rho_a(\varphi) U_3 = 0 \tag{3.4}$$

$$\rho_A(\varphi) U_4 = U_4 \rho_A{}'(\varphi). \tag{3.5}$$

https://doi.org/10.1017/S0004972700033475 Published online by Cambridge University Press

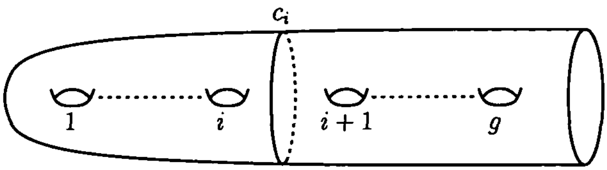

Figure 2: Simple closed curve c_i

On the other hand, we set φ_i and ν_i are the Dehn twists about simple closed curves c_i and n_i as depicted in Figure 2 and Figure 3 respectively and by direct calculation we have

$$\varphi_i(\alpha_j) = \begin{cases} \delta^{-1}\alpha_j\delta & j \leqslant i \\ \alpha_j & j > i \end{cases} \qquad \varphi_i(\beta_j) = \begin{cases} \delta^{-1}\beta_j\delta & j \leqslant i \\ \beta_j & j > i \end{cases}$$

$$\nu_i(\alpha_j) = \begin{cases} \alpha_i{\beta_i}^{-1}\alpha_{i+1}\beta_{i+1}{\alpha_{i+1}}^{-1} & j = i \\ \alpha_{i+1}{\beta_{i+1}}^{-1}{\alpha_{i+1}}^{-1}\beta_i\alpha_{i+1} & j = i+1 \\ \alpha_j & \text{otherwise} \end{cases}$$

$$\nu_i(\beta_j) = \begin{cases} \alpha_{i+1}{\beta_{i+1}}^{-1}{\alpha_{i+1}}^{-1}\beta_i\alpha_{i+1}\beta_{i+1}{\alpha_{i+1}}^{-1} & j = i \\ \beta_j & \text{otherwise} \end{cases}$$

$${\nu_i}^{-1}(\alpha_j) = \begin{cases} \alpha_i\alpha_{i+1}{\beta_{i+1}}^{-1}{\alpha_{i+1}}^{-1}\beta_i & j = i \\ {\beta_i}^{-1}\alpha_{i+1}\beta_{i+1} & j = i+1 \\ \alpha_j & \text{otherwise} \end{cases}$$

$${\nu_i}^{-1}(\beta_j) = \begin{cases} {\beta_i}^{-1}\alpha_{i+1}\beta_{i+1}{\alpha_{i+1}}^{-1}\beta_i\alpha_{i+1}{\beta_{i+1}}^{-1}{\alpha_{i+1}}^{-1}\beta_i & j = i \\ \beta_j & \text{otherwise} \end{cases}$$

Here $\delta = [\alpha_1, \beta_1][\alpha_2, \beta_2]\cdots[\alpha_i, \beta_i]$. Thus we obtain

$$\rho_a(\varphi_i) = (\underbrace{0, \ldots, 0}_{g-1}, b_1, \ldots, b_i, \underbrace{0, \ldots, 0}_{g-i}) \quad i = 1, \ldots, g$$

$$\rho_a(\nu_i\varphi_i{\nu_i}^{-1}) =$$
$$(\underbrace{0, \ldots, 0}_{i-1}, -b_ib_{i+1}, \underbrace{0, \ldots, 0}_{g-i-1}, b_1, \ldots, b_{i-1}, p_i, \underbrace{0, \ldots, 0}_{g-i}) \; i = 1, \ldots, g-1$$

where $p_i = \overline{y}_{i+1}(1 - \overline{y}_i)(\overline{y}_i + \overline{y}_{i+1} - \overline{y}_i\overline{y}_{i+1})$. Since we have the equation (3.4) for any element $\varphi \in \mathcal{I}_{g,1}$, U_3 is a zero column vector. Then we get $\det U = U_1 \cdot \det U_4 \neq 0$ and it implies that U_4 is a non-singular matrix. Therefore, by (3.5),

$$\rho_A'(\varphi) = {U_4}^{-1}\rho_A(\varphi)U_4.$$

https://doi.org/10.1017/S0004972700033475 Published online by Cambridge University Press

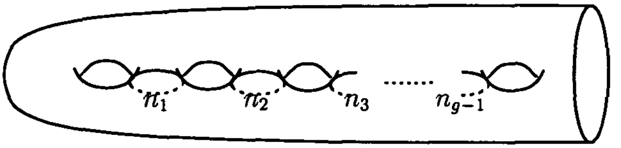

Figure 3: Simple closed curve n_i

This means that $\rho_A{}'$ is conjugate to ρ_A. By explicit calculation we get $\rho_A(\tau_\zeta) = I_{2g-1}$. Thus we can conclude $\rho_A{}'(\tau_\zeta) = I_{2g-1}$ and $\tau_\zeta \in \ker \bar{r}$. However, we can check that τ_ζ is not an element of $\ker \bar{r}$. This is a contradiction. □

4. Reducibility of the quotient representation ρ_A

The quotient representation ρ_A is irreducible for $g = 1$. However, for higher genera, the quotient representation ρ_A is reducible. That is, we have the following proposition.

Proposition 4.1. *The representation ρ_A has a $(2g-2)$-dimensional subrepresentation ρ_B for $g \geqslant 2$.*

Proof: We have defined the matrix $P_1 \in GL(2g; R)$ such that

$$P_1{}^{-1}\,\bar{r}(\varphi)\,P_1 = \left(\begin{array}{c|c} 1 & \rho_a(\varphi) \\ \hline 0 & \\ \vdots & \rho_A(\varphi) \\ 0 & \end{array}\right).$$

We remark that ρ_A is a homomorphism and that ρ_a is a crossed homomorphism. That is to say, for any elements $\varphi, \psi \in \mathcal{I}_{g,1}$

$$\rho_A(\varphi\psi) = \rho_A(\varphi)\rho_A(\psi)\ , \quad \rho_a(\varphi\psi) = \rho_a(\psi) + \rho_a(\varphi)\rho_A(\psi).$$

Let τ_ζ be the Dehn twist along a simple closed curve on $\Sigma_{g,1}$ which is parallel to the boundary as before. Since

$$\rho_A(\tau_\zeta) = I_{2g-1}\ , \quad \rho_a(\tau_\zeta) = (\underbrace{0, \ldots, 0}_{g-1}, b_1, \ldots, b_g),$$

we have

$$\begin{aligned} \rho_a(\tau_\zeta\varphi) &= \rho_a(\varphi) + \rho_a(\tau_\zeta)\rho_A(\varphi) \\ \rho_a(\varphi\tau_\zeta) &= \rho_a(\tau_\zeta) + \rho_a(\varphi)\rho_A(\tau_\zeta) \\ &= \rho_a(\tau_\zeta) + \rho_a(\varphi)\ . \end{aligned}$$

We recall $\tau_\zeta \varphi = \varphi \tau_\zeta$ for any element $\varphi \in \mathcal{I}_{g,1}$, because τ_ζ is central in $\mathcal{M}_{g,1}$. Then we get

$$\rho_a(\tau_\zeta)\rho_A(\varphi) = \rho_a(\tau_\zeta)$$
$${}^t\rho_A(\varphi){}^t\rho_a(\tau_\zeta) = {}^t\rho_a(\tau_\zeta).$$

This means that ${}^t\rho_a(\tau_\zeta)$ is an eigenvector of the matrix ${}^t\rho_A(\varphi)$ with eigenvalue 1 for any $\varphi \in \mathcal{I}_{g,1}$. Therefore there exists a non-singular $(2g-1)\times(2g-1)$-matrix Q_1 such that

$$Q_1^{-1}\,{}^t\rho_A(\varphi)\,Q_1 = \left(\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & {}^t\rho_B(\varphi) \\ 0 & \end{array}\right)$$

$${}^tQ_1\,\rho_A(\varphi)\,{}^tQ_1^{-1} = \left(\begin{array}{c|c} 1 & 0 \quad \cdots \quad 0 \\ \hline & \\ * & \rho_B(\varphi) \\ & \end{array}\right).$$

By putting $Q = {}^tQ_1{}^{-1} \cdot Q_2{}^{-1}$, where

$$Q_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

we obtain

$$Q^{-1}\,\rho_A(\varphi)\,Q = \left(\begin{array}{c|c} \rho_B(\varphi) & * \\ & \\ \hline 0 \quad \cdots \quad 0 & 1 \end{array}\right).$$

This equation means that the representation ρ_A has a $(2g-2)$-dimensional subrepresentation ρ_B. This completes the proof. ∎

REMARK 4.2. We define a non-singular matrix Q_1 by

$$Q_1 = \begin{pmatrix} 0 & I_{g-1} \\ -P_{11} & 0 \end{pmatrix}.$$

Then we can easily check that $Q_1 e_1 = {}^t\rho_a(\tau_\zeta)$ where $e_1 = {}^t(1\ 0\ \cdots\ 0)$. In this way we can construct $(2g-1)\times(2g-1)$-matrices Q_1 and Q, which appear in Proposition 4.1, explicitly.

5. Irreducibility of the representation ρ_B

Combining Corollary 3.2 and Proposition 4.1, we obtain the following main result of this paper.

Theorem 5.1. *For $g \geqslant 2$ there exsists a non-singular matrix $P \in GL(2g; R)$ such that for any element $\varphi \in \mathcal{I}_{g,1}$*

$$P^{-1}\,\overline{r}(\varphi)\,P = \left(\begin{array}{c|ccc|c} 1 & & * & & * \\ \hline 0 & & & & \\ \vdots & & \rho_B(\varphi) & & * \\ & & & & \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right).$$

Moreover,

$$\rho_B : \mathcal{I}_{g,1} \longrightarrow GL(2g-2; R)$$

is a $(2g-2)$-dimensional irreducible representation of $\mathcal{I}_{g,1}$.

We have only to set $P = P_1 \cdot (I_1 \oplus Q)$ to obtain the above decomposition. It remains to prove the irreduciblity of ρ_B.

Let $\mathbb{C}^n$ be the n dimensional complex vector space consisting of column vectors. We denote by ${}^t\mathbb{C}^n$ the transposed vector space consisting of row vectors. A matrix $X \in M(n;\mathbb{C})$ is called a *pseudoreflection* if $X - I_n$ has rank 1. If X is a pseudoreflection, then

$$X = I_n - AB,$$

for some $A \in \mathbb{C}^n$ and $B \in {}^t\mathbb{C}^n$.

Theorem 5.2. (Formanek [4]) *Let $X_1 = I_n - A_1B_1, \dots, X_n = I_n - A_nB_n$ be n invertible pseudoreflections in $M(n;\mathbb{C})$, where $n \geqslant 2$. Let Γ be the directed graph whose vertices are $1, 2, \dots, n$, and which has a directed edge from i to j ($i \neq j$) precisely when $B_iA_j \neq 0$. Let G be the subgroup of $GL(n;\mathbb{C})$ generated by $X_1, \dots, X_n$. Then the following are equivalent.*

1. $\mathrm{Span}_{\mathbb{C}}\{G\} = M(n;\mathbb{C})$
2. *For each $i \neq j$, $1 \leqslant i, j \leqslant n$, the graph Γ contains a directed path from i to j, $\{A_1, \dots, A_n\}$ is a basis for $\mathbb{C}^n$ and $\{B_1, \dots, B_n\}$ is a basis for ${}^t\mathbb{C}^n$.*
3. *For each $i \neq j$, $1 \leqslant i, j \leqslant n$, the graph Γ contains a directed path from i to j and $\det(B_iA_j)_{i,j} \neq 0$.*

This theorem gives a criterion for a certain representation to be irreducible. This is because, for a group G' and a representation $T : G' \to GL(n;\mathbb{C})$, if $\mathrm{Im}T$ spans $M(n;\mathbb{C})$, then T is an irreducible representation (see [3]). We shall prove that ρ_B is irreducible by making use of this method.

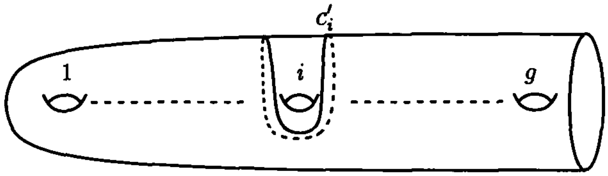

Figure 4: Simple closed curve c'_i

PROOF OF THE IRREDUCIBILITY OF ρ_B: Let $\rho_{B(z,w)}$ be the representation

$$\rho_{B(z,w)} : \mathcal{I}_{g,1} \longrightarrow GL(2g-2;\mathbb{C})$$

obtained by specialising $x_i \mapsto z_i$, $y_i \mapsto w_i$ in ρ_B, where $(z,w) = (z_1,\dots,z_g,w_1,\dots,w_g)$ are complex numbers $z_i \in \mathbb{C}\setminus\{0\}$, $w_i \in \mathbb{C}\setminus\{0,1\}$. We denote by L_g the subgroup of $\mathcal{I}_{g,1}$ generated by the $2g-2$ elements $\nu_1\varphi_1{\nu_1}^{-1}$, φ'_2, $\nu_2\varphi_2{\nu_2}^{-1}$, φ'_3, $\cdots$, $\nu_{g-1}\varphi_{g-1}{\nu_{g-1}}^{-1}$, φ'_g. Let us simply write $\psi_1,\psi_2,\dots,\psi_{2g-2}$ for them. Here φ'_i is the Dehn twist about a simple closed curve c'_i as shown in Figure 4. We note the action of φ'_i on the generators γ_j.

$$\varphi'_i(\alpha_j) = \begin{cases} [\beta_i,\alpha_i]\alpha_i[\alpha_i,\beta_i] & j = i \\ \alpha_j & \text{otherwise} \end{cases}$$

$$\varphi'_i(\beta_j) = \begin{cases} [\beta_i,\alpha_i]\beta_i[\alpha_i,\beta_i] & j = i \\ \beta_j & \text{otherwise} \end{cases}$$

Because all ψ_i are BSCC maps, that is, the Dehn twists along 0-homologous simple closed curves, the matrices $\rho_{B(z,w)}(\psi_i)$ are pseudoreflections, namely $\rho_{B(z,w)}(\psi_i) = I_{2g-2} + A_iB_i$ for some $A_i \in \mathbb{C}^{2g-2}$, $B_i \in {}^t\mathbb{C}^{2g-2}$. By direct calculation, we get

$$A_{2k-1} = {}^t\left(0,\dots,0,d'_k\overset{k}{\check{d}}_{k+1},0,\dots,0,\overset{g+k-1}{\check{-d_{k+1}-\frac{w_{k+1}}{w_k}}},-d'_k\overset{g+k}{\check{w}}_{k+1},0,\dots,0\right)$$

$$A_{2k} = {}^t(0,\dots,0,\overset{g+k}{\check{1}},\overset{g+k+1}{\check{-1}},0,\dots,0)$$

$$B_{2k-1} = \left(0,\dots,0,-w_k\overset{k-1}{\check{d}}{}'_{k+1},\overset{k}{\check{-d_k-\frac{w_k}{w_{k+1}}}},0,\dots,0,-d_k\overset{g+k-1}{\check{d}}{}'_{k+1},0,\dots,0\right)$$

$$B_{2k} = \begin{cases} (0,\dots,0,\overset{k}{\check{1}},\overset{k+1}{\check{-1}},0,\dots,0) & 1 \leqslant k \leqslant g-2 \\ (0,\dots,0,\overset{g-1}{\check{1}},0,\dots,0) & k = g-1 \end{cases}$$

where $d_k = 1 - w_k, d'_k = 1 - (1/w_k)$. We remark that z_k does not appear in A_i and B_i. Then we have the $(2g-2) \times (2g-2)$-matrix $(B_iA_j)_{i,j}$ as follows:

$$(B_iA_j)_{i,j} = \begin{pmatrix} 0 & d_1d'_2 & & & & & & 0 \\ -d'_1d_2 & 0 & d'_2d_3 & & & & & \\ & -d_2d'_3 & 0 & d_2d'_3 & & & & \\ & & -d'_2d_3 & 0 & d'_3d_4 & & & \\ & & & -d_3d'_4 & 0 & & & \\ & & & & & \ddots & & \\ & 0 & & & & & 0 & d_{g-1}d'_g \\ & & & & & & -d'_{g-1}d_g & 0 \end{pmatrix}.$$

Thus we obtain

$$\det(B_iA_j) = \frac{(1-w_1)^2(1-w_2)^4\cdots(1-w_{g-1})^4(1-w_g)^2}{w_1{w_2}^2\cdots{w_{g-1}}^2w_g}$$

and the associated graph can be expressed as

1 2 3 $2g-3$ $2g-2$

Therefore by virtue of Theorem 5.2, the representation $\rho_{B(z,w)}|_{L_g}$ is irreducible. Hence $\rho_{B(z,w)}$ and ρ_B are irreducible. □

REMARK 5.3. If there exists a single (z,w) such that $\rho_{B(z,w)}|_{L_g}$ is irreducible, then we can conclude ρ_B is irreducible. However, the above proof shows that $\rho_{B(z,w)}|_{L_g}$ is irreducible for any $z_i \in \mathbb{C} \setminus \{0\}$ and $w_i \in \mathbb{C} \setminus \{0,1\}$.

6. Some remarks and applications

This section is devoted to give some remarks and applications of the irreducible decomposition.

Let $\mathcal{I}_g$ be the Torelli group of Σ_g and $\mathcal{I}_{g,*}$ that of Σ_g relative to the base point $* \in \Sigma_g$. We have the following by explicit computations.

$$\rho_B(\tau_\zeta) = I_{2g-2} \tag{6.6}$$

$$\rho_B(\widetilde{\alpha}_i) = \overline{x}_i I_{2g-2} \tag{6.7}$$

$$\rho_B(\widetilde{\beta}_i) = \overline{y}_i I_{2g-2} \tag{6.8}$$

Here $\widetilde{\alpha}_i \in \mathcal{I}_{g,1}$ (respectively $\widetilde{\beta}_i \in \mathcal{I}_{g,1}$) is a product of the Dehn twist along α_{i+} (respectively β_{i+}) and the inverse of the Dehn twist along α_{i-} (respectively β_{i-}) as depicted in Figure 5 (respectively Figure 6). The equation (6.6) means that ρ_B factors through $\mathcal{I}_{g,*}$.

$$\rho_B : \mathcal{I}_{g,*} \longrightarrow GL(2g-2; R)$$

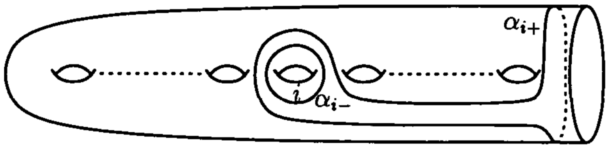

Figure 5: Simple closed curves α_{i+} and α_{i-}

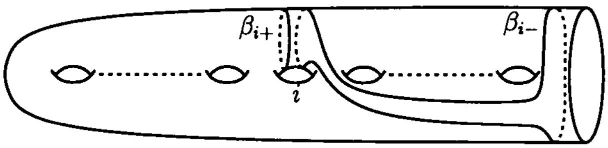

Figure 6: Simple closed curves β_{i+} and β_{i-}

Moreover, the equations (6.6), (6.7) and (6.8) show that ρ_B does not factor through $\mathcal{I}_g$. However, we write $PGL(2g-2;R)$ for the quotient group of $GL(2g-2;R)$ by all scalar matrices, then we obtain the following representation

$$\rho_B : \mathcal{I}_g \longrightarrow PGL(2g-2;R).$$

From our previous paper [**8**], we know that the kernel of the Magnus representation of the Torelli group is non-trivial for $g \geqslant 2$. As an application of the irreducible decomposition, we shall obtain additional information about the kernel. To be more precise we can mention the relation between $\ker \overline{r}$ and the lower central series of $\mathcal{I}_{g,1}$. We denote by $\mathcal{I}_{(n)}$ the n-th term in the lower central series of $\mathcal{I}_{g,1}$ so that $\mathcal{I}_{(0)} = \mathcal{I}_{g,1}$ and $\mathcal{I}_{(n)} = [\mathcal{I}_{(n-1)}, \mathcal{I}_{(0)}]$.

PROPOSITION 6.1. *There exists no natural number $n \in \mathbb{N}$ so that*

$$\mathcal{I}_{(n)} \subset \ker \overline{r}.$$

PROOF: We define f_n inductively as follows.

$$f_1 = [\varphi_1, \widetilde{\alpha}_1], f_2 = [f_1, \widetilde{\beta}_1], \ldots, f_{2k-1} = [f_{2k-2}, \widetilde{\alpha}_1], f_{2k} = [f_{2k-1}, \widetilde{\beta}_1], \ldots .$$

Here φ_1, $\widetilde{\alpha}_1$, $\widetilde{\beta}_1$ are as above. Then f_n is an element of $\mathcal{I}_{(n)}$. Now we use the irreducible decomposition. Let $\overline{r}'$ be the equivalent representation to $\overline{r}$ so that

$$\overline{r}'(\varphi) = P^{-1}\overline{r}(\varphi)P.$$

By explicit calculation and an inductive proof, we get the following (i,j)-components of the matrix $\overline{r}'(f_n)$.

$$\overline{r}'(f_{2k-1})_{i,j} = \begin{cases} 1 & i = j \\ -\overline{a}_1{}^k \overline{b}_1{}^{k-1} & i = 1, j = 2 \\ -a_1{}^k b_1{}^{k-1} & i = g+1, j = 2g \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{r}'(f_{2k})_{i,j} = \begin{cases} 1 & i = j \\ -\overline{a}_1{}^k \overline{b}_1{}^{k} & i = 1, j = 2 \\ -a_1{}^k b_1{}^{k} & i = g+1, j = 2g \\ 0 & \text{otherwise} \end{cases}$$

Thus for any natural number $n \in \mathbb{N}$, we can deduce

$$f_n \notin \ker \overline{r}.$$

This completes the proof. ∎

References

[1] M. Abdulrahim, 'Complex specialization of the reduced Gassner representation of the pure braid group', *Proc. Amer. Math. Soc.* **125** (1997), 1617–1624.

[2] J. Birman, *Braids, links and mapping class groups*, Annals of Math. Studies No. **82** (Princeton University Press, Princeton N.J., 1975).

[3] L. Dornhoff, *Group representation theory Part A* (Marcel Dekke Inc., New York, 1971).

[4] E. Formanek, 'Braid group representations of low degree', *Proc. London Math. Soc.* **73** (1996), 279–322.

[5] D. Johnson, 'A survey of the Torelli group', *Contemporary Math.* **20** (1983), 165–179.

[6] S. Morita, 'Abelian quotients of subgroups of the mapping class group of surfaces', *Duke Math. J.* **70** (1993), 699–726.

[7] S. Morita, 'Structure of the mapping class groups of surfaces: a survey and a prospect', in *Proceedings of the Kirbyfest (1999)*, Geom. Topol. Monogr. Vol. 2 (Geom. Topol. Publ., Coventyr), pp. 349–406.

[8] M. Suzuki, 'The Magnus representation of the Torelli group $\mathcal{I}_{g,1}$ is not faithful for $g \geqslant 2$', *Proc. Amer. Math. Soc.* **130** (2002), 909–914.

Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba
Meguro
Tokyo 153-8914
Japan
e-mail: macky@ms.u-tokyo.ac.jp

https://doi.org/10.1017/S0004972700033475 Published online by Cambridge University Press