Canad. Math. Bull. Vol. **18** (1), 1975

REGULAR, COMMUTATIVE, MAXIMAL SEMIGROUPS OF QUOTIENTS

BY
JURGEN ROMPKE

1. **Introduction.** A well-known theorem which goes back to R. E. Johnson [4], asserts that if R is a ring then $Q(R)$, its maximal ring of quotients is regular (in the sense of v. Neumann) if and only if the singular ideal of R vanishes. In the theory of semigroups a natural question is therefore the following: Do there exist properties which characterize those semigroups whose maximal semigroups of quotients are regular? Partial answers to this question have been given in [3], [7] and [8]. In this paper we completely solve the commutative case, i.e. we give necessary and sufficient conditions for a commutative semigroup S in order that $Q(S)$, the maximal semigroup of quotients, is regular. These conditions reflect very closely the property of being semiprime, which in the theory of commutative rings characterizes those rings which have a regular ring of quotients.

2. **Preliminaries.** All semigroups considered in this paper are commutative: they are not required to have a zero or an identity. We briefly sketch the construction of the maximal semigroups of quotients; all details and further properties are found in [1] or [6].

Let S be a semigroup. An ideal D in S is said to be dense if and only if for s, $t \in S$ the equations $sd=td$ for all $d \in D$ imply $s=t$. With D_1 and D_2 the ideals D_1D_2 and $D_1 \cap D_2$ are dense, as well as any ideal containing a dense ideal. The semigroup S is called reductive if it contains at least one dense ideal.

For each dense ideal D in S, we denote with $\mathrm{Hom}_S(D, S)$ the set of all S-homomorphisms from D into S, i.e. all those maps $f: D \to S$ satisfying $f(ds)=f(d)s$ for all $d \in D$ and all $s \in S$. Each such $f: D \to S$ is called a fraction. For a reductive semigroup S denote with $H(S)$ the union of all $\mathrm{Hom}_S(D, S)$, D a dense ideal. $H(S)$ is a semigroup by composing $f_1: D_1 \to S$ and $f_2: D_2 \to S$ in the following way: $f_1f_2: D_1D_2 \to S$ by $f_1f_2(d_1d_2)=f_1(d_1)f_2(d_2)$. $Q(S)$ the maximal semigroup of quotients is $H(S)$ modulo the congruence which identifies two fractions if and only if they agree on some dense ideal.

To each element $s \in S$ there corresponds the fraction $\hat{s}: S \to S$ defined by $\hat{s}(x)=sx$. The map which associates with each $s \in S$ the congruence class containing $\hat{s}$, is an embedding of S into $Q(S)$.

We shall largely be concerned with separative semigroups, i.e. those semigroups

Received by the editors April 27, 1973.

where the equation $x^2=xy=y^2$ implies $x=y$. Let us shortly recall their structure (see [2, ch. 4.3] for more details): Each separative semigroup S is a semilattice of cancellative, archimedean semigroups, i.e. $S=\bigcup_{\alpha\in Y} S_\alpha$, where the union is disjoint, Y is a (lower) semilattice, and $S_\alpha S_\beta \subseteq S_{\alpha\wedge\beta}$. Each S_α, $\alpha\in Y$, is a cancellative subsemigroup of S and each S_α is archimedean, i.e. each element divides some power of every other element.

Each separative semigroup is reductive; this we see as follows: assume $sx=tx$ for all $x\in S$. Then we have in particular $s^2=ts$ and $st=t^2$ and therefore $s=t$; so S is dense in itself.

We need one last definition: we call a semigroup S regular if and only if to each element $s\in S$ there exists $s'\in S$ such that $ss's=s$ and $s'ss'=s'$.

3. **Some lemmata.** If S is reductive and $Q(S)$ is regular, then $Q(S)$ is a semilattice of groups. Since S can always be embedded into $Q(S)$, we have S necessarily separative. Therefore we want to describe in this section how fractions act in separative semigroups. In the following lemmata, let $S=\bigcup_{\alpha\in Y} S_\alpha$ be a separative semigroup with archimedean components S_α and let $f\colon D\to S$ be a fraction.

LEMMA 1. *S satisfies the following cancellation law:*
If $a^n b=a^n c$ for some natural number n then $ab=ac$.

Proof. We assume that $n>1$. Then

$$(ab)^n = a^n b^n = a^n b\cdot b^{n-1} = a^n c\cdot b^{n-1} = ac(ab)^{n-1}$$
$$(ac)^n = a^n c^n = a^n c\cdot c^{n-1} = a^n b\cdot c^{n-1} = ab(ac)^{n-1}$$

The equality follows now from [3, Cor. 4.15], q.e.d.

LEMMA 2. *If $s\in D^2\cap S_\alpha$ and if $f(s)\in S_\beta$ then $\beta\le\alpha$.*

Proof. Since $s\in D^2$, we have $f[f(s)]$ well defined. Let $f[f(s)]\in S_\gamma$ then

$$s\cdot f[f(s)] = f(s)\cdot f(s)\in S_{\gamma\wedge\alpha}\cap S_\beta$$

Hence $\gamma\wedge\alpha=\beta$ or $\beta\le\alpha$, q.e.d.

LEMMA 3. *If $s\in D^2\cap S_\alpha$ and $f(s)\in S_\beta$ then $f(D^2\cap S_\alpha)\subseteq S_\beta$.*

Proof. For $n>1$ we have

$$[f(s^n)]^2 = f[f(s^{2n})] = s^{2n-2}f[f(s^2)]$$
$$= s^{2n-2}[f(s)]^2\in S_\alpha S_\beta\subseteq S_\beta$$

Hence $f(s^n)\in S_\beta$ for all natural numbers n. Now let $a\in D^2\cap S_\alpha$ be arbitrary. Since S_α is archimedean there exists $x\in S_\alpha$ such that $ax=s^m$ for some $m\in\mathbb{N}$. If $f(a)\in S_\gamma$, then we know from lemma 1 that $\gamma\le\alpha$. Hence

$$f(ax) = f(a)x = f(s^m)\in S_{\gamma\wedge\alpha}\cap S_\beta = S_\gamma\cap S_\beta$$

Therefore $\gamma=\beta$, q.e.d.

LEMMA 4. *If $f(D^2)\cap S_\alpha\neq\varnothing$ then $f(D^2\cap S_\alpha)\subseteq S_\alpha$.*

Proof. Let $a \in D^2$ and $f(a) \in S_\alpha$. By lemma 2 we have $f(a^2)=af(a) \in D^2 \cap S_\alpha$. Therefore $f[af(a)]=[f(a)]^2 \in S_\alpha$ and the rest follows from lemma 3, q.e.d.

REMARK. The restriction of D^2 rather than D appearing in the lemmata cannot be removed: Consider for example F the free abelian semigroup on the generators x and y. Then since F is cancellative, every ideal is dense, in particular (x) the principal ideal generated by x. Define $f:(x)\to S$ by $f(x)=y$. Then $f(x) \in S_\beta$, with $\beta \not\leq \alpha$, since the archimedean subsemigroup containing y is $\{y^n \mid n \in \mathbb{N}\}$.

4. **The main theorem.** We are now ready to state and prove the main theorem of this paper.

THEOREM 1. *For a (commutative) semigroup S the following are equivalent statements:*

(1) *S is reductive and $Q(S)$ is regular.*
(2) *S is separative and for $a \in S$ the ideal $\Gamma(a)=\{s \in S \mid$ there exists $b \in aS$ such that $bt=st$ for all $t \in aS\}$ is dense.*
(3) *S is separative, $S=\bigcup_{\alpha\in Y} S_\alpha$, and to every $a \in S$, $a \in S_\alpha$ say, there exists a dense ideal $D[a]$ such that for all $x \in D[a]$, $x \in S_\xi$ say, there exists $w \in S_{\alpha\wedge\xi}$ such that $xa=wa$.*

Before proving this theorem some remarks seem appropriate:

REMARK 1. S separative does not imply that $Q(S)$ is regular as one sees from the following example: Let $S=S_0 \cup S_1$, where $S_0=\{a^n \mid n \in \mathbb{N}\}$, $S_1=\{b^n \mid n \in \mathbb{N}\}$, both copies of the infinite cyclic semigroup. Define $a^n \cdot b^m=a^n$, i.e. $S_0S_1 \subseteq S_0$. Then the fraction $\hat{a}:S\to S$, defined by $\hat{a}(s)=as$, cannot have a regular inverse: Suppose $\hat{a}f\hat{a}(d)=\hat{a}(d)$ for all d in some dense ideal D. Since each dense ideal must contain some elements of S_1 we have for some $m \in \mathbb{N}$:

$$a^2f(b^m) = \hat{a}f\hat{a}(b^m) = \hat{a}(b^m) = a$$

But this equation is impossible whether $f(b^m) \in S_0$ or $f(b^m) \in S_1$. So $Q(S)$ is not regular.

REMARK 2. If S is the multiplicative semigroup of a (commutative) ring R, we have S separative if and only if R is semiprime, a property which is equivalent to having a regular quotient ring [5, §2.4]. The condition that $\Gamma(a)$ has to be dense, corresponds to the fact that in a semiprime ring all ideals of the form $K+\text{ann}(K)$, K any ideal, are dense. If again S is the multiplicative semigroup of the ring R, then we can express $\Gamma(a)=aR+\text{ann}(a)$. It is however not true that ΓI (analogously defined as $\Gamma(a)$) is dense for every ideal I in a semigroup S.

REMARK 3. If S is a semilattice of groups, i.e. S is regular already, then $\Gamma(a)$ is always dense, and we get (for the commutative case) the statements already proven in [7].

Proof of theorem 1.

(1)⇒(2): If $Q(S)$ is regular, then $\hat{a}: S \to S$ has a regular inverse f. So on some dense ideal D we have $\hat{a}f\hat{a}(d)=\hat{a}(d)$, for all $d \in D$. We show that $D \subseteq \Gamma(a)$, which then makes $\Gamma(a)$ dense. For $d \in D$ we have $af(d)a=da$. Multiplying with $s \in S$ we get $af(d) \cdot as = d \cdot as$. This means we can simulate the multiplication of d with elements of aS by $af(d)$, and element in aS, i.e. $D \subseteq \Gamma(a)$. That S is separative under the assumption we have seen earlier.

(2)⇒(3): Put $D[a]=\Gamma(a)$ which is dense. Let $x \in D[a]=\Gamma(a)$ be arbitrary. Then there exists $w \in aS$ such that $x \cdot as = w \cdot as$ for all $s \in S$. Hence $xa^2=wa^2$ and by lemma 1 we have $xa=wa$. If $a \in S_\alpha$, $x \in S_\xi$ and $w \in S_\beta$, then we have to show that $\beta = a \wedge \xi$. Since $w \in aS$ we get $\beta \leq \alpha$. We know that $xa \in S_{\alpha \wedge \xi}$ and $wa \in S_{\beta \wedge \alpha} = S_\beta$. Hence $\alpha \wedge \xi = \beta$.

(3)⇒(1): We shall construct for a fraction $f: D \to S$ a regular inverse $g: G \to S$, such that f and fgf agree on some dense ideal.

First of all we define

$$(\ker(f))^* = \{s \in S \mid sa = sb \text{ for all } (a, b) \in \ker(f)\}$$

Clearly $(\ker(f))^*$ is an ideal. It is non-empty since $f(D) \subseteq (\ker(f))^*$. Next we show that f when restricted to $D \cap (\ker(f))^*$ is monomorphic (note that $D \cap (\ker(f))^* \neq \varnothing$ since $Df(D) \subseteq D \cap f(D) \subseteq D \cap (\ker(f))^*$): Assume that $f(d_1)=f(d_2)$ with $d_1, d_2 \in D \cap (\ker(f))^*$. Then $d_1d_2=d_1d_1$ since $d_1 \in (\ker(f))^*$ and $d_2d_2=d_2d_1$ since $d_2 \in (\ker(f))^*$. Now $d_1^2=d_1d_2=d_2^2$ and since S is separative we get $d_1=d_2$.

Denote from now on $E=D \cap (\ker(f))^*$. Let $g': f(E) \to E$ be the inverse mapping of $f \mid E$. g' is clearly an S-homomorphism. We shall show later that g' can be extended to a fraction $g: G \to S$.

We claim next that $f^{-1}f(E)=\{s \in D \mid f(s) \in f(E)\}$ is a dense ideal. Let us therefore assume that $xs=ys$ for all $s \in f^{-1}f(E)$ and that moreover $x, y \in D^3$, a dense ideal. If we can show that these equations imply $x=y$, then, since D^3 is dense, it follows that $f^{-1}f(E)$ is dense. If then $x \in D^3$, $x \in S_\xi$ say, then $f(x) \in D^2$ and $f(x) \in f^{-1}f(E)$. By our hypothesis there exists for $f(x)$, $f(x) \in S_\alpha$ say, a dense ideal $D[f(x)]$ such that for $d \in D[f(x)]$, $d \in S_\delta$ say, there exists w with $dx \cdot f(x) = w \cdot f(x)$. Since $x \in D^3 \subseteq D^2$ we have by lemma 2 that $\alpha \leq \xi$. Clearly $dx \in D[f(x)] \cap S_{\xi \wedge \delta}$ and hence w can be chosen in $S_{\xi \wedge \delta \wedge \alpha} = S_{\alpha \wedge \delta}$. By lemma 4 we have $f[f(dx)] \in S_{\alpha \wedge \delta}$ since $f(dx) = df(x) \in D^2 \cap S_{\alpha \wedge \delta}$. Since $S_{\alpha \wedge \delta}$ is archimedean, there exists $a \in S_{\alpha \wedge \delta}$, $m \in \mathbb{N}$, such that $f[f(dx)] \cdot a = w^m$. Now $w^m \in f(E)$ since $f[f(x)] \in f(E)$ which is an ideal. The equation $dx \cdot f(dx) = w \cdot f(dx)$ implies:

$$f[(dx)^{m+1}] = (dx)^m f(dx) = w^m f(dx) \in f(E)$$

or $(dx)^{m+1} \in f^{-1}f(E)$.

We assumed that x and y act the same on $f^{-1}f(E)$, and therefore we get $x(dx)^{m+1} = y(dx)^{m+1}$ or by lemma 1: $x^2d = xyd$. Since this equation can be derived for every $d \in D[f(x)]$, a dense ideal, we must have $x^2 = xy$. Similarly one shows

$y^2=yx$, and since S is separative we may conclude $x=y$ and hence $f^{-1}f(E)$ is dense.

So far we have established that $fg'f$ and f agree on the dense ideal $f^{-1}f(E)$. It remains to extend g' to a fraction $g:G\to S$ in such a way that fgf and f agree on $f^{-1}f(E)$.

Define the ideal G as follows:

$$G = \Gamma(f(E)) = \{s \in S \mid \text{there exists } b \in f(E) \text{ such that } st = bt \text{ for all } t \in f(E)\}$$

Clearly G contains $f(E)$. We note that the element $b \in f(E)$ on which $s \in G$ founds its existence is unique: let $st=bt=ct$ for $b, c \in f(E)$ and all $t \in f(E)$. Then in particular $sb=b^2=cb$ and $sc=bc=c^2$. Hence $b=c$ and we denote this particular element by b_s. On G we now define $g:G\to S$ by $g(s)=g'(b_s)$. The uniqueness of b_s makes first of all g an extension of g' and secondly makes g into an S-homomorphism: If $st=b_s\cdot t$ then $xst=xb_s\cdot t$ and so $b_{xs}=xb_s$. Now

$$g(sx) = g'(b_{sx}) = g'(b_sx) = g'(b_s)x = g(s)x$$

and g is an S-homomorphism.

By showing that G is dense we complete the proof. Let $xs=ys$ for all $s \in G$. As before, it suffices to take x and y from some dense ideal. In this case we take $x, y \in f^{-1}f(E) \cap D^2$. As before to $f(x)$ exists a dense ideal $D[f(x)]$ such that for $d_1 \in D[f(x)]$ we have

$$d_1xf(d_1x) = wf(d_1x)$$

By lemma 2 and since $x \in D^2$, we can choose w to be in the same archimedean component as $f(d_1x)$. For arbitrary $d_2 \in D^2$ we have

$$d_1d_2\cdot f(d_1d_2x) = d_2w\cdot f(d_1d_2x)$$

with d_2w and $f(d_1d_2x)$ still in the same archimedean component, say S_α. Then we have $S_\alpha \cap f(D^2)\neq\varnothing$ and by lemma 4 we conclude that $f(d_2w) \in S_\alpha$ as well. Since S_α is cancellative we deduce from

$$f(d_1d_2x)f(d_1d_2x) = f(d_2w)f(d_1d_2x)$$

that $f(d_1d_2x)=f(d_2w)$; and then also that

$$f[(d_1d_2x)^n] = f[(d_2w)^n] \quad \text{for all } n \in \mathbb{N}.$$

Since S_α is archimedean, there exist $a \in S_\alpha$ and $m \in \mathbb{N}$ such that $f(d_1d_2x)\cdot a=(d_2w)^m$. Since $f(d_1d_2x) \in f(E)$ we now have $(d_2w)^m \in f(E)$.

We next show that $(d_1d_2x)^m \in G$: Let $t \in f(E)$ be arbitrary. Then

$$\begin{aligned}(d_1d_2x)^mt &= (d_1d_2x)^mfg'(t) = f[(d_1d_2x)^m]g'(t)\\ &= f[(d_2w)^m]g'(t) = (d_2w)^mfg'(t)\\ &= (d_2w)^mt\end{aligned}$$

Since $(d_2w)^m \in f(E)$ we have $(d_1d_2x)^m \in G$. Hence $x(d_1d_2x)^m = y(d_1d_2x)^m$ and by lemma 1 again $x^2d_1d_2 = xyd_1d_2$. Using the denseness of $D[f(x)]$ and D^2 we conclude that $x^2 = xy$. Similar $y^2 = yx$ and since S is separative, we get $x = y$. Therefore G is dense, q.e.d.

References

1. P. Berthiaume, *Generalized Semigroups of Quotients*, Glasgow Math. J. **12** (1971), 150–161.

2. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. I, 1961, Amer. Math. Soc., Providence.

3. C. V. Hinkle, Jr., *Semigroups of Right Quotients of a Semigroup which is a Semilattice of Groups*, Semigroup Forum **5** (1972), 167–173.

4. R. E. Johnson, *The Extended Centralizer of a Ring over a Module*, Proc. Amer. Math. Soc. **2** (1951), 891–895.

5. J. Lambek, *Lectures on Rings and Modules*, 1966, Waltham, Mass.

6. F. R. McMorris, *The Maximal Quotient Semigroup*, Semigroup Forum **4** (1972), 360–364.

7. F. R. McMorris, *The Quotient Semigroup of a Semigroup that is a Semilattice of Groups*, Glasgow Math. J. **12** (1971), 18–23.

8. F. R. McMorris, *The Singular Congruence and the Maximal Quotient Semigroup*, Canad. Math. Bull. **15** (1972) 301–303.

McMaster University,
Hamilton, Ontario, Canada

8000 München 40
Germania Str. 5
Fed. Rep. of Germany