CURVES ON SURFACES OF CONSTANT WIDTH

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Introduction. A surface S of constant width is the boundary of a convex set K of constant width in euclidean 3-dimensional space E^3. (See [1] pp. 127-139.)

Our first result concerns the interdependence of five properties which a curve on such a surface may possess. Let S be a surface of constant width $D > 0$ which satisfies the smoothness condition that it be a 2-dimensional submanifold of E^3 of class C^2. We use the symbols P, E, G, L, *, A to refer to properties of a curve C on S as follows:

Property P: C is planar, i.e. C is the intersection with S of some plane M in E^3 which passes through an interior point of K. Since M is not the unique tangent plane to S at any point of C, C is a simple closed curve of class C^2.

Property E: C is the locus of points of S where the outwardly directed surface normal vector $\underset{\sim}{N}$ satisfies an equation $\underset{\sim}{N} \cdot \underset{\sim}{u} = 0$, for some fixed unit vector $\underset{\sim}{u}$. We claim that C is a simple closed curve, which we shall call an equator of S. For, consider the projection of S onto a plane perpendicular to $\underset{\sim}{u}$. C is the inverse image of the continuous curve C_1 which is the boundary of the image of S. Now a surface S of constant width cannot contain any straight line segments, since for each pair of points of S (or indeed of the corresponding convex body K) there is contained in K a "spindle" formed by intersecting all balls of radius D containing

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the two points ([1] p. 128). Hence each point of C_1 is the image of exactly one point of C. Furthermore, by the same property of S, the natural map from C_1 to C is continuous, which proves that C is a simple closed curve.

Property G: C is a *geodesic*, which we can characterize as a curve of class C^2 on S with $\underset{\sim}{t}'$ parallel to $\underset{\sim}{N}$, where $\underset{\sim}{t}$ is the unit tangent vector to C and $'$ denotes differentiation with respect to arc-length s. Curves with property G we suppose to be already prolonged indefinitely in both directions or to be closed. Any segment of a geodesic can be so prolonged in the case of a compact surface of class C^2 in E^3 such as we have before us. (See [3], and [4] p. 133.)

Property L: C is a *line of curvature*. We characterize these as being curves of class C^1 having $\underset{\sim}{N}'$ parallel to $\underset{\sim}{t}$ at each point.

Property *: C is a *self-antipodal curve*. Let us first define what we mean by the antipodal curve to a given one. We take any curve C of class C^k $(k \leq 2)$ on S, represented in terms of arc-length by a C^k-function $\underset{\sim}{r}(s)$ defined on $(-\infty, \infty)$ with values in S. Composing $\underset{\sim}{r}(s)$ with the antipodal mapping $\underset{\sim}{r} \to \underset{\sim}{r}^*$ of S (where $\underset{\sim}{r}^* = \underset{\sim}{r} - D\underset{\sim}{N}$) which is of class C^1, we get $\underset{\sim}{r}(s)^*$ which represents a curve of class C^m, $m = \min\{k, 1\}$, on S. This antipodal curve to C can be reparametrized in terms of its arc-length s^*, and $s^* = f(s)$ is of class C^m. Now for a self-antipodal curve, we require that we can choose a function $f(s)$ so that $\underset{\sim}{r}(s)^* = \underset{\sim}{r}(f(s))$ for all real s. By changing the sense of C if necessary, we can arrange for $f(s) > s$. A self-antipodal curve is closed, for

$$\underset{\sim}{r}(s) = \underset{\sim}{r}(f(s))^* = \underset{\sim}{r}(f(f(s))) ,$$

and hence $f(f(s)) = s + \ell$, where $\ell > 0$ is a constant. Thus $\underset{\sim}{r}$ is periodic and C is closed.

Property A: C has *all* of the properties P, E, G, L, *. Clearly C will then be a simple closed curve of class C^2 on S, and the plane of C will contain the surface normal vectors $\underset{\sim}{N}$ along C.

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THEOREM I: If a curve C on a C^2 surface of constant width has any pair of the properties P, E, G, L, *, except for the pairs (P, L), (E, *), (L, *), then it has A.

We shall show in section 1 after the proof of theorem I that for our class of surfaces S, exclusion of the pairs (P, L), (E, *), (L, *) is really necessary.

Our second result concerns the inner metric on a surface S of constant width with no smoothness restrictions. The inner distance $\rho_i(p, q)$ between two points p and q of S is the infimum of the lengths of rectifiable curves lying in S and connecting p and q. The maximum of the inner distances taken over all pairs of points of S is the inner diameter D_i of S. (See [2] p. 73 ff.)

THEOREM II: Let S be a surface of constant width D in E^3. Then

(a) if S is a surface of revolution

$$D_i = \pi D/2 ,$$

(b) if S is not a surface of revolution

$$\pi D/3 < D_i < \pi D/2 .$$

The methods of proof for theorems I and II are elementary.

1. <u>Proof of theorem I</u>. We do not always mention the differentiability of C in this proof, but it is easy to check that at each stage the differentiability is enough for the operations carried out.

1.) (P, *) $\Rightarrow$ A: Let $\underset{\sim}{u}$ be perpendicular to the plane of C. By *, for each point on C given by a position vector $\underset{\sim}{r}$, the point $\underset{\sim}{r} - D\underset{\sim}{N}$ is on C, so that $\underset{\sim}{N}$ is in the plane of C, and $\underset{\sim}{N} \cdot \underset{\sim}{u} \equiv 0$.

Therefore E holds. $\underset{\sim}{N}$ is perpendicular to $\underset{\sim}{t}$ and in a fixed plane with $\underset{\sim}{t}$, so $\underset{\sim}{t}'$ is parallel to $\underset{\sim}{N}$ and $\underset{\sim}{N}'$ to $\underset{\sim}{t}$, giving G, L, and hence A.

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2.) (P, E) $\Rightarrow$ A, since E $\Rightarrow$ * .

3.) (P, G) $\Rightarrow$ A: P $\Rightarrow t' \cdot u \equiv 0$ for some unit vector u. From G, it follows that $N \cdot u \equiv 0$, hence E and A hold.

4.) (E, G) $\Rightarrow$ A: Let u be a unit vector such that $N \cdot u \equiv 0$. Then by G $(r \cdot u)'' \equiv t' \cdot u \equiv 0$; and therefore $r \cdot u \equiv as + b$ (a, b constants). But by E, r is periodic, so $a = 0$ and $r \cdot u \equiv b$, i.e. P holds and hence A.

5.) (E, L) $\Rightarrow$ A: $N \cdot u \equiv 0 \Rightarrow N' \cdot u \equiv 0$, hence by L $t \cdot u \equiv 0$, $r \cdot u \equiv \text{const}$. Therefore P holds and hence A.

6.) (G, L) $\Rightarrow$ A: $N \times t' \equiv N' \times t \equiv 0$ so $N \times t \equiv u$ for some unit vector u. Therefore E, P, and A hold.

7.) (G, *) $\Rightarrow$ A: (This part is more difficult than the others.) If r is a point on C, then the principal normal line to C at r is, by G, the same as the surface normal line at r ; similarly at r^*. But the same line is the surface normal at r and r^* ([1], p. 127). C is therefore a Bertrand curve with respect to itself as mate.

Let $t^*(s) = t(f(s))$, $\kappa^*(s) = \kappa(f(s))$, $n^*(s) = n(f(s))$ be respectively the unit tangent vector, curvature, principal normal vector to C, where f is the function used in defining property *. Now t' and $t^{*\prime} = f' \kappa^* n^*$ are, by G, both in the direction of N. Hence $t' \cdot t^* = t \cdot t^{*\prime} = 0$. This leads to the well-known result concerning Bertrand curves $(t \cdot t^*)' \equiv 0$, i.e. t and t^* remain at a fixed angle α from each other, $0 \leq \alpha < 2\pi$, $t^* = \cos\alpha\, t + \sin\alpha\, b$.

We conclude the proof of 7.) by splitting it into two cases:

Case a.) $\alpha \neq \pi$: Let R be the position vector of a fixed point on C. We wish to show

$$[(R - r) \cdot (t + t^*)]' \leq 0 \qquad \text{for all } r \text{ on } C . \tag{1}$$

Assuming (1) is proved, we use the fact that $(R-r) \cdot (t+t^*)$ is periodic and vanishes for $r = R$ to get $(R-r) \cdot (t+t^*) \equiv 0$. Since this holds for arbitrary R on C, and since $\alpha \neq \pi$ implies $t + t^* \neq 0$, we see that C is planar. But then $\alpha = \pi$, a contradiction showing that case a.) cannot occur.

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Proof of (1): We use the Serret-Frenet formulas ([4] p. 14) on $\underset{\sim}{r}^* = \underset{\sim}{r} + D\underset{\sim}{n}$ to get

$$(2) \qquad f' \, \underset{\sim}{t}^* = \underset{\sim}{t} + D(\tau \underset{\sim}{b} - \kappa \underset{\sim}{t}) \, .$$

Remark: $\underset{\sim}{n}(s) = -\underset{\sim}{N}(s)$ is of class C^1, since $\underset{\sim}{N}$ is a class C^1 function on S and $\underset{\sim}{r}(s)$ is of class C^2 by G. Furthermore, the binormal vector $\underset{\sim}{b}(s) = \underset{\sim}{t}(s) \times \underset{\sim}{n}(s)$ to C is also C^1.

Taking the scalar product of (2) with $\underset{\sim}{t}$ and $\underset{\sim}{b}$ successively, we get

$$(3) \qquad f' \cos \alpha = 1 - D\kappa$$

$$(4) \qquad f' \sin \alpha = \tau D \, .$$

By the smoothness condition on S $1 - D\kappa < 0$. For κ, being the curvature of a geodesic, is a normal curvature of S at $\underset{\sim}{r}$, which is at least equal to the lesser κ_a of the two principal curvatures at $\underset{\sim}{r}$. The sum $R_a + R_a^*$ of the corresponding principal radii of curvature at $\underset{\sim}{r}$ and $\underset{\sim}{r}^*$ is D, and $R_a, R_a^* > 0$, so $0 < R_a < D$ and $\kappa > D^{-1}$.

So we get $\cos \alpha < 0$ from (3), and

$$(5) \qquad \tau = \tan \alpha \, (D^{-1} - \kappa) \, .$$

Differentiating the scalar product in (1) we obtain for the left member

$$(\underset{\sim}{R} - \underset{\sim}{r}) \cdot \underset{\sim}{n} \; [(1 + \cos \alpha)\kappa - \sin \alpha \tau] - 1 - \cos \alpha \, .$$

Since $\underset{\sim}{R}$ lies on S, $0 \leq (\underset{\sim}{R} - \underset{\sim}{r}) \cdot \underset{\sim}{n} \leq D$, and so (1) will hold if

$$D[(1 + \cos \alpha)\kappa - \sin \alpha \tan \alpha \, (D^{-1} - \kappa)] - 1 - \cos \alpha$$

$$= (1 + \cos \alpha) \, (\cos \alpha)^{-1} \, (D\kappa - 1) \leq 0 \, ,$$

which is indeed the case!

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Case b.) $\alpha = \pi$, $\underset{\sim}{t}^* = -\underset{\sim}{t}$: Let $\underset{\sim}{R}$ be fixed on C. We calculate easily that

$$[(\underset{\sim}{R}-\underset{\sim}{r})\cdot\underset{\sim}{t}\times(\underset{\sim}{r}-\underset{\sim}{r}^*)]' = 0 \text{ , hence}$$

$$(\underset{\sim}{R}-\underset{\sim}{r})\cdot\underset{\sim}{t}\times(\underset{\sim}{r}-\underset{\sim}{r}^*) = 0 .$$

Since $\underset{\sim}{R}$ is arbitrary on C, P holds and hence A also.

This completes the proof of theorem I.

Let us now turn our attention to the exceptional cases in theorem I.

8.) (P, L): If S is a surface of revolution, all of the parallels are lines of curvature and planar, but only the one of largest diameter will have A.

9.) (E, *): There are equators which are non-planar on every surface of constant width except a sphere, by a theorem of Blaschke ([1] p. 142).

10.) (L, *): On any part of a surface of constant width which is spherical, any C^1 curve has L, and it is easy to construct curves with (L, *) but not A, since the antipodal curve to any curve with L also has L.

2. <u>Proof of theorem II.</u> If $\rho_i(p, q) = D_i$, consider a plane M through p and q. The circumference of the perpendicular projection of S on M is πD by Barbier's principle, so the curve $M \cap S$ contains p and q and has length $\leq \pi D$. (See [1] p. 47.) This implies $D_i \leq \pi D/2$.

If $D_i = \pi D/2$, then every plane M through p and q must intersect S in an equator. Consider a plane Q perpendicular to the line pq and intersecting pq in c. The curve $Q \cap S$ has at each one of its points d a support line in Q which is perpendicular to cd. Hence $Q \cap S$ must be a circle with centre c, i.e. S is a surface of revolution with axis pq. (To see that $Q \cap S$ is a circle, let n rays radiate from c at equal angles in Q and observe how the distances along these rays to $Q \cap S$ can be estimated. Then let $n \to \infty$.)

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To show that $D_i \geq \pi D/3$, we need only recall that if p and q are any antipodal points of S, then the "spindle" formed by intersecting all balls of radius D containing p and q is contained in K.

If equality were attained in $D_i \geq \pi D/3$, then we would have for each pair of antipodal points p, q of S a plane M through p and q such that $M \cap S$ would be a Reuleaux triangle of width D with vertices p, q, r. But it is easy to see that then there can be no such triangle with two of its vertices being r and the midpoint of the side p, q of the original Reuleaux triangle. This is a contradiction. Thus $D_i > \pi D/3$.

This completes the proof of theorem II.

3. Questions.

1.) Can the inequality $D_i > \pi D/3$ be improved?

2.) One can show (by putting "bumps" and antipodal "flattenings" on a sphere) that there are surfaces of constant width which have non-closed geodesics. Is the sphere the only surface of constant width all of whose geodesics are closed?

3.) Using the inequality

$$(\underset{\sim}{r} \cdot \underset{\sim}{u})'' = \kappa \underset{\sim}{n} \cdot \underset{\sim}{u} \geq -D^{-1} \underset{\sim}{N} \cdot \underset{\sim}{u}$$

one can show that every geodesic ray cuts every equator $\underset{\sim}{N} \cdot \underset{\sim}{u} = 0$ on a C^2 surface of constant width. In fact, given any planar equator, any geodesic segment of length πD must cut that equator. Are there stronger results than the above?

4.) Is there a simple "inner" criterion that a surface be of constant width?

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