ANZIAM J. 61(2019), 99–117
doi:10.1017/S1446181119000014

OPTIMAL INVESTMENT AND CONSUMPTION WITH STOCHASTIC FACTOR AND DELAY

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(Received 19 March, 2018; accepted 2 December, 2018; first published online 18 February 2019)

Abstract

We analyse an optimal portfolio and consumption problem with stochastic factor and delay over a finite time horizon. The financial market includes a risk-free asset, a risky asset and a stochastic factor. The price process of the risky asset is modelled as a stochastic differential delay equation whose coefficients vary according to the stochastic factor; the drift also depends on its historical performance. Employing the stochastic dynamic programming approach, we establish the associated Hamilton–Jacobi–Bellman equation. Then we solve the optimal investment and consumption strategies for the power utility function. We also consider a special case in which the price process of the stochastic factor degenerates into a Cox–Ingersoll–Ross model. Finally, the effects of the delay variable on the optimal strategies are discussed and some numerical examples are presented to illustrate the results.

2010 *Mathematics subject classification*: primary 91G80; secondary 91G10.

Keywords and phrases: stochastic differential delay equation, power utility function, stochastic factor, Cox–Ingersoll–Ross model.

1. Introduction

Merton was the pioneer who solved the continuous-time portfolio optimization problem [15]. In the classical Merton-type problem, it is generally assumed that the market includes a risk-free asset with constant rate of return and n risky assets whose price processes are described by Markovian stochastic processes with deterministic coefficients such as geometric Brownian processes. The investor aims to choose the optimal investment and/or consumption controls to maximize the total expected utility (see, for example, the papers by Merton [15, 16], Cox and Huang [7], Bielecki and Pliska [4] and the references therein for more information). Nowadays, Merton's model has been generalized in many different directions. This paper is related to the following two strands of literature.

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First, the portfolio problem with stochastic factor has been widely studied in the literature. The stochastic factor is used to describe the evolution of macroeconomic or microeconomic factors, such as the stock price indices that influence the market. Zariphopoulou [22] considered optimal investment and consumption models with nonlinear stock dynamics and analysed the utility-based prices and hedging strategies. Zariphopoulou [23] also investigated a class of stochastic optimization models when the coefficients of the risky asset price depend on a correlated stochastic factor. Further, Delong and Klüppelberg [8] studied an optimal investment and consumption problem with stochastic factor driven by a Lévy process. Moreover, stochastic factors are frequently used to model the predictability of stock returns, stochastic volatility and stochastic interest rate. For example, Fouque et al. [11] discussed a portfolio optimization problem with stochastic volatility. Fleming and Hernández-Hernández [10] introduced a consumption model with stochastic volatility. Chacko and Viceira [5] examined the dynamic consumption and portfolio choice problem for long-horizon investors with volatility risk and recursive utility preference. Furthermore, Hernández-Hernández and Schied [12] solved a robust portfolio problem in an incomplete market model whose volatility and interest rate processes were driven by a stochastic factor. For further overview of the literature, we refer to the survey paper of Zariphopoulou [24].

Second, for the above Merton-type models, historical information is not taken into consideration for the risky asset price process. That is, the price of the risky asset is described by a Markovian process with constant coefficients or stochastic coefficients driven by a stochastic factor; the future price's variation of the risky asset is only based on the current information and is irrelevant of the historical performance. Nevertheless, many natural and social phenomena display that the future variation of the state process depends not only on its current state but also essentially on its previous information. That is, the behaviour of the investor is influenced by past information. Investors tend to make their decisions based on the historical performance of the risky asset or their portfolio in the real finance world. Therefore, it is natural to model the price process of the risky asset by a stochastic differential equation with delay. Øksendal and Sulem [17] and Agram et al. [3] studied the maximum principles of the optimal control for stochastic delay systems with financial application to the portfolio problem. In addition, Shen et al. [20] had an application for mean-field jump-diffusion stochastic delay differential equations. Using the dynamic programming principle approach, Chang et al. [6] considered a stochastic portfolio optimization model with bounded memory. Pang and Hussain [18, 19] investigated the models with delay over a finite time horizon and an infinite time horizon, respectively. What is more, recent literature has expanded the range of research to the insurance investment problem. For example, A and Li [1] considered an optimal investment and excess-of-loss reinsurance problem with delay for an insurer under Heston's stochastic volatility (SV) model. A and Shao [2] discussed the portfolio optimization problem with delay under the Cox–Ingersoll–Ross (CIR) model. Shen and Zeng [21] developed an optimal investment and reinsurance problem with bounded delay under the mean-variance criterion using a maximum principle approach.

In this paper, we provide an integrated framework with stochastic factor and delay for studying a Merton-type optimal investment and consumption problem. To our best knowledge, there is little work in the literature on the portfolio optimization problem when some delay factors are added to the stochastic factor framework. We take into account a new revised portfolio optimization problem in which we formulate the wealth dynamics as a stochastic differential delay equation with stochastic factor. The main contribution of this work is a method to connect the stochastic factor with delay variables. We derive explicit solutions for the value function and the optimal investment models when the risky asset price is affected by a correlated stochastic factor.

Finally, our results may be applied to a different direction of valuation models. For this, we view a special case in which the stochastic factor satisfies a CIR stochastic volatility model. Under some assumptions, the closed-form expressions for the optimal controls and the value function for constant relative risk aversion (CRRA) or power utility are explicitly derived.

The rest of this paper is organized as follows. In Section 2, some properties of the state variable $X(t)$ and the delay variables $Y(t)$ and $Z(t)$ are provided. Also, the rigorous mathematical formulation of our problem is presented. In Section 3, by using the dynamic programming principle, we derive the corresponding Hamilton–Jacobi–Bellman (HJB) equation. Meanwhile, under certain conditions, employing a power transformation, we express the value function in terms of the generic solution. In Section 4, using the above basis, we provide a special case in which the stochastic factor satisfies a CIR stochastic volatility model. In Section 5, some numerical examples and sensitivity analyses are provided to show our results. Section 6 summarizes this paper and provides its links to some other topics.

2. Problem formulation and main results

We consider that the financial market includes a risk-free asset, a risky asset and a stochastic factor. A bank account or a bond can be referred to as the risk-free asset which has a fixed interest rate $r > 0$. The risky asset can be a stock whose price is affected by a nontraded stochastic factor [23]. We assume that it is free to transfer money between the risk-free asset account and the risky asset account at any time. Suppose that $K(t)$ and $L(t)$ denote the amounts which are invested in the risky asset and the risk-free asset at time t, respectively. Then $X(t) \equiv K(t) + L(t)$ is the total wealth.

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $W_1(t)$ and $W_2(t)$ be the given one-dimensional Brownian motions defined on the space, which are interrelated with the correlation coefficient ρ $(-1 \leq \rho \leq 1)$. Inspired by Chang et al. [6], Pang and Hussain [18, 19] and Zariphopoulou [23], we assume that $K(t)$ and $L(t)$ are described by the stochastic differential equations

$$dK(t) = [K(t)(\mu_1(\eta(t), t) + \mu_2 Y(t) + \mu_3 Z(t)) + I(t)]\,dt + \sigma(\eta(t), t)K(t)\,dW_1(t),$$

https://doi.org/10.1017/S1446181119000014 Published online by Cambridge University Press

and

$$dL(t) = (rL(t) - I(t) - C(t))\,dt.$$

The process $\eta(t)$ is constructed as the "stochastic factor" and it is assumed to satisfy

$$d\eta(t) = b(\eta(t), t)\,dt + a(\eta(t), t)\,dW_2(t),$$

where μ_1, σ, b and a are functions of the factor $\eta(t)$, μ_2 and μ_3 are real constants, $I(t)$ is the total money amount transferred from risk-free asset to risky asset up to time $t\,(t \in [0, T])$ and $C(t)$ is the consumption rate. In addition, the investor's investment performance is influenced by the past information; further, $K(t)$ depends on the delay variables $Y(t)$ and $Z(t)$, defined by

$$Y(t) = \int_{-h}^{0} e^{\lambda\theta} X(t+\theta)\,d\theta, \tag{2.1}$$

$$Z(t) = X(t-h), \quad t \in [0, T], \tag{2.2}$$

where $\lambda > 0$ is a constant. By using $X(t) = K(t) + L(t)$, we have the total wealth process $X(t)$ satisfying

$$\begin{aligned} dX(t) = {} & [K(t)(\mu_1(\eta(t), t) + \mu_2 Y(t) + \mu_3 Z(t)) + rL(t) - C(t)]\,dt \\ & + \sigma(\eta(t), t)K(t)dW_1(t), \quad t \in [0, T]. \end{aligned} \tag{2.3}$$

The initial condition is

$$X(t) = \varphi(t), \quad t \in [-h, 0],$$

where $\varphi \in \mathbb{J}$, $\mathbb{J} \equiv C[-h, 0]$ and

$$\|\varphi\| = \sup_{\theta \in [-h,0]} |\varphi(\theta)|.$$

Based on describing the portfolio problem between risky asset and risk-free asset, we take into account $K(t)$ and $C(t)$ as our control variables. From (2.3), we observe that the wealth process $X(t)$ depends on delay variables $Y(t)$ and $Z(t)$. For convenience of using technology, we modify (2.3) to the following model:

$$\begin{aligned} dX(t) = {} & [K(t)\mu_1(\eta(t), t) + \mu_2 Y(t) + \mu_3 Z(t) + rL(t) - C(t)]\,dt \\ & + \sigma(\eta(t), t)K(t)\,dW_1(t), \quad t \in [0, T]. \end{aligned} \tag{2.4}$$

Remark 2.1. If we assume that $K(t) > 0$ almost surely, the following delay variables:

$$\tilde{Y}(t) = \frac{1}{K(t)} \int_{-h}^{0} e^{\lambda\theta} X(t+\theta)\,d\theta,$$

$$\tilde{Z}(t) = \frac{1}{K(t)} Z(t) = \frac{1}{K(t)} X(t-h), \quad t \in [0, T],$$

can be used instead of (2.1)–(2.2); then we can get (2.4) (see the paper by Chang et al. [6] for the details).

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By using $L(t) = X(t) - K(t)$, we represent the dynamics of the wealth process $X(t)$ as the stochastic delay differential equation

$$\begin{aligned} dX(t) = [rX(t) + (\mu_1(\eta(t), t) - r)K(t) + \mu_2 Y(t) + \mu_3 Z(t) - C(t)]\, dt \\ + \sigma(\eta(t), t)K(t)\, dW_1(t), \quad t \in [0, T]. \end{aligned} \tag{2.5}$$

The initial condition is

$$X(t) = \varphi(t), \quad t \in [-h, 0], \tag{2.6}$$

where $\varphi \in \mathbb{J}$ and $\varphi(\theta) > 0$ for all $\theta \in [-h, 0]$.

Definition 2.2 (Admissible control space). An $\mathcal{F}_t$-adapted control strategy $(K(t), C(t))$ is said to be admissible in Π if:

(i) $(K(t), C(t))$ is $\mathcal{F}_t$-measurable for any $t \in [0, T]$;
(ii) for any $t \in [0, T]$, $C(t) \geq 0$,
(iii)

$$|K(t)| \leq \Lambda |X(t) + Y(t)|, \quad t \in [0, T],$$
$$|C(t)| \leq \Lambda |X(t) + Y(t)|, \quad t \in [0, T].$$

Here $\Lambda > 0$ is a constant and Π denotes the admissible control space.

Assumption 2.3. The coefficients $\mu_1, \sigma, b, a : R \times [0, T] \to R$ satisfy the global Lipschitz and linear growth conditions

$$|l(\eta, t) - l(\bar{\eta}, t)| \leqslant M|\eta - \bar{\eta}|,$$
$$l^2(\eta, t) \leqslant M^2(1 + \eta^2)$$

for every $t \in [0, T]$, $\eta, \bar{\eta} \in R$, M being a positive constant and l standing for μ_1, σ, b and a.

The investor's objective is to maximize his/her discounted expected utility of consumption and terminal utility function, which depends on both $X(T)$ and $Y(T)$. We consider the problem of optimal portfolio on a finite time horizon $[0, T]$ with the objective function

$$J(t, \varphi, \eta, K, C) = E_{t,\varphi,\eta,K,C}\Big[\alpha \int_0^T e^{-\beta t} U_1(C(t))\, dt + (1 - \alpha)e^{-\beta T} U_2(X(T), Y(T))\Big].$$

The value function of the investor is

$$V(t, \varphi, \eta) = \sup_{K,C \in \Pi} J(t, \varphi, \eta, K, C); \tag{2.7}$$

note that $V(t, \varphi, \eta) = V(t, x, y, z, \eta)$.

In addition, we have an assumption that V only depends on (t, x, y, η), that is, $V(t, \varphi, \eta) = V(t, x, y, z, \eta) = V(t, x, y, \eta)$. In the following section, we will derive some details. Meanwhile, it has the boundary condition $V(T, x, y, \eta) = (1 - \alpha)e^{-\beta T} U_2(x, y)$.

The goal herein is to analyse the value function and to derive the optimal investment and consumption controls when the utility accords with the power form. Our results are based on more than one skilful transformation. Without beginning the necessary technical assumptions and the properties of the relevant solutions, we list the main results below.

PROPOSITION 2.4. (i) *The value function V is given by*

$$V(t,x,y,\eta) = e^{-\beta t}\frac{u^{\delta}}{\delta}\Big\{\alpha^{1/(1-\delta)}\int_t^T \exp\{\phi(s)\eta+\Psi(s)\}\,ds + (1-\alpha)^{1/(1-\delta)}\exp\{\phi(t)\eta+\Psi(t)\}\Big\}^{1-\delta},$$

where $u = x + \mu_3 e^{\lambda h} y$ *and* $\phi, \Psi : [0,T] \to R$ *are solutions of the equation*

$$0 = \phi'(t)\eta + \frac{\delta(\mu_1(\eta,t)-r)^2}{2(\delta-1)^2\sigma^2(\eta,t)} + \Big(b(\eta,t) - \frac{\delta\rho a(\eta,t)(\mu_1(\eta,t)-r)}{(\delta-1)\sigma(\eta,t)}\Big)\phi(t) + \frac{1}{2}a^2(\eta,t)[1+\delta(\rho^2-1)]\phi^2(t) + \Psi'(t) + \frac{\delta(r+\mu_3 e^{\lambda h})-\beta}{1-\delta}$$

with boundary conditions $\phi(T) = \Psi(T) = 0$.

(ii) *The optimal strategy* $(K^*(t,x,y,\eta), C^*(t,x,y,\eta))$ *is given in the form*

$$K^*(t,x,y,\eta) = \Big[\frac{\mu_1(\eta,t)-r}{(1-\delta)\sigma^2(\eta,t)} + \frac{\rho a(\eta,t)}{\sigma(\eta,t)}\frac{g_\eta(\eta,t)}{g(\eta,t)}\Big](x+\mu_3 e^{\lambda h}y)$$

and

$$C^*(t,x,y,\eta) = \alpha^{1/(1-\delta)} g^{-1}(\eta,t)(x+\mu_3 e^{\lambda h}y),$$

where

$$g(\eta,t) = \alpha^{1/(1-\delta)}\int_t^T \exp\{\phi(s)\eta+\Psi(s)\}\,ds + (1-\alpha)^{1/(1-\delta)}\exp\{\phi(t)\eta+\Psi(t)\}.$$

3. The HJB equation and optimal strategies

In this section, the HJB equation is established. Moreover, we derive the closed form solutions for the CRRA utility function. By employing the dynamic programming approach, the problem about optimal policies is equivalent to looking for the solutions to the HJB equation. To begin with, we present an important lemma.

LEMMA 3.1 (Delayed Itô's formula). *Let* $h \in C^{1,2,1,2}(\mathbb{R}^4)$ *and* $G = h(t,x,y,\eta)$; *then*

$$dG = \pounds h\,dt + \sigma(\eta(t),t)Kh_x\,dW_1(t) + a(\eta,t)h_\eta\,dW_2(t) + (x-\lambda y - e^{-\lambda h}z)h_y\,dt,$$

where

$$\pounds h = h_t + [rx + (\mu_1(\eta,t)-r)K + \mu_2 y + \mu_3 z - C]h_x + \tfrac{1}{2}\sigma^2(\eta,t)K^2 h_{xx} + b(\eta,t)h_\eta + \tfrac{1}{2}a^2(\eta,t)h_{\eta\eta} + \rho a(\eta,t)\sigma(\eta,t)Kh_{x\eta}$$

and $x = X(t)$, $y = y(X_t) = \int_{-h}^0 e^{\lambda\theta}X(t+\theta)\,d\theta$, $z = z(X_t) = X(t-h)$, $K = K(t)$, $C = C(t)$ *and* $\eta = \eta(t)$.

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PROOF. By using the Leibnitz formula [6, 9],

$$\begin{aligned}\frac{d}{dt}y(X_t(\cdot)) &= \frac{d}{dt}\Big[\int_{-h}^{0} e^{\lambda\theta}X(t+\theta)\,d\theta\Big]\\ &= \frac{d}{dt}\Big[\int_{t-h}^{t} e^{\lambda(u-t)}X(u)\,du\Big]\\ &= X(t) - e^{-\lambda h}X(t-h) - \lambda\int_{t-h}^{t} e^{\lambda(u-t)}X(u)\,du\\ &= X(t) - e^{-\lambda h}X(t-h) - \lambda\int_{-h}^{0} e^{\lambda\theta}X(t+\theta)\,d\theta\\ &= X(t) - e^{-\lambda h}Z(t) - \lambda Y(t)\\ &= x - \lambda y - e^{-\lambda h}z.\end{aligned}$$

Since $G = h(t, x, y, \eta)$, by using the classical Itô's formula, we can obtain the result. □

Using the delayed Itô's formula in Lemma 3.1, which is to show the HJB equation, the value function is expected to solve the following equation:

$$\begin{aligned}&\sup_{K,C\in\Pi}\{V_t + [rx + (\mu_1(\eta,t) - r)K + \mu_2 y + \mu_3 z - C]V_x + \tfrac{1}{2}\sigma^2(\eta,t)K^2V_{xx} + b(\eta,t)V_\eta\\ &\qquad + \tfrac{1}{2}a^2(\eta,t)V_{\eta\eta} + \rho a(\eta,t)\sigma(\eta,t)KV_{x\eta} + (x - \lambda y - e^{-\lambda h}z)V_y + \alpha e^{-\beta t}U_1(C)\}\\ &\quad = 0. \end{aligned} \tag{3.1}$$

Let $V(t, x, y, \eta) \in C^{1,2,1,2}(\mathbb{R}^4)$ be a solution of (3.1). Moreover, V satisfies the boundary condition $V(T, x, y, \eta) = (1-\alpha)e^{-\beta T}U_2(x, y)$. According to equation (3.1), we have, respectively, the maximizing conditions for the optimal investment and consumption controls

$$K^*(t, x, y, \eta) = -\frac{(\mu_1(\eta,t) - r)V_x + \rho a(\eta,t)\sigma(\eta,t)V_{x\eta}}{\sigma^2(\eta,t)V_{xx}} \tag{3.2}$$

and

$$U_1'(C^*(t, x, y, \eta)) = \frac{V_x}{\alpha e^{-\beta t}}. \tag{3.3}$$

Substituting (3.2)–(3.3) into (3.1) yields

$$\begin{aligned}&V_t + (rx + \mu_2 y + \mu_3 z)V_x + b(\eta,t)V_\eta + \frac{1}{2}a^2(\eta,t)V_{\eta\eta} + (x - \lambda y - e^{-\lambda h}z)V_y\\ &\quad - \frac{(\mu_1(\eta,t) - r)^2}{2\sigma^2(\eta,t)}\frac{V_x^2}{V_{xx}} - \frac{\rho a(\eta,t)(\mu_1(\eta,t) - r)}{\sigma(\eta,t)}\frac{V_xV_{x\eta}}{V_{xx}} - \frac{\rho^2a^2(\eta,t)}{2}\frac{V_{x\eta}^2}{V_{xx}}\\ &\quad - C^*V_x + \alpha e^{-\beta t}U_1(C^*)\\ &= 0. \end{aligned} \tag{3.4}$$

To find a possible solution of (3.4), we consider a particular case of power utility functions of CRRA type, which is given as

$$U_1(x) = U_2(x) = \frac{x^\delta}{\delta},$$

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where $\delta < 1, \delta \neq 0$ and $1-\delta$ is the relative risk-aversion coefficient of the investor [14]. Therefore, the HJB equation (3.4) can be rewritten as

$$\begin{aligned} &V_t + (rx + \mu_2 y + \mu_3 z)V_x + b(\eta,t)V_\eta + \frac{1}{2}a^2(\eta,t)V_{\eta\eta} + (x - \lambda y - e^{-\lambda h}z)V_y \\ &\quad - \frac{(\mu_1(\eta,t)-r)^2}{2\sigma^2(\eta,t)}\frac{V_x^2}{V_{xx}} - \frac{\rho a(\eta,t)(\mu_1(\eta,t)-r)}{\sigma(\eta,t)}\frac{V_x V_{x\eta}}{V_{xx}} - \frac{\rho^2 a^2(\eta,t)}{2}\frac{V_{x\eta}^2}{V_{xx}} \\ &\quad - C^* V_x + \alpha e^{-\beta t}\frac{(C^*)^\delta}{\delta} \\ &= 0. \end{aligned} \tag{3.5}$$

From (3.3),

$$C^*(t,x,y,\eta) = \Big(\frac{V_x}{\alpha e^{-\beta t}}\Big)^{1/(\delta-1)}. \tag{3.6}$$

Using the form of (3.6), equation (3.5) becomes

$$\begin{aligned} &V_t + (rx + \mu_2 y + \mu_3 z)V_x + b(\eta,t)V_\eta + \frac{1}{2}a^2(\eta,t)V_{\eta\eta} + (x - \lambda y - e^{-\lambda h}z)V_y \\ &\quad - \frac{(\mu_1(\eta,t)-r)^2}{2\sigma^2(\eta,t)}\frac{V_x^2}{V_{xx}} - \frac{\rho a(\eta,t)(\mu_1(\eta,t)-r)}{\sigma(\eta,t)}\frac{V_x V_{x\eta}}{V_{xx}} - \frac{\rho^2 a^2(\eta,t)}{2}\frac{V_{x\eta}^2}{V_{xx}} \\ &\quad + \Big(\frac{1}{\delta} - 1\Big)(\alpha e^{-\beta t})^{1/(1-\delta)} V_x^{\delta/(\delta-1)} \\ &= 0. \end{aligned} \tag{3.7}$$

Now we have the following transformation. Let

$$u \equiv x + \mu_3 e^{\lambda h} y$$

and assume that

$$V(t,x,y,\eta) = e^{-\beta t}\frac{u^\delta}{\delta}f(t,\eta), \tag{3.8}$$

where the terminal condition $f(T,\eta) = 1-\alpha$. Then

$$\begin{aligned} &V_t = -\beta e^{-\beta t}\frac{u^\delta}{\delta}f + e^{-\beta t}\frac{u^\delta}{\delta}f_t, \quad V_x = e^{-\beta t}u^{\delta-1}f, \quad V_\eta = e^{-\beta t}\frac{u^\delta}{\delta}f_\eta, \quad V_{\eta\eta} = e^{-\beta t}\frac{u^\delta}{\delta}f_{\eta\eta}, \\ &V_y = \mu_3 e^{\lambda h}e^{-\beta t}u^{\delta-1}f, \quad V_{xx} = (\delta-1)e^{-\beta t}u^{\delta-2}f, \quad V_{x\eta} = e^{-\beta t}u^{\delta-1}f_\eta. \end{aligned}$$

Substituting the above derivatives into (3.7) and eliminating $e^{-\beta t}$,

$$\begin{aligned} &-\beta\frac{u^\delta}{\delta}f + \frac{u^\delta}{\delta}f_t + [(r + \mu_3 e^{\lambda h})x + (\mu_2 - \lambda\mu_3 e^{\lambda h})y]u^{\delta-1}f + \frac{1}{2}a^2(\eta,t)\frac{u^\delta}{\delta}f_{\eta\eta} \\ &\quad + b(\eta,t)\frac{u^\delta}{\delta}f_\eta - \frac{(\mu_1(\eta,t)-r)^2}{2\sigma^2(\eta,t)}\frac{u^\delta}{\delta-1}f - \frac{\rho a(\eta,t)(\mu_1(\eta,t)-r)}{\sigma(\eta,t)}\frac{u^\delta}{\delta-1}f_\eta \\ &\quad - \frac{\rho^2 a^2(\eta,t)}{2}\frac{u^\delta}{\delta-1}\frac{f_\eta^2}{f} + \Big(\frac{1}{\delta}-1\Big)\alpha^{1/(1-\delta)}f^{\delta/(\delta-1)}u^\delta \\ &= 0. \end{aligned}$$

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Moreover, assuming that

$$\mu_2 - \lambda\mu_3 e^{\lambda h} = (r + \mu_3 e^{\lambda h})\mu_3 e^{\lambda h}, \tag{3.9}$$

$$\begin{aligned}\frac{u^\delta}{\delta}\Big[&-\beta f + f_t + \delta(r+\mu_3 e^{\lambda h})f + b(\eta,t)f_\eta + \frac{1}{2}a^2(\eta,t)f_{\eta\eta} - \frac{\delta(\mu_1(\eta,t)-r)^2}{2(\delta-1)\sigma^2(\eta,t)}f \\ &- \frac{\delta\rho a(\eta,t)(\mu_1(\eta,t)-r)}{(\delta-1)\sigma(\eta,t)}f_\eta - \frac{\delta\rho^2 a^2(\eta,t)}{2(\delta-1)}\frac{f_\eta^2}{f} + (1-\delta)\alpha^{1/(1-\delta)}f^{\delta/(\delta-1)}\Big] \\ &= 0.\end{aligned}$$

Eliminating the dependence on u,

$$\begin{aligned}f_t &+ \Big[\delta(r+\mu_3 e^{\lambda h}) - \beta - \frac{\delta(\mu_1(\eta,t)-r)^2}{2(\delta-1)\sigma^2(\eta,t)}\Big]f + \Big[b(\eta,t) - \frac{\delta\rho a(\eta,t)(\mu_1(\eta,t)-r)}{(\delta-1)\sigma(\eta,t)}\Big]f_\eta \\ &+ \frac{1}{2}a^2(\eta,t)f_{\eta\eta} - \frac{\delta\rho^2 a^2(\eta,t)}{2(\delta-1)}\frac{f_\eta^2}{f} + (1-\delta)\alpha^{1/(1-\delta)}f^{\delta/(\delta-1)} \\ &= 0.\end{aligned} \tag{3.10}$$

Now we make the following transformation. Suppose that

$$f(t,\eta) = g(t,\eta)^{1-\delta}, \quad g(T,\eta) = (1-\alpha)^{1/(1-\delta)}.$$

Then the partial derivatives are

$$f_t = (1-\delta)g^{-\delta}g_t, \quad f_\eta = (1-\delta)g^{-\delta}g_\eta, \quad f_{\eta\eta} = (1-\delta)(-\delta)g^{-\delta-1}g_\eta^2 + (1-\delta)g^{-\delta}g_{\eta\eta}$$

and substituting these into (3.10) yields

$$\begin{aligned}(1-\delta)g^{-\delta}\Big[&g_t + \Big\{\frac{\delta(r+\mu_3 e^{\lambda h})-\beta}{1-\delta} + \frac{\delta(\mu_1(\eta,t)-r)^2}{2(\delta-1)^2\sigma^2(\eta,t)}\Big\}g + \frac{1}{2}a^2(\eta,t)g_{\eta\eta} + \alpha^{1/(1-\delta)} \\ &+ \Big\{b(\eta,t) - \frac{\delta\rho a(\eta,t)(\mu_1(\eta,t)-r)}{(\delta-1)\sigma(\eta,t)}\Big\}g_\eta + \frac{\delta a^2(\eta,t)}{2}(\rho^2-1)\frac{g_\eta^2}{g}\Big] \\ &= 0.\end{aligned} \tag{3.11}$$

In addition, solving (3.11) yields another expression for g as

$$\begin{aligned}g_t &+ \Big[\frac{\delta(r+\mu_3 e^{\lambda h})-\beta}{1-\delta} + \frac{\delta(\mu_1(\eta,t)-r)^2}{2(\delta-1)^2\sigma^2(\eta,t)}\Big]g + \Big[b(\eta,t) - \frac{\delta\rho a(\eta,t)(\mu_1(\eta,t)-r)}{(\delta-1)\sigma(\eta,t)}\Big]g_\eta \\ &+ \frac{1}{2}a^2(\eta,t)g_{\eta\eta} + \frac{\delta a^2(\eta,t)}{2}(\rho^2-1)\frac{g_\eta^2}{g} + \alpha^{1/(1-\delta)} \\ &= 0.\end{aligned} \tag{3.12}$$

As far as we know, (3.12) is difficult to solve directly, because there exists the term $\alpha^{1/(1-\delta)}$. Inspired by the paper of Liu [13], we assume that g is of the form

$$g(t,\eta) = \alpha^{1/(1-\delta)}\int_t^T \hat{g}(s,\eta)\,ds + (1-\alpha)^{1/(1-\delta)}\hat{g}(t,\eta).$$

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Define ∇ as

$$\nabla g = \Big[\frac{\delta(r+\mu_3 e^{\lambda h})-\beta}{1-\delta} + \frac{\delta(\mu_1(\eta,t)-r)^2}{2(\delta-1)^2\sigma^2(\eta,t)}\Big]g + \Big[b(\eta,t) - \frac{\delta\rho a(\eta,t)(\mu_1(\eta,t)-r)}{(\delta-1)\sigma(\eta,t)}\Big]g_\eta + \frac{1}{2}a^2(\eta,t)g_{\eta\eta} + \frac{\delta a^2(\eta,t)}{2}(\rho^2-1)\frac{g_\eta^2}{g} = 0.$$

According to the above equation, (3.12) can be rewritten as

$$\frac{\partial g}{\partial t} + \nabla g + \alpha^{1/(1-\delta)} = 0, g(T,\eta) = (1-\alpha)^{1/(1-\delta)}.$$

We find that

$$\begin{aligned}\frac{\partial g}{\partial t} + \nabla g &= \frac{\partial}{\partial t}\Big(\alpha^{1/(1-\delta)}\int_t^T \hat{g}(s,\eta)\,ds\Big) + \nabla\Big(\alpha^{1/(1-\delta)}\int_t^T \hat{g}(s,\eta)\,ds\Big) \\ &\quad + (1-\alpha)^{1/(1-\delta)}\Big(\frac{\partial}{\partial t}\hat{g}(t,\eta) + \nabla\hat{g}(t,\eta)\Big) \\ &= -\alpha^{1/(1-\delta)}\hat{g}(t,\eta) + \alpha^{1/(1-\delta)}\int_t^T \nabla\hat{g}(s,\eta)\,ds \\ &\quad + (1-\alpha)^{1/(1-\delta)}\Big(\frac{\partial}{\partial t}\hat{g}(t,\eta) + \nabla\hat{g}(t,\eta)\Big) \\ &= -\alpha^{1/(1-\delta)}.\end{aligned}$$

Therefore,

$$\begin{cases}\hat{g}(t,\eta) - \displaystyle\int_t^T \nabla\hat{g}(s,\eta)\,ds = 1, \\ \dfrac{\partial}{\partial t}\hat{g}(t,\eta) + \nabla\hat{g}(t,\eta) = 0.\end{cases}$$

This shows that (3.12) is reduced to

$$\hat{g}_t + \Big[\frac{\delta(r+\mu_3 e^{\lambda h})-\beta}{1-\delta} + \frac{\delta(\mu_1(\eta,t)-r)^2}{2(\delta-1)^2\sigma^2(\eta,t)}\Big]\hat{g} + \Big[b(\eta,t) - \frac{\delta\rho a(\eta,t)(\mu_1(\eta,t)-r)}{(\delta-1)\sigma(\eta,t)}\Big]\hat{g}_\eta + \frac{1}{2}a^2(\eta,t)\hat{g}_{\eta\eta} + \frac{\delta a^2(\eta,t)}{2}(\rho^2-1)\frac{\hat{g}_\eta^2}{\hat{g}} = 0 \tag{3.13}$$

with the boundary condition $\hat{g}(T,\eta) = 1$.

For equation (3.13), we assume that the function $\hat{g}(t,\eta)$ is given in the form

$$\hat{g}(t,\eta) = \exp\{\phi(t)\eta + \Psi(t)\}$$

with the boundary conditions $\phi(T) = \Psi(T) = 0$. From the above equation,

$$\hat{g}_t = [\phi'(t)\eta + \Psi'(t)]\hat{g}, \quad \hat{g}_\eta = \phi(t)\hat{g}, \quad \hat{g}_{\eta\eta} = \phi^2(t)\hat{g}.$$

Direct substitution into (3.13) and cancelling the term $\hat{g}$ on both sides yields

$$\phi'(t)\eta + \frac{\delta(\mu_1(\eta,t)-r)^2}{2(\delta-1)^2\sigma^2(\eta,t)} + \Big[b(\eta,t) - \frac{\delta\rho a(\eta,t)(\mu_1(\eta,t)-r)}{(\delta-1)\sigma(\eta,t)}\Big]\phi(t) + \frac{1}{2}a^2(\eta,t)[1+\delta(\rho^2-1)]\phi^2(t) + \Psi'(t) + \frac{\delta(r+\mu_3 e^{\lambda h})-\beta}{1-\delta} = 0 \tag{3.14}$$

with the terminal conditions

$$\phi(T) = \Psi(T) = 0.$$

Using (3.2) and (3.6) and considering the representation formula (3.8) as the value function, one obtains the candidates for optimal controls

$$K^*(t,x,y,\eta) = \Big[\frac{\mu_1(\eta,t)-r}{(1-\delta)\sigma^2(\eta,t)} + \frac{\rho a(\eta,t)}{\sigma(\eta,t)}\frac{g_\eta(\eta,t)}{g(\eta,t)}\Big](x+\mu_3 e^{\lambda h}y)$$

and

$$C^*(t,x,y,\eta) = \alpha^{1/(1-\delta)} g^{-1}(\eta,t)(x+\mu_3 e^{\lambda h}y),$$

where

$$g(\eta,t) = \alpha^{1/(1-\delta)}\int_t^T \exp\{\phi(s)\eta + \Psi(s)\}\,ds + (1-\alpha)^{1/(1-\delta)}\exp\{\phi(t)\eta + \Psi(t)\}.$$

Remark 3.2. It is interesting that our results are similar to the results of Zariphopoulou [23]. Zariphopoulou studied a class of stochastic optimization models of expected utility in the market with stochastically changing investment opportunities. The prices of the primitive assets are modelled as diffusion factors. Here are some comparisons between them.

(i) In our results, the amount of the risky asset depends on wealth $X(t)$, delay variables $Y(t)$ and $Z(t)$ and stochastic factor $\eta(t)$ at time t. However, in Zariphopoulou [23] the optimal strategy only depends on wealth $X(t)$ and stochastic factor $\eta(t)$. Moreover, the consumption rate $C(t)$ is taken into account in our risk-free asset. In the meantime, Zariphopoulou [23] did not consider it in his paper.

(ii) Comparing with Zariphopoulou [23], the value function $V(t,\varphi,\eta)$ of (2.7) is based on the parameter α for distinguishing the proportion between discounted expected utility of consumption and terminal utility function, which depends on both $X(T)$ and $Y(T)$. There exist some differences. Nonetheless, our results are consistent with the results of Zariphopoulou [23] when the delay variables and parameters α and β are not considered in our model (that is, $\mu_2 = \mu_3 = \alpha = \beta = 0$).

(iii) In Zariphopoulou [23], the power exponent depends on the coefficients of correlation and risk aversion for expressing the value function in terms of the solution of a linear parabolic equation when they employ a power transformation. It is a pretty treatment method whereas we express directly a power exponent depending only on risk aversion. At the same time, the value function is not expressed especially as the solution of a linear parabolic equation at all. Based on it, we derive the explicit form of the value function.

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REMARK 3.3. For the sake of using technology, equation (3.9) is provided to achieve this optimal value. See Elsanosi et al. [9] for the details. In allusion to this equation, Pang and Hussain [19] also have given a summary in their conclusion. Meanwhile, Chang et al. [6] have discussed some examples for a better study of the equation.

4. A special case

Now we take into account a special case in our model. Suppose that the dynamics of the stochastic factor degenerates into a CIR stochastic volatility model. The results of our model will be expressed as the following special case.

Consider the following problem under the CIR model:

$$\sup_{K,C\in\Pi} E\Big[\alpha\int_0^T e^{-\beta t}U_1(C(t))\,dt + (1-\alpha)e^{-\beta T}U_2(X(T),Y(T))\Big]$$

such that

$$dX(t) = [rX(t) + k\eta(t)K(t) + \mu_2 Y(t) + \mu_3 Z(t) - C(t)]\,dt + \sqrt{\eta(t)}K(t)\,dW_1(t),$$
$$d\eta(t) = \kappa(\vartheta - \eta(t))\,dt + \sigma\sqrt{\eta(t)}\,dW_2(t),$$

where $U_1(x) = U_2(x) = x^\delta/\delta$. The value function of the investor is modelled by

$$\begin{aligned} V(t,\varphi,\eta) &= \sup_{K,C\in\Pi} J(t,\varphi,\eta,K,C) \\ &= \sup_{K,C\in\Pi} E_{t,\varphi,\eta,K,C}\Big[\alpha\int_0^T e^{-\beta t}U_1(C(t))\,dt + (1-\alpha)e^{-\beta T}U_2(X(T),Y(T))\Big] \end{aligned} \tag{4.1}$$

and the associated HJB equation is given by

$$\begin{aligned} &V_t + (rx + \mu_2 y + \mu_3 z)V_x + \kappa(\vartheta-\eta)V_\eta + \frac{1}{2}\sigma^2\eta V_{\eta\eta} + (x - \lambda y - e^{-\lambda h}z)V_y - \frac{k^2\eta}{2}\frac{V_x^2}{V_{xx}} \\ &\quad - \rho\sigma k\eta\frac{V_xV_{x\eta}}{V_{xx}} - \frac{\rho\sigma^2\eta}{2}\frac{V_{x\eta}^2}{V_{xx}} + \Big(\frac{1}{\delta}-1\Big)(\alpha e^{-\beta t})^{1/(1-\delta)}V_x^{\delta/(\delta-1)} \\ &= 0. \end{aligned} \tag{4.2}$$

Then equation (3.14) is translated as

$$\begin{aligned} &\eta\Big[\phi'(t) + \frac{\delta k^2}{2(\delta-1)^2} - \Big(\kappa + \frac{\delta k\rho\sigma}{\delta-1}\Big)\phi(t) + \frac{1}{2}\sigma^2(\delta\rho^2 + 1 - \delta)\phi^2(t)\Big] + \Psi'(t) \\ &\quad + \kappa\vartheta\phi(t) + \frac{\delta(r+\mu_3 e^{\lambda h}) - \beta}{1-\delta} = 0. \end{aligned} \tag{4.3}$$

In order to eliminate the dependence on η, we split equation (4.3) into two parts

$$\phi'(t) + \frac{\delta k^2}{2(\delta-1)^2} - \Big(\kappa + \frac{\delta k\rho\sigma}{\delta-1}\Big)\phi(t) + \frac{1}{2}\sigma^2(\delta\rho^2 + 1 - \delta)\phi^2(t), \quad \phi(T) = 0 \tag{4.4}$$

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and

$$\Psi'(t) + \kappa\vartheta\phi(t) + \frac{\delta(r + \mu_3 e^{\lambda h}) - \beta}{1 - \delta} = 0, \quad \Psi(T) = 0. \tag{4.5}$$

Rewriting equation (4.4) yields

$$\begin{aligned}\phi'(t) = -\frac{1}{2}\sigma^2(\delta\rho^2 + 1 - \delta)\Big[\phi^2(t) - \frac{2}{\sigma^2(\delta\rho^2 + 1 - \delta)}\Big(\kappa + \frac{\delta k\rho\sigma}{\delta - 1}\Big)\phi(t) \\ + \frac{\delta k^2}{(\delta - 1)^2\sigma^2(\delta\rho^2 + 1 - \delta)}\Big].\end{aligned} \tag{4.6}$$

Let Δ_ϕ denote the discriminant of the quadratic equation

$$\phi^2(t) - \frac{2}{\sigma^2(\delta\rho^2 + 1 - \delta)}\Big(\kappa + \frac{\delta k\rho\sigma}{\delta - 1}\Big)\phi(t) + \frac{\delta k^2}{(\delta - 1)^2\sigma^2(\delta\rho^2 + 1 - \delta)} = 0; \tag{4.7}$$

then

$$\begin{aligned}\Delta_\phi &= \frac{4}{\sigma^4(\delta\rho^2 + 1 - \delta)^2}\Big(\kappa + \frac{\delta k\rho\sigma}{\delta - 1}\Big)^2 - \frac{4\delta k^2}{(\delta - 1)^2\sigma^2(\delta\rho^2 + 1 - \delta)} \\ &= \frac{4}{\sigma^4(\delta\rho^2 + 1 - \delta)^2}\Big[\frac{-\kappa^2}{\delta - 1} + \frac{\delta}{\delta - 1}\{(k\sigma + \rho\kappa)^2 + \kappa^2(1 - \rho^2)\}\Big].\end{aligned}$$

Suppose that $\Delta_\phi > 0$, that is,

$$\delta < \frac{\kappa^2}{(k\sigma + \rho\kappa)^2 + \kappa^2(1 - \rho^2)} < 1. \tag{4.8}$$

Under condition (4.8), integrating both sides of (4.6) with respect to t,

$$\frac{1}{\lambda_1 - \lambda_2}\int_t^T \Big(\frac{1}{\phi(t) - \lambda_1} - \frac{1}{\phi(t) - \lambda_2}\Big)\, d\phi(t) = -\frac{1}{2}\sigma^2(\delta\rho^2 + 1 - \delta)(T - t), \tag{4.9}$$

where λ_1 and λ_2 are two real roots of (4.7), namely,

$$\begin{aligned}\lambda_{1,2} = \frac{1}{\sigma^2(\delta\rho^2 + 1 - \delta)}\Big(\kappa + \frac{\delta k\rho\sigma}{\delta - 1}\Big) \\ \pm \frac{1}{\sigma^2(\delta\rho^2 + 1 - \delta)}\sqrt{\frac{-\kappa^2}{\delta - 1} + \frac{\delta}{\delta - 1}[(k\sigma + \rho\kappa)^2 + \kappa^2(1 - \rho^2)]}.\end{aligned}$$

Solving (4.9) with the boundary condition $\phi(T) = 0$,

$$\phi(t) = \frac{\lambda_1\lambda_2[1 - \exp\{-(\sigma^2/2)(\delta\rho^2 + 1 - \delta)(\lambda_1 - \lambda_2)(T - t)\}]}{\lambda_1 - \lambda_2 \exp\{-(\sigma^2/2)(\delta\rho^2 + 1 - \delta)(\lambda_1 - \lambda_2)(T - t)\}}. \tag{4.10}$$

For equation (4.5),

$$\Psi(t) = \kappa\vartheta \int_t^T \phi(s)\, ds + \frac{\delta(r + \mu_3 e^{\lambda h}) - \beta}{1 - \delta}(T - t). \tag{4.11}$$

Finally, using (3.2) and (3.6) and the representation formula (3.8) for the value function, one obtains the optimal investment and consumption strategies under the power utility function. The following is the main result in this section.

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Proposition 4.1. *If the utility function $U_1(x) = U_2(x) = x^\delta/\delta$ with the condition* (4.8) *and $\delta \neq 0$, the candidates for optimal controls of the problem* (4.1) *are given by*

$$\begin{aligned} K^*(t) &= \frac{k}{1-\delta}(X(t) + \mu_3 e^{\lambda h} Y(t)) + \rho\sigma \frac{g_\eta}{g}(X(t) + \mu_3 e^{\lambda h} Y(t)) \\ &= \Big(\frac{k}{1-\delta} + \rho\sigma\frac{g_\eta}{g}\Big)\{X(t) + \mu_3 e^{\lambda h} Y(t)\} \end{aligned} \tag{4.12}$$

and

$$C^*(t) = \alpha^{1/(1-\delta)} g^{-1}(X(t) + \mu_3 e^{\lambda h} Y(t)), \tag{4.13}$$

where

$$g = g(t, \eta) = \alpha^{1/(1-\delta)} \int_t^T \exp\{\phi(s)\eta + \Psi(s)\}\, ds + (1-\alpha)^{1/(1-\delta)} \exp\{\phi(t)\eta + \Psi(t)\}.$$

In addition, $\phi(t)$ and $\Psi(t)$ are determined by (4.10) *and* (4.11)*, respectively.*

According to Chang et al. [6], we may adopt a similar approach to prove a verification theorem; we present the following verification theorem without proof.

Theorem 4.2 (Verification theorem). *Assume that $X(t)$ is a strong solution of* (2.5)–(2.6) *and $Y(t)$ and $Z(t)$ are given by* (2.1) *and* (2.2)*, respectively. Let $V(t, x, y, \eta) \in C^{1,2,1,2}([0, T] \times R \times R \times R)$ be a solution of the HJB equation given by* (4.2) *such that*

$$E\Big[\int_0^T [K(t)V_x(t, X(t), Y(t), \eta(t))]^2 dt\Big] < \infty$$

for every $(K, C) \in \Pi$. Then

$$V(t, x, y, \eta) \geq J(t, \varphi, \eta, K, C) \quad \textit{for all } (K, C) \in \Pi,$$

where $J(\cdot)$ is given by (4.1)*. In addition, assume that the utility function is given by $U_1(x) = U_2(x) = x^\delta/\delta$; we have* (4.12)–(4.13)*. If $(K^*, C^*) \in \Pi$, then (K^*, C^*) is the optimal control strategy. In this case,*

$$V(t, x, y, \eta) = J(t, \varphi, \eta, K^*, C^*).$$

Remark 4.3. Without considering historical information (that is, $\mu_2 = \mu_3 = 0$) and choosing the parameters $\alpha = \beta = 0$ in our model, the efficient strategies for problem (4.1) are similar to the results of Liu and Pan [14]. They assumed that the asset price is generated by a diffusion process without delay variables. However, many phenomena cannot be explained by Liu and Pan's model. Therefore, extensions to delay variables are being gradually studied by a growing number of scholars. In some sense, this special case extends their results.

5. Sensitivity analysis and numerical experiment

In this section, we consider the effects of parameters which include delay variable and wealth on the optimal investment and the optimal consumption rate. Numerical examples are provided to illustrate these effects. Here, unless otherwise stated, we use the values of the basic parameters as $r = 0.05,\ k = 0.6,\ \eta(0) = 0.36,\ \sigma = 0.8,\ \kappa\vartheta = 1.0,$ $\kappa = 0.6,\ t = 0,\ T = 1,\ \alpha = 0.4,\ \beta = 0.1,\ \delta = -1,\ \lambda = 0.3,\ h = 1,\ \rho = 1.$

5.1. The effect of delay variable $Y(t)$ on the optimal controls From (4.12) and (4.13), it follows that the optimal investment and consumption depend on the delay variable $Y(t)$. Consequently,

$$\frac{\partial K^*}{\partial Y} = \mu_3 e^{\lambda h}\Big(\frac{k}{1-\delta} + \rho\sigma\frac{g_\eta}{g}\Big)$$
$$\begin{cases} = 0, & \mu_3 = 0, \\ > 0, & \mu_3 > 0, \\ < 0, & \mu_3 < 0 \end{cases}$$

and

$$\frac{\partial C^*}{\partial Y} = \mu_3 e^{\lambda h}\alpha^{1/(1-\delta)} g^{-1}$$
$$\begin{cases} = 0, & \mu_3 = 0, \\ > 0, & \mu_3 > 0, \\ < 0, & \mu_3 < 0. \end{cases}$$

In other words, we have the following:

(i) if $\mu_3 = 0$, then $\partial K^*/\partial Y = \partial C^*/\partial Y = 0$, which is the case without delay; the optimal investment and consumption strategies K^*, C^* do not depend on Y;
(ii) if $\mu_3 > 0$, then $\partial K^*/\partial Y > 0, \partial C^*/\partial Y > 0$, which means that the delay factor takes a positive effect on the optimal investment and consumption strategies;
(iii) if $\mu_3 < 0$, then $\partial K^*/\partial Y < 0, \partial C^*/\partial Y < 0$, which means that the delay factor takes a passive effect on the optimal investment and consumption strategies.

Figures 1 and 2 show the effect of delay variable $Y(t)$ on the optimal investment and optimal consumption rate. The parameters are described earlier in Section 5. Here we set $Y = 2$, $Y = 20$ and $Y = 40$, respectively. As is shown in Figures 1(a) and 2(a), let $\mu_3 = 0$, which can be interpreted as the case without delay; the optimal investment and optimal consumption rate are not affected by the value of Y. In Figures 1(b) and 2(b), let $\mu_3 = 0.01 > 0$, which means that the delay factor takes a positive effect on the optimal investment and optimal consumption strategies. From the results of Figures 1(b) and 2(b), the bigger the delay variable, the greater are the optimal investment and optimal consumption rate. We can see that, if a stock price is increasing for some time, the investor will be willing to invest more money to that stock. In Figures 1(c) and 2(c), let $\mu_3 = -0.01 < 0$, which means that the delay factor takes a passive effect on the optimal investment and optimal consumption strategies. On another note, once the price of a stock decreases, the investor may give it up and buy other stocks. This is coincident with the result of Figures 1(c) and 2(c).

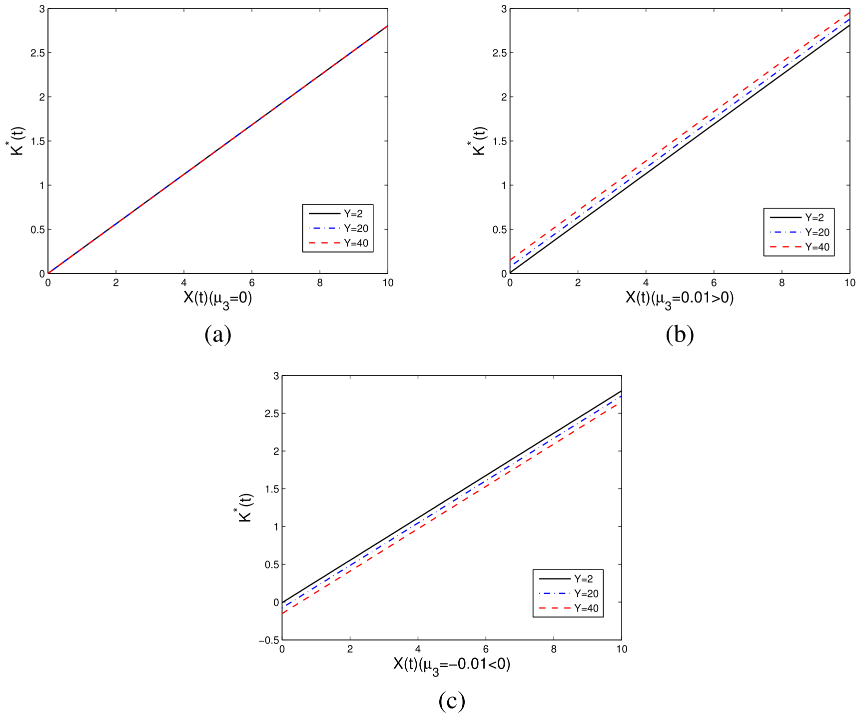

FIGURE 1. Optimal investment K^* as a linear function of wealth X and delay variable Y.

5.2. The effect of wealth $X(t)$ on the optimal investment and consumption

Note that

$$\frac{\partial K^*}{\partial X} = \frac{k}{1-\delta} + \rho\sigma\frac{g_\eta}{g} > 0,$$
$$\frac{\partial C^*}{\partial X} = \alpha^{1/(1-\delta)} g^{-1} > 0.$$

Therefore, the optimal investment and consumption controls K^* and C^* both increase with respect to wealth $X(t)$.

Figures 3 and 4 illustrate the effect of the wealth $X(t)$ on the optimal controls. The curves show that the optimal controls K^* and C^* increase with the aggregation of the wealth X.

6. Conclusion

We considered an optimal portfolio problem with delay in the stochastic factor framework. We established the associated HJB equation by employing the stochastic dynamic programming approach and a power transformation. Furthermore, a general expression form for a CRRA utility function was derived. On this basis, we provided

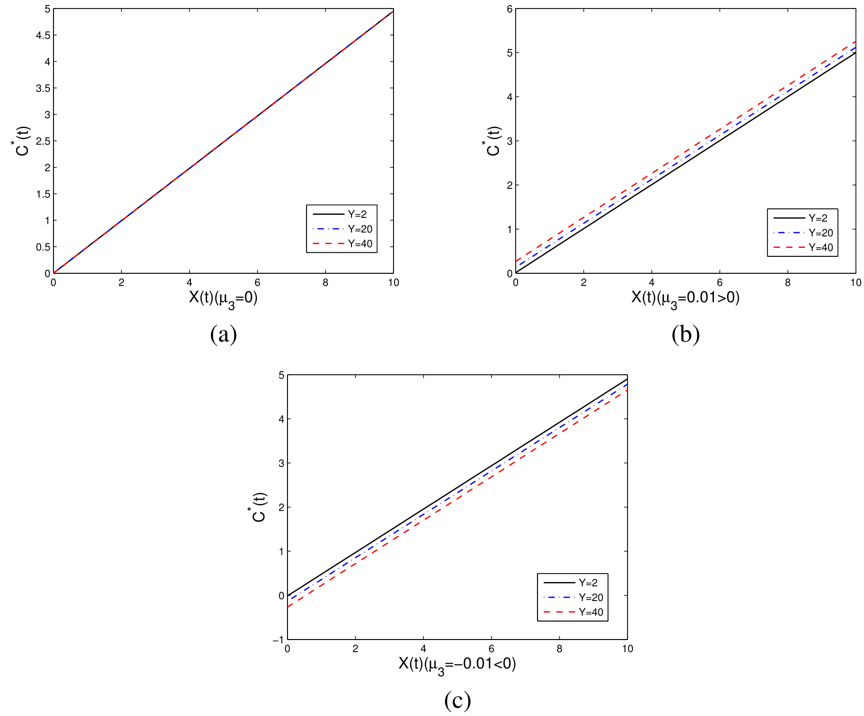

FIGURE 2. Optimal consumption rate C^* as a linear function of wealth X and delay variable Y.

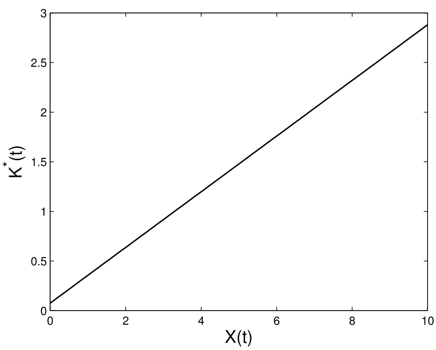

FIGURE 3. Optimal investment K^* as a linear function of X.

a special case in which the dynamics of the stochastic factor degenerates a CIR stochastic volatility model. We also discussed the effects of the delay variable $Y(t)$ on the optimal investment and consumption strategies with respect to different values of μ_3.

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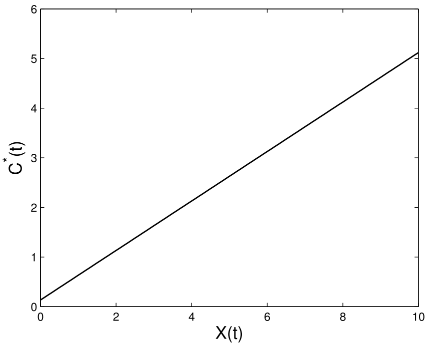

FIGURE 4. Optimal consumption rate C^* as a linear function of X.

There are many related topics which may be worthy of research in future. As illustrated in this paper, we studied the optimal portfolio problem in which the stochastic factor has a general expression by the dynamic programming principle and a power transformation method. However, it will be an interesting case that has multiple stochastic factors. Another interesting topic that deserves investigation is the correlation between the problems with one stochastic factor and multiple stochastic factors.

Acknowledgements

We thank the anonymous reviewers for their careful reading and helpful comments on an earlier version of this paper. This work was jointly supported by the National Natural Science Foundation of China (No. 61304065) and the Qing Lan Project of Jiangsu Province.

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