BULL. AUSTRAL. MATH. SOC. 47H10, 47H09
VOL. 45 (1992) [25–36]

A NONLINEAR ERGODIC THEOREM FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

KOK-KEONG TAN AND HONG-KUN XU

Let X be a real uniformly convex Banach space satisfying the Opial's condition, C a bounded closed convex subset of X, and $T\colon C \to C$ an asymptotically nonexpansive mapping. Then we show that for each x in C, the sequence $\{T^n x\}$ almost converges weakly to a fixed point y of T, that is,

$$\text{weak-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x = y \quad \text{uniformly in } k \geqslant 0.$$

This implies that $\{T^n x\}$ converges weakly to y if and only if T is weakly asymptotically regular at x, that is, weak-$\lim_{n\to\infty}\left(T^{n+1}x - T^n x\right) = 0$. We also present a weak convergence theorem for asymptotically nonexpansive semigroups.

1. INTRODUCTION

Let C be a closed convex subset of a Banach space X and T be a mapping from C into itself. Then T is said to be a Lipschitzian mapping if there exists, for each integer $n \geqslant 1$, a corresponding real number $\lambda_n > 0$ such that

$$\|T^n x - T^n y\| \leqslant \lambda_n \|x - y\|$$

for all $x, y \in C$. A Lipschitzian mapping T is called nonexpansive if $\lambda_n = 1$ for all $n \geqslant 1$ and asymptotically nonexpansive if $\lim_{n\to\infty} \lambda_n = 1$, respectively. We denote by $F(T)$ the set of fixed points of T. The first nonlinear ergodic theorem for nonexpansive mappings was proved in 1975 by Baillon [1]: Let C be a bounded closed convex subset of a Hilbert space H and T be a nonexpansive mapping from C into itself. Then for each $x \in C$, the Cesaro means

$$S_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$$

converge weakly to some $y \in F(T)$. In 1979, Reich [13] and Bruck [2] independently generalised Baillon's theorem to a setting of a uniformly convex Banach space with a

Received 8 January 1991

Frechet differentiable norm. (See Hirano [7] for another proof.) In 1982, Hirano [8] proved that the conclusion of Baillon's theorem is valid in a uniformly convex Banach space satisfying the Opial's condition. On the other hand, Hirano and Takahashi [9] proved that in a Hilbert space setting, Baillon's theorem holds true for asymptotically nonexpansive mappings. (This is in fact true [16] even for a wider class of mappings of asymptotically nonexpansive type [10].) However, whether Baillon's theorem is valid for asymptotically nonexpansive mappings in a Banach space setting remained open for a few years. Recently, the authors [14] have provided an affirmative answer to this question in a uniformly convex Banach space which has a Frechet differentiable norm. The purpose of this paper is to prove a counterpart to the result in [14]. That is, we show that if X is a uniformly convex Banach space satisfying the Opial's condition, C a bounded closed convex subset of X, and $T\colon C \to C$ an asymptotically nonexpansive mapping, then for each $x \in C$, the sequence $\{T^n x\}$ almost converges weakly to a fixed point of T, that is, there is a $y \in F(T)$ such that

$$\text{weak-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x = y \quad \text{uniformly in } k \geqslant 0.$$

This not only gives the above question another positive answer, but also implies that $\{T^n x\}$ converges weakly to y if and only if T is weakly asymptotically regular at x, that is, weak-$\lim\limits_{n\to\infty} \left(T^{n+1}x - T^n x\right) = 0$. We also present a weak convergence theroem for asymptotically nonexpansive semigroups. Our results generalise those of Hirano [8] and our proofs employ ideas of Hirano [8], Tan and Xu [14], and a technique of Bruck [2, 3].

2. Preliminaries and Lemmas

Recall that a Banach space X is said to satisfy the Opial's condition ([2]) if for any sequence $\{x_n\}$ in X, the condition $x_n \to x_0 \in X$ weakly implies that $\liminf\limits_{n\to\infty} \|x_n - x_0\| < \liminf\limits_{n\to\infty} \|x_n - x\|$, or equivalently $\limsup\limits_{n\to\infty} \|x_n - x_0\| < \limsup\limits_{n\to\infty} \|x_n - x\|$ for all $x \neq x_0$. It is known [12] that all Hilbert spaces and $\ell^p (1 < p < \infty)$ satisfy the Opial's condition. However, the $L^p (1 < p < \infty)$ spaces do not unless $p = 2$. A deeper result, shown by van Dulst [4], is that every separable Banach space can be equivalently renormed so that it possesses the Opial's condition.

Let F be a closed convex subset of a Banach space X and $\{x_n\}$ be a bounded sequence in X. Then we let

$$r(\{x_n\}, y) = \limsup_{n\to\infty} \|x_n - y\|,$$

and

$$r(\{x_n\}, F) = \min\{r(\{x_n\}, y)\colon y \in F\}.$$

We note here that, in Edelstein's terminology [5], the number $r(\{x_n\}, F)$ is called the asymptotic radius of the sequence $\{x_n\}$ with respect to the set F. We now establish some lemmas for later use. The following two lemmas are easy to prove (see Hirano [8]).

Lemma 2.1. *Let F be a closed convex subset of a reflexive Banach space X and $\{x_n\}$ be a bounded sequence in X such that for each $y \in F$, $\lim_n \|x_n - y\|$ exists. Then there is a $y_0 \in F$ such that*

$$\lim_n \|x_n - y_0\| = \min\{\lim_n \|x_n - y\| : y \in F\}.$$

Lemma 2.2. *Let F be a closed convex subset of a uniformly convex Banach space X and Λ be a set of bounded sequences in X which satisfies the following conditions:*

(i) *if $\{x_n\} \in \Lambda$, then for each $y \in F$, $\lim_n \|x_n - y\|$ exists;*

(ii) *if $\{x_n\}, \{y_n\} \in \Lambda$, then there exists $\{z_n\} \in \Lambda$ such that $r(\{z_n\}, y) \leqslant r(\{x_n\}, y)$ and $r(\{z_n\}, y) \leqslant r(\{y_n\}, y)$ for every $y \in F$.*

Let $r = \inf\{r(\{x_n\}, F) : \{x_n\} \in \Lambda\}$ and $\{\{x_n^{(i)}\} : i \geqslant 1\}$ be a sequence in Λ such that $\lim_i r\left(\{x_n^{(i)}\}, F\right) = r$. Then there exists a sequence $\{z_i\} \subset F$ such that $r\left(\{x_n^{(i)}\}, F\right) = r\left(\{x_n^{(i)}\}, z_i\right)$ for all $i \geqslant 1$ and $\{z_i\}$ converges strongly to a point in F.

Lemma 2.3. *Let X be a uniformly convex Banach space satisfying the Opial's condition, C a bounded closed convex subset of X, and $\{x_n\}$ a sequence in C such that $\limsup_{m\to\infty} \left(\limsup_{n\to\infty} \|T^m x_n - x_n\|\right) = 0$ and $\lim_{n\to\infty} \|x_n - y\|$ exists for each $y \in F(T)$. Then $\{x_n\}$ converges weakly to a point z in $F(T)$ such that $r(\{x_n\}, z) = r(\{x_n\}, F(T))$.*

Proof: We first note that by Geobel and Kirk [6], $F(T)$ is closed convex and nonempty. By Lemma 2.3 of [14], every weak limit point of the sequence $\{x_n\}$ is a fixed point of T. Suppose now $x_{n_i} \to u$ and $x_{m_j} \to v$ weakly; then $u, v \in F(T)$. If $u \neq v$, then the Opial's condition of X implies that

$$\begin{aligned} \lim_n \|x_n - u\| &= \lim_i \left\|x_{n_i} - u\right\| \\ &< \lim_i \left\|x_{n_i} - v\right\| = \lim_j \left\|x_{m_j} - v\right\| \\ &< \lim_j \left\|x_{m_j} - u\right\| = \lim_n \|x_n - u\|. \end{aligned}$$

This is a contradition, proving that $\{x_n\}$ converges weakly to some $z \in F(T)$. The equality $r(\{x_n\}, z) = r(\{x_n\}, F(T))$ now follows directly from the Opial's condition of X. The proof is complete. ∎

LEMMA 2.4. ([14]). *Let C be a bounded closed convex subset of a uniformly convex Banach space X, and $T: C \to C$ an asymptotically nonexpansive mapping. Then for each $x \in C$, each integer $n \geqslant 1$, and arbitrary $\varepsilon > 0$, there exist integers i_n and k_ε depending only on n and ε, respectively, such that*

$$\left\|T^k S_n T^i x - S_n T^k T^i x\right\| \leqslant (1+\varepsilon) g^{-1}\left(\frac{1}{n} + \varepsilon M\right) \tag{2.1}$$

for all $k \geqslant k_\varepsilon$ and $i \geqslant i_n$, where $S_n = (1/n)\left(I + T + \ldots + T^{n-1}\right)$ with I the identity operator of X, $M = \operatorname{diam}(T^n x)$, and $g: [0, \infty) \to [0, \infty)$ is a strictly increasing convex continuous function such $g(0) = 0$.

COROLLARY 2.1. *Let C, T, and i_n be as in Lemma 2.4. Then for all sequences $\{j_n\}$, $\{k_n\}$ of integers such that $j_n \geqslant i_n$ for all $n \geqslant 1$ and $\lim_n k_n = \infty$, we have*

$$\lim_n \left\|T^{k_n} S_n T^{j_n} x - S_n T^{k_n + j_n} x\right\| = 0. \tag{2.2}$$

In the sequel, we always assume that the integers $\{i_n\}$ in Lemma 2.4 are chosen so that $i_1 < i_2 < \ldots < i_n < \ldots \to \infty$.

LEMMA 2.5. *Let C be a bounded closed convex subset of a uniformly convex Banach space X with the Opial's condition, $T: C \to C$ an asymptotically nonexpansive mapping, and x an element of C. Suppose $\{k_n\}$ is a sequence of integers such that $k_n > i_{2^n}$ and $k_{n+1} > k_n + i_{2^n}$ for all $n \geqslant 1$ (where i_{2^n} is as selected in Lemma 2.4.) Then for each $y \in F(T)$, $\lim_n \left\|S_{2^n} T^{k_n} x - y\right\|$ exists and $\{S_{2^n} T^{k_n} x\}$ converges weakly to some $z \in F(T)$.*

PROOF: For a fixed $y \in F(T)$, let $r := \liminf_{n \to \infty} \left\|S_{2^n} T^{k_n} x - y\right\|$. It follows from (2.1) that

$$\left\|T^k S_{2^n} T^{k_n} x - S_{2^n} T^k T^{k_n} x\right\| \leqslant (1+\varepsilon) g^{-1}\left(\frac{1}{2^n} + \varepsilon M\right)$$

for each $n \geqslant 1$ and all $k \geqslant k_\varepsilon$. Since T is asymptotically nonexpansive, we have an integer $\overline{k} > k_\varepsilon$ such that

$$\lambda_k < 1 + \varepsilon \quad \text{for} \quad k \geqslant \overline{k}. \tag{2.3}$$

We then have an integer n large enough so that

$$\left\|S_{2^n} T^{k_n} x - y\right\| < r + \varepsilon, \quad k_{n+1} - k_n > \overline{k}, \quad \text{and} \quad 2^{-n} < \varepsilon. \tag{2.4}$$

It follows from (2.1), (2.3) and (2.4) that

$$\begin{aligned}
&\left\|S_{2^{n+1}}T^{k_{n+1}}x - y\right\| \\
&\quad= \left\|\left(T^{k_{n+1}}x + T^{k_{n+1}+1}x + \ldots + T^{k_{n+1}+2^{n+1}-1}x\right)/2^{n+1} - y\right\| \\
&\quad= \left\|\frac{1}{2}\left(S_{2^n}T^{k_{n+1}}x + S_{2^n}T^{k_{n+1}+2^n}x\right) - y\right\| \\
&\quad\leqslant \left(\left\|S_{2^n}T^{k_{n+1}}x - T^{k_{n+1}-k_n}S_{2^n}T^{k_n}x\right\| + \left\|T^{k_{n+1}-k_n}S_{2^n}T^{k_n}x - y\right\|\right)/2 \\
&\qquad+ \left(\left\|S_{2^n}T^{k_{n+1}+2^n}x - T^{k_{n+1}-k_n+2^n}S_{2^n}T^{k_n}x\right\|\right. \\
&\qquad+ \left.\left\|T^{k_{n+1}-k_n+2^k}S_{2^n}T^{k_n}x - y\right\|\right)/2 \\
&\quad\leqslant (1+\varepsilon)g^{-1}\left(2^{-n} + \varepsilon M\right) + (1+\varepsilon)(r+\varepsilon) \\
&\quad\leqslant (1+\varepsilon)g^{-1}(\varepsilon(1+M)) + (1+\varepsilon)(r+\varepsilon).
\end{aligned}$$

In the same way, we can prove

$$\left\|S_{2^{n+i}}T^{k_{n+i}}x - y\right\| \leqslant (1+\varepsilon)g^{-1}(\varepsilon(1+M)) + (1+\varepsilon)(r+\varepsilon)$$

for all $i \geqslant 1$, from which it follows that

$$\begin{aligned}
\limsup_{i\to\infty}\left\|S_{2^i}T^{k_i}x - y\right\| &= \limsup_{i\to\infty}\left\|S_{2^{n+i}}T^{k_{n+i}}x - y\right\| \\
&\leqslant (1+\varepsilon)g^{-1}(\varepsilon(1+M)) + (1+\varepsilon)(r+\varepsilon).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\limsup_{n\to\infty}\left\|S_{2^n}T^{k_n}x - y\right\| \leqslant \liminf_{n\to\infty}\left\|S_{2^n}T^{k_n}x - y\right\|,$$

showing $\lim_n \left\|S_{2^n}T^{k_n}x - y\right\|$ exists. Noticing for each $u \in C$ and each fixed integer $m \geqslant 1$,

$$\|S_n(T^m u) - S_n(u)\| \leqslant \frac{m}{n}\operatorname{diam}(C) \to 0 \quad \text{as} \quad n \to \infty,$$

we get by Lemma 2.4 that

$$\begin{aligned}
&\limsup_{m\to\infty}\left(\limsup_{n\to\infty}\left\|T^m S_{2^n}T^{k_n}x - S_{2^n}T^{k_n}x\right\|\right) \\
&\quad\leqslant \limsup_{m\to\infty}\left(\limsup_{n\to\infty}\left\{\left\|T^m S_{2^n}T^{k_n}x - S_{2^n}T^mT^{k_n}\right\|\right.\right. \\
&\qquad\left.\left. + \left\|S_{2^n}T^{m+k_n}x - S_{2^n}T^{k_n}x\right\|\right\}\right) \\
&\quad\leqslant (1+\varepsilon)g^{-1}(\varepsilon M) \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Now applying Lemma 2.3, we complete the proof of the lemma. ∎

3. The nonlinear ergodic theorem

In this section, we prove the main result of the paper. We begin by recalling the notion of almost convergence due to Lorentz [11].

Definition: Let X be a Banach space. A sequence $\{x_n\}_{n=0}^{\infty}$ in X is said to be weakly almost convergent to an element y of X if

$$\text{weak-}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}x_{i+k}=y \quad \text{uniformly in } k\geqslant 0.$$

Theorem 3.1. *Let X be a uniformly convex Banach space satisfying the Opial's condition, C a bounded closed convex subset of X, and $T\colon C\to C$ an asymptotically nonexpansive mapping. Then for each $x\in C$, the sequence $\{T^n x\}$ is weakly almost convergent to a fixed point of T. That is, there is a $z\in F(T)$ such that*

$$\text{weak-}\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}T^{i+k}x=z \quad \text{uniformly in } k\geqslant 0.$$

Proof: We first observe that T has a fixed point by Goebel and Kirk [6]. For a fixed $x\in C$, let

$$\Lambda=\{\{S_{2^n}T^{h_n}x\}: h_n>i_{2^n} \text{ and } h_{n+1}>h_n+i_{2^n} \text{ for all } n\geqslant 1\},$$

where i_{2^n} is chosen as in Lemma 2.4. Then each $\{S_{2^n}T^{h_n}x\}$ in Λ is bounded since C is bounded and by Lemma 2.5, $\lim_n\left\|S_{2^n}T^{h_n}x-y\right\|$ exists for every $y\in F(T)$ and $\{S_{2^n}T^{h_n}x\}$ converges weakly to a fixed point of T. Now let $\{S_{2^n}T^{h_n}x\}$ and $\{S_{2^n}T^{r_n}x\}$ be in Λ and let $p_n=\max(h_n,r_n)+n$. Then it is readily seen that $p_n>i_{2^n}$ and $p_{n+1}>p_n+i_{2^n}$ for all n and hence $\{S_{2^n}T^{p_n}x\}\in\Lambda$. Moreover, in view of Lemma 2.5 and Corollary 2.1, we derive for each $y\in F(T)$ that

$$\begin{aligned}
&\lim_n\|S_{2^n}T^{p_n}x-y\|\\
&\leqslant\lim_n\left(\left\|S_{2^n}T^{p_n}x-T^{p_n-h_n}S_{2^n}T^{h_n}x\right\|+\left\|T^{p_n-h_n}S_{2^n}T^{h_n}x-y\right\|\right)\\
&\leqslant\lim_n\left(\left\|S_{2^n}T^{p_n-h_n}T^{h_n}x-T^{p_n-h_n}S_{2^n}T^{h_n}x\right\|+\lambda_{p_n-h_n}\left\|S_{2^n}T^{h_n}x-y\right\|\right)\\
&=\lim_n\left\|S_{2^n}T^{h_n}x-y\right\|,
\end{aligned}$$

that is,

$$\lim_n\|S_{2^n}T^{p_n}x-y\|\leqslant\lim_n\left\|S_{2^n}T^{h_n}x-y\right\|. \tag{3.1}$$

Similarly, we have

$$\lim_n\|S_{2^n}T^{p_n}x-y\|\leqslant\lim_n\|S_{2^n}T^{r_n}x-y\|.$$

It then follows that Λ satisfies the hypotheses of Lemma 2.2 with $F = F(T)$. Set

$$r = \inf\{r(\{S_{2^n}T^{h_n}x\}, F(T)) : \{S_{2^n}T^{h_n}\} \in \Lambda\}$$

and choose a sequence $\{\{S_{2^n}T^{h_n^{(j)}}x\}\}_{j\geqslant 1}$ in Λ such that $\lim_j r\left(\{S_{2^n}T^{h_n^{(j)}}x\}, F(T)\right) = r$. Then by Lemma 2.2, there exists a sequence $\{y_j\}$ in $F(T)$ which satisfies the equality $r\left(\{S_{2^n}T^{h_n^{(j)}}x\}, F(T)\right) = r\left(\{S_{2^n}T^{h_n^{(j)}}x\}, y_j\right)$ for all $j \geqslant 1$, and converges strongly to some $y \in F(T)$. Define $h_n = \max\left(h_n^{(j)} : 1 \leqslant j \leqslant n\right) + n$ for all $n \geqslant 1$. Then it is easily seen that $\{S_{2^n}T^{h_n}x\} \in \Lambda$. Similarly to (3.1), we can prove that

$$\begin{aligned} r(\{S_{2^n}T^{h_n}x\}, y) &= \lim_j r(\{S_{2^n}T^{h_n}x\}, y_j) \\ &\leqslant \lim_j r\left(\{S_{2^n}T^{h_n^{(j)}}x\}, y_j\right) \\ &= r. \end{aligned}$$

It thus follows that

$$r(\{S_{2^n}T^{h_n}x\}, F(T)) = r(\{S_{2^n}T^{h_n}x\}, y) = r \tag{3.2}$$

and $\{S_{2^n}T^{h_n}x\}$ converges weakly to y by the Opial's condition and Lemma 2.5. We now prove the following

CLAIM: Each $\{S_{2^n}T^{t_n}x\} \in \Lambda$, with $t_n \geqslant h_n + n$ for all n, must converge weakly to y.

In fact, by Lemma 2.5, $\{S_{2^n}T^{t_n}x\}$ converges weakly to a point, say z, in $F(T)$. If $z \neq y$, then it follows from (3.1) and the Opial's condition of X that

$$\begin{aligned} r &\leqslant \lim_n \|S_{2^n}T^{t_n}x - z\| < \lim_n \|S_{2^n}T^{t_n}x - y\| \\ &\leqslant \lim_n \|S_{2^n}T^{h_n}x - y\| = r. \end{aligned}$$

This contradiction proves the claim. Since $r\left(\{S_{2^n}T^{h_n+k_n2^n+j_n}(x)\}, y\right) = r$ for any sequences $\{k_n\}$ and $\{j_n\}$, by the same way as above, we can prove that $\{S_{2^n}T^{h_n+k2^n+j}(x)\}$ converges weakly to y as $n \to \infty$ uniformly in $k, j \geqslant 0$. We are now in a position to complete the proof of the theorem. For any integers n and m with $m > h_n$, we have

$$\begin{aligned} S_mT^ix &= \frac{1}{m}\sum_{k=0}^{m-1} T^{k+i}x \\ &= \frac{1}{m}\left\{\sum_{k=0}^{h_n-1} T^{k+i}x + 2^n\left(\sum_{k=0}^{j-1} S_{2^n}T^{h_n+k2^n+i}x\right) + \sum_{k=h_n+j2^n}^{m-1} T^{k+i}x\right\}, \end{aligned}$$

https://doi.org/10.1017/S0004972700036972 Published online by Cambridge University Press

where $m = j2^n + h_n + r$, $0 \leqslant r < 2^n$. Since $\{S_{2^n}T^{h_n+k2^n+i}x\}$ converges weakly to y as $n \to \infty$ uniformly in $k, i \geqslant 0$, we conclude that $\{S_mT^ix\}$ converges weakly to y as $m \to \infty$ uniformly in $i \geqslant 0$. This completes the proof. ∎

Recall that T is said to be weakly asymptotically regular at $x \in C$ if $\text{weak-}\lim_n (T^{n+1}x - T^nx) = 0$.

THEOREM 3.2. *Let C, X and T be as in Theorem 3.1. Then for each $x \in C$, the sequence $\{T^nx\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x.*

PROOF: Necessity is trivial. Sufficiency follows from Theorem 3.1 and the fact that the weak asymptotic regularity of T at x is a Tauberian condition for weak almost convergence of $\{T^nx\}$ (see Lorentz [11]). ∎

4. NONLINEAR SEMIGROUPS

Let C be a closed convex subset of a Banach space X. A one parameter family $\mathcal{F} = \{T(t) : t \geqslant 0\}$ of mappings from C into itself is said to be a Lipschitzian semigroup on C if the following conditions are satisfied:

(1) $T(0)x = x$ for $x \in C$;
(2) $T(t+s)x = T(t)T(s)x$ for $x \in C$ and $t, s \geqslant 0$;
(3) for each $x \in C$, the mapping $T(t)x$ is continuous for $t \in [0, \infty)$;
(4) for each $t > 0$, there exists a real number $\lambda_t > 0$ such that

$$\|T(t)x - T(t)y\| \leqslant \lambda_t \|x - y\| \quad \text{for} \quad x, y \in C.$$

A Lipschitzian semigroup $\mathcal{F}$ is said to be nonexpansive if $\lambda_t = 1$ for all $t > 0$ and asymptotically nonexpansive if $\lim_{t\to\infty} \lambda_t = 1$, respectively. We denote by $F(\mathcal{F})$ the set of common fixed points of the semigroup $\mathcal{F}$, that is, $F(\mathcal{F}) = \bigcap_{t>0} F(T(t))$. If C is assumed to be a bounded closed convex subset of a uniformly convex Banach space and if $\mathcal{F} = \{T(t) : t \geqslant 0\}$ is an asymptotically nonexpansive semigroup on C, then it has been shown (see [15]) that $F(\mathcal{F})$ is closed, convex and nonempty. In this case, the metric projection P from X onto $F(\mathcal{F})$ is well-defined. If we assume, in addition, that $\mathcal{F} = \{T(t) : t \geqslant 0\}$ is nonexpansive and the space X either has a Frechet differentiable norm or satisfies the Opial's condition, then it has also been shown (see [2], [3], [8]) that for each $x \in C$, $\{T(t)x\}$ converges weakly to a common fixed point of $\mathcal{F}$ if and only if $\mathcal{F}$ is weakly asymptotically regular at x, that is, $\text{weak-}\lim_{t\to\infty} (T(t+h)x - T(t)x) = 0$ for all $h > 0$. The same conclusion was recently shown true by the authors [15] for an asymptotically nonexpansive semigroup $\mathcal{F}$ on C in the case when X has a Frechet differentiable norm. The object of this section is to show a counterpart in the case, when X satisfies the Opial's condition.

Theorem 4.1. *Let X be a uniformly convex Banach space satisfying the Opial's condition, C a bounded closed convex subset of X, and $\mathcal{F} = \{T(t) : t \geqslant 0\}$ an asymptotically nonexpansive semigroup on C. Then for each $x \in C$, $\{T(t)x\}$ converges weakly to a member of $F(\mathcal{F})$ if and only if $\mathcal{F}$ is weakly asymptotically regular at x, that is, weak-$\lim_{t\to\infty}(T(t+h)x - T(t)x) = 0$ for all $h > 0$.*

Proof: It suffices to show the sufficiency part. We first show that if $u =$ weak-$\lim_k T(t_k)x$ for some sequence $\{t_k\}$ of real numbers such that $\lim_k t_k = \infty$, then $u \in F(\mathcal{F})$. Under this assumption, since $\mathcal{F}$ is weakly asymptotically regular at x, we see that weak-$\lim_k T(t_k + s)x = u$ for all $s \geqslant 0$. Let

$$r_s = \limsup_{k\to\infty} \|T(t_k + s)x - u\|.$$

Using the Opial's condition, we get for all $s, t \geqslant 0$,

$$\begin{aligned} r_{s+t} &= \limsup_{k\to\infty} \|T(t_k + s + t)x - u\| \\ &\leqslant \limsup_{k\to\infty} \|T(t)T(t_k + s)x - T(t)u\| \\ &\leqslant \lambda_t \limsup_{k\to\infty} \|T(t_k + s)x - u\| \\ &= \lambda_t r_s, \quad \text{namely,} \end{aligned}$$

$$r_{s+t} \leqslant \lambda_t r_s \quad \text{for all} \quad s, t \geqslant 0. \tag{4.1}$$

From this, it follows that $\lim_{t\to\infty} r_t =: r$ exists and $r \leqslant r_s$ for all $s \geqslant 0$. If $r = 0$, then it is immediate that $u \in F(\mathcal{F})$. So, we assume $r > 0$. In this case, we show that $T(t)u \to u$ strongly as $t \to \infty$. Suppose not; then there is a sequence $\{\bar{t}_n\}$ for which $\lim_n \bar{t}_n = \infty$ such that

$$\|T(\bar{t}_n)u - u\| \geqslant \varepsilon_0, \qquad n = 1, 2, \ldots \tag{4.2}$$

for some $\varepsilon_0 > 0$. Choose $0 < \eta < \varepsilon_0$ so small that

$$(r + \eta)(1 - \delta(\varepsilon_0/(r + \eta))) < r, \tag{4.3}$$

where δ is the modulus of convexity of X. Choose N and s_0 so that

$$\lambda_{\bar{t}_N} r_{s_0} < r + \eta,$$

where $\lambda_{\bar{t}_N}$ is the Lipschitz constant of $T(\bar{t}_N)$. Using the Opial's condition of X and combining (4.1), (4.2) and (4.4), it yields

$$\begin{aligned} r \leqslant r_{s_0+\bar{t}_N} &= \limsup_{k\to\infty} \left\|T(t_k + s_0 + \bar{t}_N)s - u\right\| \\ &\leqslant \limsup_{k\to\infty} \left\|T(t_k + s_0 + \bar{t}_N)x - \frac{1}{2}\big(T(\bar{t}_N)u + u\big)\right\| \\ &\leqslant \lambda_{\bar{t}_N} r_{s_0}\Big(1 - \delta\Big(\varepsilon_0/\Big(\lambda_{\bar{t}_N} r_{s_0}\Big)\Big)\Big) \\ &\leqslant (r+\eta)(1 - \delta(\varepsilon_0/(r+\eta))), \end{aligned}$$

which contradicts (4.3) and therefore, $T(t)u \to u$ strongly as $t \to \infty$. This implies that $u \in F(\mathcal{F})$ by continuity of $\mathcal{F}$. Now we set

$$d(t) = \|T(t)x - PT(t)x\|, \quad t \geqslant 0,$$

where P is the metric projection of X onto $F(\mathcal{F})$. Since

$$\begin{aligned} d(t+s) &= \|T(t+s)x - PT(t+s)x\| \\ &\leqslant \|T(t+s)x - PT(t)x\| \\ &= \|T(s)T(t)x - T(s)PT(t)x\| \\ &\leqslant \lambda_s \|T(t)x - PT(t)x\| \\ &= \lambda_s d(t) \end{aligned}$$

for all t, $s \geqslant 0$, it follows that $d := \lim_{t\to\infty} d(t)$ exists and $d \leqslant d(t)$ for all $t \geqslant 0$. We now claim that $\{PT(t)x\}$ is norm Cauchy. This is trivially valid if $d = 0$. Suppose now $d > 0$. For any $\varepsilon > 0$, choose first $\eta > 0$ such that

$$(d+\eta)(1 - \delta(\varepsilon/(d+\eta))) < d \tag{4.5}$$

and then t_0 such that

$$d(t) < d + \frac{1}{2}\eta \quad \text{and} \quad \lambda_t\left(d + \frac{1}{2}\eta\right) < d + \eta \tag{4.6}$$

for all $t \geqslant t_0$. Now let t_1, $t_2 \geqslant t_0$ be arbitrary but fixed. If $\|PT(t_1)x - PT(t_2)x\| \geqslant \varepsilon$, then, since

$$\begin{aligned} \|T(t_0+t_1+t_2)x - PT(t_1)x\| &= \|T(t_0+t_2)T(t_1)x - T(t_0+t_2)PT(t_1)x\| \\ &\leqslant \lambda_{t_0+t_2} \|T(t_1)x - PT(t_1)x\| \\ &= \lambda_{t_0+t_2} d(t_1) < \lambda_{t_0+t_2}\left(d + \frac{1}{2}\eta\right) < d + \eta, \end{aligned}$$

we get

$$\begin{aligned} d \leqslant d(t_0 + t_1 + t_2) &= \|T(t_0 + t_1 + t_2)x - PT(t_0 + t_1 + t_2)x\| \\ &\leqslant \left\| T(t_0 + t_1 + t_2)x - \frac{1}{2}(PT(t_1)x + PT(t_2)x) \right\| \\ &\leqslant (d + \eta)(1 - \delta(\varepsilon/(d + \eta))), \end{aligned}$$

a contradiction to (4.5). This shows $\|PT(t_1)x - PT(t_2)x\| < \varepsilon$ and hence $\{PT(t)x\}$ is norm Cauchy. Let $y = \lim_{t \to \infty} PT(t)x$ and $u = \text{weak-}\lim_k T(t_k)x$ be an arbitrary weak limit point of $\{T(t)x\}$. If $u \neq y$, using the Opial's condition of X, we then obtain

$$\begin{aligned} \varliminf_k \|T(t_k)x - y\| &= \varliminf_k \|T(t_k)x - PT(t_k)x\| \\ &\leqslant \varliminf_k \|T(t_k)x - u\| \\ &< \varliminf_k \|T(t_k)x - y\|. \end{aligned}$$

This is a contradiction. We have therefore $u = y$ and $\{T(t)x\}$ converges weakly to y. The proof is complete. ∎

References

[1] J.B. Baillon, 'Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert', *C.R. Acad. Sci. Paris* **280** (1975), 1511–1514.

[2] R.E. Bruck, 'A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces', *Israel J. Math.* **32** (1979), 107–116.

[3] R.E. Bruck On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces, *Israel J. Math.* **34** (1981), 304–314.

[4] D. van Dulst, 'Equivalent norms and the fixed point property for nonexpansive mappings', *J. London Math. Soc.* **25** (1982), 139–144.

[5] M. Edelstein, 'The construction of an asymptotic center with a fixed-point property', *Bull. Amer. Math. Soc.* **78** (1972), 206–208.

[6] K. Goebel and W.A. Kirk, 'A fixed point theorem for asymptotically nonexpansive mappings', *Proc. Amer. Math. Soc.* **35** (1972), 171–174.

[7] N. Hirano, 'A proof of the mean ergodic theorem for nonexpansive mappings in Banach spaces', *Proc. Amer. Math. Soc.* **78** (1980), 361–365.

[8] N. Hirano, 'Nonlinear ergodic theorems and weak convergence theorems', *J. Math. Soc. Japan* **34** (1982), 35–46.

[9] N. Hirano and W. Takahashi, 'Nonlinear ergodic theorems for nonexpansive mappings in Hilbert spaces', *Kodai Math. J.* **2** (1979), 11–25.

[10] W.A. Kirk, 'Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type', *Israel J. Math.* **17** (1974), 339–346.

[11] G.G. Lorentz, 'A contribution to the theory of divergent series', *Acta Math.* **80** (1948), 167–190.

[12] Z. Opial, 'Weak convergence of the sequence of successive approximations for nonexpansive mappings', *Bull. Amer. Math. Soc.* **73** (1967), 591–597.

[13] S. Reich, 'Weak convergence theorems for nonexpansive mappings in Banach spaces', *J. Math. Anal. Appl.* **57** (1979), 274–276.

[14] K.K. Tan and H.K. Xu, 'The nonlinear ergodic theorem for asymptotically nonexpansive mappings in Banach spaces', *Proc. Amer. Math. Soc.* (to appear).

[15] K.K. Tan and H.K. Xu, 'An ergodic theorem for nonlinear semigroups of Lipschitzian mappings in Banach spaces', *Nonlinear Anal.* (to appear).

[16] Z.Y. You and H.K. Xu, 'An ergodic convergence theorem for mappings of asymptotically nonexpansive type', *Chinese Ann. of Math.* **11A** (1990), 519–523. (in Chinese) .

Department of Mathematics,
Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia
Canada B3H 3J5

Institute of Applied Mathematics
East China University of Chemical Technology
Shanghai 200237
People's Republic of China