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RADIALLY GEOMETRIC STABLE DISTRIBUTIONS AND PROCESSES

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Abstract

Motivated by the investigation of probability distributions with finite variance but heavy tails, we study infinitely divisible laws whose Lévy measure is characterized by a radial component of geometric (tempered) stable type. We closely investigate the univariate case: characteristic exponents and cumulants are calculated, as well as spectral densities; absolute continuity relations are shown, and short- and long-time scaling limits of the associated Lévy processes analyzed. Finally, we derive some properties of the involved probability density functions.

Keywords: Geometric stable distributions; tempered stable distributions; Lévy processes; Mittag-Leffler function; generalized hypergeometric functions

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1. Introduction

In physics and natural sciences, systems evolving according to Lévy stable laws have long been observed (see [9] for a survey). Modeling these phenomena as "Lévy flights/walks", i.e. random processes with stationary independent stable increments, however has the serious drawback of producing dynamic probabilistic representations with infinite variance, which is problematic both theoretically and in practice. In the seminal works of [43] and [30], a remedy was proposed by introducing a truncation procedure that, while inducing a minimal perturbation of the central part of distribution, impacted the tails in such a way as to recover finiteness of variance, and thus ultimately the Gaussian behavior of the process at large times. Since the observed convergence is very slow, for most practical purposes the Lévy stable empirical paradigm is robust to this modification. This idea proved be very successful and enjoyed a vast range of applications even outside physics, most notably in economics and finance: see [6, 8, 58]. In particular, Koponen's idea of using a negative exponential cutoff function, yielding to a tractable analytic structure of the law, proved to be of consequence. Such procedure began to be known as "tempering".

For some applications, however, exponential tempering is not fully satisfactory. For example, a large number of independent statistical studies from around the turn of the century (e.g. [40, 42, 18, 19, 48]) have reported strong evidence of power-law decay (and scaling) of market returns that survive even at long lags, and with typical Pareto exponents of about 3. Therefore,

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according to these estimates, the existence of the variance in financial data is not called into question, while the existence of higher moments remains a more delicate issue. It would then seem appropriate to seek a dynamic return model whose distribution is of stable type in its central part and exhibits heavy power-law tails at longer lags, albeit with finite variance, and hence retains Gaussian limiting properties. Bringing these features together is not easy. As argued in [9] and [58], abrupt density truncation, or exponential tempering as proposed by [30, 43], respectively, determine too much of a sharp cutoff and do not maintain the power-law decay for all the orders of magnitude for which it is observed in the empirical data, typically ranging from milliseconds to several days. One possible solution is proposed in [57], where truncation is achieved by means of a model whose density function solves a Fokker–Planck equation of fractional order. In [59] it is instead proposed to directly choose a Lévy measure in rational form. Neither of these approaches seem to yield full analytical tractability of the related densities and characteristic functions.

With the aim of capturing both finite variance and heavy tails, in this work we introduce a novel class of tempered distributions, whose radial component is given as an exponentially dampened negative Mittag-Leffler function. The Mittag-Leffler function is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad z \in \mathbb{C}, \quad \alpha \in (0, 1], \tag{1}$$

where Γ stands for Euler's gamma function. For $r > 0$ we thus consider distributions whose Lévy measures are given in polar coordinates, with radial component of the form

$$\frac{e^{-\theta r} E_\alpha(-\lambda r^\alpha)}{r^{1+\gamma}} dr, \qquad r, \lambda, \theta > 0, \ \alpha \in (0, 1), \ \gamma \in [0, 1). \tag{2}$$

In the above we see a combination of a stable inverse-power component, a classic exponential tempering function and a negative Mittag-Leffler factor. The latter is a type of generalized exponential function with rapid decay around zero and heavy tails.

The genesis of the proposed distribution can be retraced in the concept of geometric stability, introduced in [29], answering the problem of finding a class of random variables that are infinitely divisible under random geometric summation. A geometric stable law X can be defined as the distribution whose characteristic exponent ψ_X can be written as a log-transform of a stable characteristic exponent ψ_Z, i.e.

$$\psi_X(z) = -\log(1 - \psi_Z(z)) \tag{3}$$

for some stable distribution Z. Such laws are naturally infinitely divisible, because the characteristic exponent above corresponds to the unit time law of the Lévy process obtained by subordinating the Lévy process stemming from Z to a unit scale and shape gamma process. Theory and applications have been developed by, among others, [29, 32, 34, 38, 45, 52]. Known probability distributions such as the Laplace distribution [47] and Mittag-Leffler law are included in this class. Introducing a generalizing shape parameter, symmetric geometric strictly stable laws lead to the Linnik distribution, as discussed in [46] (see also [10, 39]), and the Erlang/gamma distribution as special cases. Tempered versions of the positive Linnik law (TPL) have been considered in [1, 2, 61].

Geometric stability inspires the framework we present in that for univariate distributions, the Lévy measure of a positive geometric strictly stable law (Pillai's) is known to be of the form (2) with $\gamma = 0$. The presence of $\theta > 0$ signifies that the strictly stable law can itself be

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subject to exponential tempering. A radially geometric (tempered) strictly stable distribution (RG(T)S) is then a probability distribution whose radial component in the polar representation of its Lévy measure follows a geometric (tempered) stable law. Considering further, $\gamma > 0$ yields the full generalized radially geometric tempered stable (GRGTS) class we introduce. The interest in such a generalization is that it determines the distributions belonging to the family of tempered stable distributions studied in [51], where γ retains the interpretation of the stability parameter of some other stable distribution. A large number of analytical tools are then available from [51], facilitating the analysis.

We find the characteristic exponents of GRGTS laws and their RGTS and RGS sub-cases, and discuss their analytical properties. Such exponents involve two interesting special functions, namely Dotsenko's ${}_2R_1$ generalized hypergeometric function [14, 62] and Lerch's transcendent Φ. We determine cumulants and spectral densities, as well as short- and long-time scaling limiting behavior. In particular, in the pure Mittag-Leffler case we observe that the large parameter scaling can follow a classic Gaussian limit, but can also converge to stable process, depending on the stability and Mittag-Leffler parameters. Moreover, we analyze the absolute continuity conditions of GRGTS laws within their own class as well as with respect to stable laws. We conclude the paper with an analysis of the probability densities of the GRGTS distribution class. By appealing to some classic results in the theory of self-decomposable distributions, most notably [54, 63, 64], we establish regularity, unimodality, and the asymptotic order of the tails of the probability density functions.

We review the basic notions needed and fix the notation in Section 2. In Section 3, GRGTS distributions are introduced. Their characteristic functions and cumulants are discussed in Section 4. In Section 5, we analyze spectral measures, short- and long-time limits, and absolute continuity properties. Section 6 is devoted to the properties of the probability densities. In Section 7 we conclude and discuss possible developments.

2. Preliminaries

We begin by establishing the notation and recalling some concepts and notions required throughout the paper.

2.1. Infinitely divisible laws and Lévy processes

Throughout the paper we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ to which all the processes we mention are adapted. For a random variable (r.v.) $X : \Omega \to \mathbb{R}^d$ on such a space we denote by $\Psi_X : \mathbb{R}^d \to \mathbb{C}$ its characteristic function

$$\Psi_X(z) = E[e^{i\langle z, X\rangle}]. \tag{4}$$

An infinitely divisible (i.d.) r.v. is fully characterized by its characteristic exponent $\psi_X(z)$, i.e. a function $\psi_X : D \subseteq \mathbb{R}^d \to \mathbb{C}$ such that

$$\Psi_X(z) := E[e^{i\langle z, X\rangle}] = e^{\psi_X(z)}. \tag{5}$$

The exponent ψ_X can be written in terms of a characteristic triplet (μ, Σ, ν), with $\mu \in \mathbb{R}^d$, Σ a positive definite $d \times d$ matrix, ν a measure on $\mathbb{R}^d$ such that $\nu(0) = 0$, $\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty$, with $|\cdot|$ indicating the Euclidean norm, and

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$$\psi_X(z) = i\langle z, \mu\rangle - \langle z, \Sigma z\rangle/2 + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbb{1}_{\{|x|<1\}}\right) \nu(dx), \tag{6}$$

where $\langle\cdot, \cdot\rangle$ is the standard inner product on $\mathbb{R}^d$. The measure ν is called the Lévy measure. When ν is absolutely continuous its corresponding density is the Lévy density. The function $x\mathbb{1}_{\{|x|<1\}}$ ensures the convergence of the integral in 0, and is called a truncation function. Equation (6) goes under the name of Lévy–Khintchine representation. When $d = 1$ and X is a positively supported i.d. r.v., one can also use the Laplace exponent ϕ, defined to be

$$L_X(s) := E[e^{-sX}] = e^{-\phi_X(s)}, \qquad s > 0, \tag{7}$$

with

$$\phi_X(s) = bs + \int_{\mathbb{R}_+} (1 - e^{-sx})\nu(dx) \tag{8}$$

for a pair (b, ν), with $b \geq 0$ and a positively-supported Lévy measure ν satisfying $\int_{\mathbb{R}_+} (x \wedge 1)\nu(dx) < \infty$. One generic function f enjoying representation (8) is called a Bernstein function. A completely monotone (c.m.) function on $\mathbb{R}_+$ is a function of class C^∞ such that

$$(-1)^n f^{(n)}(x) \geq 0. \tag{9}$$

A classic reference for Bernstein and c.m. functions is [55].

An i.d. real-valued r.v. X is said to be self-decomposable if, for all $\alpha \in (0, 1)$, X can be written in law as $X = \alpha X + R_\alpha$, for some r.v., R_α independent of X. Equivalently, a real-valued self-decomposable distribution is characterized by the property that its Lévy measure is absolutely continuous and its density $\nu(x)$ can be expressed as $\nu(x) = k(x)/|x|$, with $k(x)$ a positive function increasing on $\mathbb{R}_-$ and decreasing on $\mathbb{R}_+$. The function k is called the canonical density of X.

A Lévy process $X = (X_t)_{t\geq 0}$ on $\mathbb{R}^d$ is a stochastically continuous process with independent and stationary increments. For a Lévy process X, X_t is i.d. for all $t > 0$. Conversely, given an i.d. r.v. $\mathcal{X}$ there exists a unique (in law) Lévy process X such that $X_1 = \mathcal{X}$ [53, Theorem 7.10]. The characteristic exponent ψ_X of a Lévy process X is by definition ψ_{X_1}. With abuse of terminology we shall refer to a Lévy process by the name of its unit time law.

2.2. Stable and tempered stable laws and processes

A stable r.v. X on $\mathbb{R}$ is an i.d. r.v. such that there exist $\alpha \in (0, 2]$ and $c \in \mathbb{R}$ for which $\Psi_X(z)^a = \Psi_X(a^{1/\alpha} z)e^{izc}$ for all $a > 0$. If $c = 0$ the r.v. is said to be strictly stable. The characteristic exponent of X depends on four parameters and has the explicit form

$$\psi_X(z) = \begin{cases} -\lambda|z|^\alpha \left(1 - i\beta \operatorname{sgn}(z) \tan(\pi\alpha/2)\right) + iz\mu & \text{for } \alpha \neq 1, \\ -\lambda|z| \left(1 + i\beta \frac{2}{\pi} \operatorname{sgn}(z) \log|z|\right) + iz\mu & \text{for } \alpha = 1, \end{cases} \tag{10}$$

where $\beta \in [-1, 1]$, $\lambda > 0$, $\alpha \in (0, 2]$, $\mu, z \in \mathbb{R}$. The corresponding stable distribution class (or α-stable when emphasizing the stability parameter) is denoted $S_\alpha(\lambda, \beta; \mu)$. For $\beta = \mu = 0$ the distribution is symmetric, for $\beta = \pm 1$ it is totally skewed, respectively, to the right and left. Strict stability is equivalent to $\mu = 0$ if $\alpha \neq 1$ and $\beta = 0$ if $\alpha = 1$. Stable laws are i.d., and a stable Lévy process $X = (X_t)_{t\geq 0}$ is one for which X_1 is stable (equivalently, X_t is stable for all t).

On $\mathbb{R}^d$, $d \geq 1$, it is instead easier to define a stable process by means of its Lévy triplet $(\mu, 0, \nu)$ where, for $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, we have the polar representation

$$\nu(B) = \int_{S^{d-1}} \sigma(du) \int_{\mathbb{R}_+} \mathbb{1}_B(ru) \frac{dr}{r^{1+\alpha}}, \tag{11}$$

with S^d indicating the d-dimensional sphere. We denote the class of α-stable distributions on $\mathbb{R}^d$ with $\mathrm{S}_\alpha(\sigma, \mu)$. On $\mathbb{R}$, sometimes cases with different stability indices for the two distinct points in S^0 are taken into account, when the Lévy measure ν has an absolutely continuous density $\nu(x)$ of the form

$$\nu(x) = \frac{\delta_+}{x^{1+\alpha_+}} \mathbb{1}_{\{x>0\}} + \frac{\delta_-}{|x|^{1+\alpha_-}} \mathbb{1}_{\{x<0\}}, \tag{12}$$

with $\alpha_+, \alpha_- \in (0, 2]$. Under this parametrization we denote the class of the stable distributions by $\mathrm{S}_{\boldsymbol{\alpha}}(\boldsymbol{\delta}, \mu)$, with $\boldsymbol{\alpha} = (\alpha_+, \alpha_-) \in (0, 2] \times (0, 2]$, $\boldsymbol{\delta} = (\delta_+, \delta_-) \in \mathbb{R}_+^2$, $\mu \in \mathbb{R}$. For a detailed survey on stable distributions see [53, Chapter 3].

A (classical) tempered stable distribution is an i.d. distribution with Lévy measure ν given by

$$\nu(dx) = \delta_+ \frac{e^{-\theta_+ x}}{x^{1+\alpha_+}} \mathbb{1}_{\{x>0\}} + \delta_- \frac{e^{-\theta_- |x|}}{|x|^{1+\alpha_-}} \mathbb{1}_{\{x<0\}} \tag{13}$$

for $\alpha_+, \alpha_- \in (0, 2)$, $\delta_+, \delta_- \geq 0$ (see e.g. [6, 8] for applications, and [36] for a theoretical analysis). We shall use the notation $\mathrm{CTS}_{\boldsymbol{\alpha}}(\boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$ with $\boldsymbol{\alpha} = (\alpha_+, \alpha_-) \in (0, 2) \times (0, 2)$, $\boldsymbol{\theta} = (\theta_+, \theta_-) \in \mathbb{R}_+^2$, $\boldsymbol{\delta} = (\delta_+, \delta_-) \in \mathbb{R}_+^2$, $\mu \in \mathbb{R}$. The stable distributions are when $\theta_\pm = 0$. The bilateral Gamma $\mathrm{BG}(\boldsymbol{\theta}, \boldsymbol{\delta})$ law (e.g. [35]) is obtained as a weak limit for $\alpha_+, \alpha_- \to 0$, and the ordinary positively supported Gamma $\mathrm{G}(\theta, \delta)$ law by additionally setting $\delta_- = 0$.

A general approach to tempering stable laws is considered in [51]. Let σ be a finite measure on S^{d-1} and consider a function $q: \mathbb{R}_+ \times S^{d-1} \to \mathbb{R}^d$, $(r, u) \mapsto q(r, u)$, c.m. in r for all u. Then for $\alpha \in (0, 2)$, a tempered α-stable process on $\mathbb{R}^d$ is one with Lévy triplet $(\mu, 0, m)$ where m is given in polar coordinates by

$$m(dr, du) = \frac{q(r, u)}{r^{1+\alpha}} \sigma(du) dr. \tag{14}$$

Since q is c.m. in the first variable, by the Bernstein theorem [55, Theorem 1.4] there exists a family of probability measures $(Q_u)_{u \in S^{d-1}}$ supported on $\mathbb{R}_+$ for which

$$q(r, u) = \int_0^\infty e^{-rs} Q_u(ds). \tag{15}$$

Therefore, for $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ we can introduce two measures Q and R, respectively, by

$$Q(B) = \int_{S^{d-1}} \sigma(du) \int_{\mathbb{R}_+} \mathbb{1}_B(ru) Q_u(dr) \tag{16}$$

and

$$R(B) = \int_{\mathbb{R}^d} \mathbb{1}_B \left(\frac{x}{\|x\|^2} \right) \|x\|^\alpha Q(dx). \tag{17}$$

It turns out that R and Q are dual in the sense that

$$Q(B) = \int_{\mathbb{R}^d} \mathbb{1}_B \left(\frac{x}{\|x\|^2} \right) \|x\|^\alpha R(dx). \tag{18}$$

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We call Q the spectral measure and R the Rosińsky measure. A given tempered α-stable distribution is fully characterized by its measures Q or R, and a drift vector $\mu \in \mathbb{R}^d$. The measure Q is useful for computer simulations [11], whereas R is pivotal to describe the analytical properties of the law.

With abuse of notation, we denote the class of tempered stable distributions in the sense of Rosińsky by $\mathrm{TS}_\gamma(\sigma, q, \mu)$, $\mathrm{TS}_\gamma(Q, \mu)$, or $\mathrm{TS}_\gamma(R, \mu)$, which covers most of the specifications illustrated above.

2.3. Geometric stability, Linnik distributions and their tempered versions

Geometric stable (GS) laws were introduced in [29] as a solution to the problem of characterizing i.d. laws whose infinite divisibility property holds true with respect to geometric summation. A geometric (strictly) stable law X on $\mathbb{R}^d$ is one such that

$$\psi_X(z) = -\log(1 - \psi_Z(z)), \tag{19}$$

where ψ_Z is the characteristic exponent of some (strictly) stable r.v. Z on $\mathbb{R}^d$. In the case $d = 1$, when Z is strictly stable and such that $\beta = 0$ and further introducing a shape parameter δ in (10), we obtain a symmetric law with characteristic exponent

$$\psi_X(z) = \delta \log\left(1 + \lambda |z|^\alpha\right), \qquad \delta, \lambda > 0, \quad \alpha \in (0, 2], \quad z > 0. \tag{20}$$

The arising probability distribution is often called the Linnik distribution. When $\alpha = 2$, $\delta = \lambda = 1$, this reduces to the well-known Laplace distribution.

Geometric stable processes and their applications have been studied in e.g. [31–34, 45, 56]. Positively-supported Linnik $\mathrm{PL}(\alpha, \lambda, \delta)$ laws can be seen as extension of the Mittag-Leffler $\mathrm{ML}(\alpha, \lambda)$ law introduced in [47]. We have, for the Lévy measure of a $\mathrm{PL}(\alpha, \lambda, \delta)$ r.v. X the Lévy density

$$\nu_X(x) = \frac{\delta\alpha}{x} E_\alpha\left(-\frac{x^\alpha}{\lambda}\right), \qquad \delta, \lambda > 0\ \alpha \in (0, 1),\ x > 0, \tag{21}$$

and $\mathrm{PL}(\alpha, \lambda, 1) \equiv \mathrm{ML}(\alpha, \lambda)$ (see [3]).

In [2] a tempered version of the Linnik positive laws, denoted $\mathrm{TPL}(\alpha, \lambda, \theta, \delta)$ was investigated, and their associated processes later studied in [37, 61]. The resulting operation leads to the Laplace exponent

$$\phi_X(s) = \delta \log\left(1 + \lambda((\theta + s)^\alpha - \theta^\alpha)\right), \qquad \delta, \lambda > 0,\ \theta \geq 0,\ \alpha \in (0, 1),\ x > 0. \tag{22}$$

Recalling that the Laplace exponent of a $\mathrm{CTS}_\alpha^+(\theta, \lambda; 0)$ law is $\lambda((\theta + s)^\alpha - \theta^\alpha)$, we see that in the case $\delta = 1$, (22) is equivalent to requiring that (19) now holds for a Laplace exponent of some *tempered* strictly stable positive law Z, leading to a notion of geometric tempered stability. The expression for the Lévy density is

$$\nu_X(x) = \delta\alpha \frac{e^{-\theta x}}{x} E_\alpha\left(\frac{\lambda\theta^\alpha - 1}{\lambda} x^\alpha\right), \qquad \delta, \lambda > 0,\ \theta \geq 0,\ \alpha \in (0, 1),\ x > 0. \tag{23}$$

TPL distributions seem to have first appeared in [44, Example 5.7] to describe the waiting times of a Poisson process subordinated to an inverse tempered stable subordinator (an increasing Lévy process). A closed form expression in terms of the three-parameter Mittag-Leffler function is also available for the p.d.f. See Section 6 and [2, 61] for details.

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2.4. Special functions

We denote by $E^c_{a,b}(z)$, $z \in \mathbb{C}$, the [50] three-parameter Mittag-Leffler function given by

$$E^c_{a,b}(z) = \sum_{k=0}^{\infty} \frac{(c)_k z^k}{k!\Gamma(ak+b)}, \qquad \mathrm{Re}(a) > 0, \quad \mathrm{Re}(b) > 0, \quad c, z \in \mathbb{C}, \tag{24}$$

where $\Gamma(\cdot)$ is the Euler's Gamma function and $(c)_k = \Gamma(c+k)/\Gamma(c)$ the Pochhammer symbol. The standard and two-parameter Mittag-Leffler functions E_a and $E_{a,b}$ coincide with $E^1_{a,1}$ and $E^1_{a,b}$, respectively. All these functions are entire.

The following leading asymptotic order of the three-parameter Mittag-Leffler function (e.g. [17]), as $x \in \mathbb{R}$, $x \to \infty$ will be useful:

$$E^c_{a,ac}(-x^a) \sim -c\frac{x^{-a(c+1)}}{\Gamma(-a)}, \qquad a \in (0, 1), \quad c > 0, \tag{25}$$

from which we can recover the well-known formula (e.g. [24])

$$E_\alpha(-x^\alpha) \sim \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \quad x \to \infty, \qquad \alpha \in (0, 1). \tag{26}$$

From the definition it is also clear that

$$E_\alpha(-x^\alpha) \sim 1 - \frac{x^\alpha}{\Gamma(1+\alpha)} \sim \exp\left(-\frac{x^\alpha}{\Gamma(1+\alpha)}\right), \quad x \to 0. \tag{27}$$

See [24] or [20] for a comprehensive introduction on Mittag-Leffler functions.

The ${}_2R_1$ generalized hypergeometric Wright-type function studied in [14] and [62], is given for $\tau > 0$, $a, b, c \in \mathbb{C}$, $c \notin \mathbb{Z}_{\leq 0}$ by

$${}_2R_1(a, b, c, \tau; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+\tau k)}{\Gamma(c+\tau k)} \frac{z^k}{k!}. \tag{28}$$

The power series expansion above converges absolutely for all a,b,c whenever $|z| < 1$ and convergence can be extended to the boundary if $\mathrm{Re}(c-b-a) > 0$. The ${}_2R_1$ function can be continued analytically outside the unit disc: the expression of the continuation is given by e.g. [28, Equation (5.2)]. Notice that ${}_2R_1(a, b, c, 1; z) = {}_2F_1(a, b, c; z)$, Gauss' hypergeometric function, so that in particular ${}_2R_1(a, b, b, 1; z) = (1-z)^{-a}$. Furthermore, the generalized hypergeometric ${}_2R_1$ can be represented in terms of the normalized Fox–Wright function ${}_p\Psi^*_q$ [66] as

$${}_2R_1(a, b, c, \tau; z) = {}_2\Psi^*_1 \left[\begin{matrix} (a, 1) & (b, \tau) \\ (c, \tau) & \end{matrix} ; z \right]. \tag{29}$$

See also [7, 27] for details on the analytic continuation of the ${}_p\Psi_q$ function.

The Lerch transcendent function Φ is defined as the convergent series

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}, \qquad z, s \in \mathbb{C}, |z| < 1, a \notin \mathbb{Z}_{\leq 0}. \tag{30}$$

It relates to the Polilogarithm $\mathrm{Li}_s(z)$ through $\mathrm{Li}_s(z)/z = \Phi(z, s, 1)$ and extends the Hurwitz/Riemann Zeta functions, in that $\zeta(s, a) = \Phi(1, s, a)$, $\zeta(s) = \Phi(1, s, 0)$. Again, even though the series representation above is only valid for $|z| < 1$, and for $|z| = 1$ if $\mathrm{Re}(s) > 0$, analytic continuations are available, and have recently been studied [16].

3. Radially geometric stable distributions

We introduce here the GRGTS class by providing the polar representation [4, Lemma 2.1] of their Lévy measure. Let $d \in \mathbb{N}$ and θ, λ:$S^{d-1} \to \mathbb{R}_+$, $\alpha : S^{d-1} \to (0, 1]$ be continuous functions. Define $q(\cdot, \cdot; \alpha, \lambda, \theta)$: $\mathbb{R}_+ \times S^{d-1}$ as given by

$$q(r, u; \alpha, \lambda, \theta) = e^{-\theta(u)r} E_{\alpha(u)} \left(-\lambda(u) r^{\alpha(u)} \right). \tag{31}$$

In order to avoid degeneracies, the following technical assumption is needed:

(**A**) either θ, or both α and λ, are bounded away from zero.

We need to show that the function above naturally induces a Lévy measure on $\mathbb{R}^d$.

Proposition 1. *Let σ be a finite measure on S^{d-1} and $r > 0$. For all $\gamma \in [0, 2)$ we have that*

$$m_\gamma(dr, du; \sigma, \alpha, \lambda, \theta) := \frac{q(r, u; \alpha, \lambda, \theta)}{r^{1+\gamma}} \sigma(du)dr \tag{32}$$

is a Lévy measure in polar coordinates on $\mathbb{R}^d$. Moreover, if $\gamma \neq 0$, the resulting distribution belongs to $\mathrm{TS}_\gamma(\sigma, q, \mu)$.

Proof. The functions $\exp(-\cdot)$ and $E_\alpha(-\ \cdot)$ are both c.m. for positive arguments; for the latter see [49]. Furthermore the function $x \to \lambda x^\alpha$, $\alpha \in (0, 1)$, is positive with a c.m. derivative, so that its composition with $E_\alpha(-\ \cdot)$ is also c.m. Therefore $q(\cdot, u)$ is c.m., it being a product of c.m. functions [55, Corollary 1.6] for all possible values of u. Observe also that $q(0+, u) = 1$, for all u. Hence, for $\gamma > 0$ that m_γ is a Lévy measure follows from [51]. To complete the proof we thus need to show that $\int_0^\infty \int_{S^{d-1}} ((ru)^2 \wedge 1) m_0(dr, du) < \infty$. From $\exp(-\cdot)$, $E_\alpha(-\ \cdot) \le 1$ we have

$$\int_{S^{d-1}} \int_0^1 r^2 \frac{q(r, u; \alpha, \lambda, \theta)}{r} dr\sigma(du) < \sigma(S^{d-1}) \int_0^1 rdr < \infty. \tag{33}$$

Also, by assumption (**A**), it follows that if $\theta_* = \min_{u \in S^{d-1}} \theta(u) = 0$, then both $\lambda_* = \min_{u \in S^{d-1}} \lambda(u)$, and $\alpha_* = \min_{u \in S^{d-1}} \alpha(u)$ are strictly positive. Then in view of (26) it holds that

$$\begin{aligned} \int_{S^{d-1}} \int_1^\infty \frac{q(r, u; \alpha, \lambda, \theta)}{r} dr\sigma(du) &< \sigma(S^{d-1}) \int_1^\infty r^{-1} E_{\alpha_*}(-\lambda_* r^{\alpha_*}) dr \\ &\sim \frac{\sigma(S^{d-1})}{\lambda_* \Gamma(1 - \alpha_*)} \int_1^\infty r^{-1-\alpha_*} dr < \infty. \end{aligned} \tag{34}$$

Similarly, if $\lambda_*, \alpha_* = 0$ then $\theta_* > 0$ whence

$$\int_{S^{d-1}} \int_1^\infty \frac{q(r, u; \alpha, \lambda, \theta)}{r} dr\sigma(du) < \sigma(S^{d-1}) \int_1^\infty r^{-1} e^{-r\theta_*} dr < \infty. \tag{35}$$

Recalling (21), we can see that for fixed $u \in S^{d-1}$, $q(r, u; \alpha(u), \lambda(u), \theta(u))/r$ is the Lévy measure of a $\mathrm{TPL}(\alpha(u), \lambda(u), \theta(u), 1/\alpha(u))$ distribution, which is to say that the radial component of the Lévy measure introduced is itself the Lévy measure of a univariate geometric tempered stable distribution. When the generalizing parameter γ is nonzero, the radial Lévy

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measure is still well defined, and the complete monotonicity of q in r takes us into the territory of [51]. We then define a general d-dimensional GRGTS variable as follows.

Definition 1. *Let* $d \geq 1$, $\gamma \in [0, 2)$, σ *be a finite measure on* S^{d-1}, *and* $\mu \in \mathbb{R}^d$. *A* generalized radially geometric tempered stable *distribution* $\mathrm{GRGTS}_\gamma(\sigma, \alpha, \lambda, \theta; \mu)$ *on* $\mathbb{R}^d$ *is the i.d. distribution class determined by the Lévy triplet* $(\mu, 0, m_\gamma(dr, du; \sigma, \alpha, \lambda, \theta))$. *The classes of the generalized radially geometric stable distribution* $\mathrm{GRGS}_\gamma(\sigma, \alpha, \lambda; \mu)$, *radially geometric tempered stable distribution* $\mathrm{RGTS}(\sigma, \alpha, \lambda, \theta; \mu)$, *and radially geometric stable distribution* $\mathrm{RGS}(\sigma, \alpha, \lambda; \mu)$ *correspond to* $\mathrm{GRGTS}_\gamma(\sigma, \alpha, \lambda, 0; \mu)$, $\mathrm{GRGTS}_0(\sigma, \alpha, \lambda, \theta; \mu)$, *and* $\mathrm{GRGTS}_0(\sigma, \alpha, \lambda, 0; \mu)$, *respectively.*

There are thus two "stability" parameters associated with GRGTS distributions when $\gamma \neq 0$. One is the parameter α of the underlying geometric stable law, the other is the stability index γ of the stable law in [51] with respect to which we apply the tempering q.

In this paper we will explore in detail the case $d = 1$, which is easier as $S^0 = \{-1, 1\}$, and a great deal can be said about the analytic structure of the distributions. In such cases the spherical measure reduces to

$$\sigma(du) = \delta_+ \delta_1(du) + \delta_- \delta_{-1}(du) \tag{36}$$

for $\delta_+, \delta_- \geq 0$, and δ_x is the Dirac delta measure concentrated in $x \in \mathbb{R}$. We have the following expression for the Lévy density, with $\alpha(u) = \alpha_\pm$, $\theta(u) = \theta_\pm$, $\lambda(u) = \lambda_\pm$, $\alpha_\pm, \theta_\pm, \lambda_\pm \in \mathbb{R}$:

$$\begin{aligned} m_\gamma(dr, du; \sigma, \alpha, \lambda, \theta) = {} & \delta_+ \frac{e^{-\theta_+ r}}{r^{1+\gamma}} E_{\alpha_+}(-\lambda_+ r^{\alpha_+}) dr \delta_1(du) \\ & + \delta_- \frac{e^{-\theta_- r}}{r^{1+\gamma}} E_{\alpha_-}(-\lambda_- r^{\alpha_-}) dr \delta_{-1}(du). \end{aligned} \tag{37}$$

One can easily extend this definition by assuming the stability indices for the negative and positive parts are not necessarily equal, i.e. if $\gamma_+ \neq, \gamma_- \in [0, 2)$, using the vector notation

$$\begin{aligned} \boldsymbol{\alpha} &= (\alpha_+, \alpha_-) \in (0, 1] \times (0, 1], \\ \boldsymbol{\lambda} &= (\lambda_+, \lambda_-) \in (0, \infty) \times (0, \infty), \\ \boldsymbol{\theta} &= (\theta_+, \theta_-) \in [0, \infty) \times [0, \infty), \\ \boldsymbol{\delta} &= (\delta_+, \delta_-) \in [0, \infty) \times [0, \infty) \setminus \{(0, 0)\}, \\ \boldsymbol{\gamma} &= (\gamma_+, \gamma_-), \in [0, 2) \times [0, 2), \end{aligned} \tag{38}$$

in Cartesian coordinates we have the expression for the Lévy density

$$m_{\boldsymbol{\gamma}}(x; \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}) = \delta_+ \frac{e^{-\theta_+ x}}{x^{1+\gamma_+}} E_{\alpha_+}(-\lambda_+ x^{\alpha_+}) \mathbb{1}_{\{x>0\}} + \delta_- \frac{e^{-\theta_- |x|}}{|x|^{1+\gamma_-}} E_{\alpha_-}(-\lambda_- |x|^{\alpha_-}) \mathbb{1}_{\{x<0\}}. \tag{39}$$

When $\alpha_+ = \alpha_-$, $\lambda_+ = \lambda_-$, $\gamma_+ = \gamma_-$, $\theta_+ = \theta_-$, $\delta_+ = \delta_-$, then $m_{\boldsymbol{\gamma}}(B) = m_{\boldsymbol{\gamma}}(-B)$, for all $B \in \mathcal{B}(\mathbb{R})$, i.e. $m_{\boldsymbol{\gamma}}$ is symmetric. If $\delta_- = 0$ or $\delta_- = 0$, then it has positive/negative support. In such cases we write, respectively, $m^s_\gamma(x; \alpha, \lambda, \theta, \delta)$, $m^+_\gamma(x; \alpha, \lambda, \theta, \delta)$, $m^-_\gamma(x; \alpha, \lambda, \theta, \delta)$, where the constants denote the only surviving or relevant value for each parameter vector. In these two latter instances the corresponding GRGTS laws are said to be spectrally positive (respectively, negative). It is important to recall the spectral positive/negativity is not the same as the corresponding probability laws being positively/negatively supported, although this equivalence holds true when $\gamma \in [0, 1)$ and $\mu = \int_{\{|x|<1\}} m^\pm_\gamma(x; \alpha, \lambda, \theta, \delta)$ (see [53, Theorem 24.10]). Again, $\int_{\mathbb{R}} (x^2 \wedge 1) m_{\boldsymbol{\gamma}}(x; \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}) dx < \infty$, so we can introduce the main definition on the real line.

Definition 2. *Let* $\mu \in \mathbb{R}$ *and* $m_{\boldsymbol{\gamma}}(x; \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta})$ *be given by (39). A* generalized radially geometric tempered stable distribution $\mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$ *on* $\mathbb{R}$ *is an i.d. distribution*

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whose Lévy triplet is given by $(\mu, 0, m_{\boldsymbol{\gamma}}(x; \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta})dx)$. *Generalized radially geometric stable* $\mathrm{GRGS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\delta}; \mu)$, *radially geometric tempered stable* $\mathrm{RGTS}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$, *and radially geometric stable* $\mathrm{RGS}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \boldsymbol{\delta}; \mu)$ *distributions are defined correspondingly as* $\mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, (0, 0), \boldsymbol{\delta}; \mu)$, $\mathrm{GRGTS}_{(0,0)}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$, *and* $\mathrm{GRGTS}_{(0,0)}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, (0, 0), \boldsymbol{\delta}; \mu)$ *laws. Symmetric, spectrally positive, and spectrally negative versions of these classes are denoted, respectively, with the superscripts* s, $+$, *and* $-$.

Remark 1. It is clear from (39) that distributions in the $\mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$ class are self-decomposable with canonical density

$$k_{\boldsymbol{\gamma}}(x; \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}) = \delta_+ \frac{e^{-\theta_+ x}}{x^{\gamma_+}} E_{\alpha_+}(-\lambda_+ x^{\alpha_+}) \mathbb{1}_{\{x>0\}} + \delta_- \frac{e^{-\theta_- |x|}}{|x|^{\gamma_-}} E_{\alpha_-}(-\lambda_- |x|^{\alpha_-}) \mathbb{1}_{\{x<0\}}. \quad (40)$$

Remark 2. With the numerical constants denoting the corresponding constant functions, we have the following particular cases.

(i) Let $\alpha_n \to 0$ be any positive sequence of real numbers; then $X_n \sim \mathrm{GRGTS}_\gamma(\sigma, \alpha_n, \lambda, \theta, \mu)$ is such that $X_n \to^d X$ with $X \sim \mathrm{TS}_\gamma(\sigma^\lambda, \exp(-r\theta), \mu)$, where $\sigma^\lambda(du) = \frac{\sigma(du)}{1+\lambda(u)}$, provided that $|\lambda(u)| < 1$ for all u.

(ii) $\mathrm{GRGTS}_\gamma(\sigma, 1, \lambda, \theta; \mu) = \mathrm{TS}_\gamma(\sigma, \exp(-r(\theta + \lambda)), \mu)$.

(iii) $\mathrm{GRGS}_\gamma(\sigma, 1, \lambda; \mu) = \mathrm{TS}_\gamma(\sigma, \exp(-r\lambda), \mu)$.

(iv) If $\lambda_n \to 0$ is a positive sequence and $\gamma > 0$, then $X_n \sim \mathrm{GRGTS}_\gamma(\sigma, \alpha, \lambda_n, \theta; \mu)$ is such that $X_n \to^d X$, with $X \sim \mathrm{TS}_\gamma(\sigma, \exp(-r\theta), \mu)$.

(v) If $\lambda_n \to 0$ is a positive sequence and $\gamma > 0$, then $X_n \sim \mathrm{GRGS}_\gamma(\sigma, \alpha, \lambda_n; \mu)$ is such that $X_n \to^d X$, with $X \sim \mathrm{S}_\gamma(\sigma, \mu)$.

(vi) $\mathrm{RGTS}^+_\alpha(\lambda, \theta, \delta; \mu_0) = \mathrm{TPL}\left(\alpha, \frac{1}{\lambda+\theta^\alpha}, \theta, \frac{\delta}{\alpha}\right)$ where $\mu_0 = \int_{\{x<1\}} x m^+_0(x; \alpha, \lambda, \theta, \delta)dx$.

Furthermore we have (up to a possible location shift) the various special cases within the GRGTS class on $\mathbb{R}$:

(vii) $\mathrm{GRGTS}_{\boldsymbol{\gamma}}((1, 1), \boldsymbol{\lambda}, \boldsymbol{\theta}, (\delta, \delta); \mu)$ are distributions in the general [30] and KoBol [6] classes.

(viii) $\mathrm{GRGTS}_{(\gamma,\gamma)}((1, 1), \boldsymbol{\lambda}, \boldsymbol{\theta}, (\delta, \delta); \mu)$ are CGMY distributions [8].

(ix) $\mathrm{RGTS}_{(1,1)}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu) = \mathrm{BG}(\boldsymbol{\lambda} + \boldsymbol{\theta}, \boldsymbol{\delta})$ studied in [35].

(x) $\mathrm{GRGTS}_{(\gamma,\gamma)}((1, 1), \boldsymbol{\lambda}, (\theta, \theta), \boldsymbol{\delta}; \mu)$ are the i.d. innovations of truncated Lévy flights [43].

Proof. Using continuity of the Mittag-Leffler function, $E_1(-\lambda(u)x) = \exp(-\lambda(u)x)$, $x \in \mathbb{R}$, $E_\alpha(0) = 1$ for all $\alpha \in (0, 1]$ and $E_\alpha(-\lambda(u)) \to 1/(1 + \lambda(u))$ as $\alpha \to 0$ (see [24]), together with dominated convergence as needed, (i)–(vi) follow operating on the Lévy measures. For (vi) write

$$m^+(x; \alpha, \lambda, \theta, \delta) = \delta \frac{e^{-\theta x}}{x} E_\alpha(-\lambda x^\alpha) = \alpha c_1 \frac{e^{-\theta x}}{x} E_\alpha\left(\frac{c_2\theta^\alpha - 1}{c_2} x^\alpha\right) \quad (41)$$

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with $c_1 = \delta/\alpha$, $c_2 = 1/(\lambda + \theta^\alpha)$, which matches the expression of the TPL Lévy measure in [61, Proposition 2.1], given there for the Laplace exponent. Finally, (vii)–(x) are simple parameter specifications of the prior cases.

Remark 2 clarifies that GRGTS distributions specialize to TS_γ, TPL, and BG distributions. In particular, they include stable, gamma, and positive/symmetric geometric stable distributions.

4. Characteristic exponents

We begin a theory of one-dimensional GRGTS distributions by determining their characteristic exponents. We have a divide depending on whether $\theta_+, \theta_- > 0$ or $\theta_+ = \theta_- = 0$, i.e. the (G)RGTS and (G)RGS cases must be treated separately. The former case is analytical. The latter corresponds to a radial Lévy density with heavy tails and uses a limiting argument on the analytic continuation of the positive θ case. The mixed cases $\theta_+ > 0, \theta_- = 0$, $\theta_+ = 0, \theta_- > 0$ are of course possible and can be obtained by combining negative and positive parts as necessary, and thus will not be considered. Cumulants are also discussed at the end of the section.

Theorem 1. *Let X be a* $\mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$ *r.v. with* $\theta_+, \theta_- > 0$ *and* $\alpha_+, \alpha_- \in (0, 1)$*, let m be the Lévy density of X, and set* $\mu_0 = \int_{\{|x|<1\}} xm(x)dx$ *and* $\mu_1 = \int_{\{|x|>1\}} xm(x)dx$*. Then*

(i) *if* $X \in \mathrm{RGTS}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$*, then*

$$\psi_X(z) = \frac{\delta_+}{\alpha_+} \log\left(\frac{\theta_+^{\alpha_+} + \lambda_+}{(\theta_+ - iz)^{\alpha_+} + \lambda_+}\right) + \frac{\delta_-}{\alpha_-} \log\left(\frac{\theta_-^{\alpha_-} + \lambda_-}{(\theta_- + iz)^{\alpha_-} + \lambda_-}\right) + iz(\mu - \mu_0); \tag{42}$$

(ii) *if* $\gamma_+, \gamma_- \in (0, 2)$*,* $\gamma_\pm \neq 1$*, then*

$$\begin{aligned}
\psi_X(z) = &\Gamma(-\gamma_+)\delta_+ \left((\theta_+ - iz)^{\gamma_+}\, {}_2R_1\left(1, -\gamma_+, 1, \alpha_+; \frac{-\lambda_+}{(\theta_+ - iz)^{\alpha_+}}\right)\right.\\
&+ iz\, \gamma_+ \theta_+^{\gamma_+ - 1}\, {}_2R_1\left(1, 1-\gamma_+, 1, \alpha_+; -\frac{\lambda_+}{\theta_+^{\alpha_+}}\right)\\
&\left. - \theta_+^{\gamma_+}\, {}_2R_1\left(1, -\gamma_+, 1, \alpha_+; -\frac{\lambda_+}{\theta_+^{\alpha_+}}\right)\right) +\\
&\Gamma(-\gamma_-)\delta_- \left((\theta_- + iz)^{\gamma_-}\, {}_2R_1\left(1, -\gamma_-, 1, \alpha_-; \frac{-\lambda_-}{(\theta_- + iz)^{\alpha_-}}\right)\right.\\
&- iz\gamma_-\, \theta_-^{\gamma_- - 1}\, {}_2R_1\left(1, 1-\gamma_-, 1, \alpha_-; -\frac{\lambda_-}{\theta_-^{\alpha_-}}\right)\\
&\left. - \theta_-^{\gamma_-}\, {}_2R_1\left(1, -\gamma_-, 1, \alpha_-; -\frac{\lambda_-}{\theta_-^{\alpha_-}}\right)\right) + iz(\mu_1 + \mu);
\end{aligned} \tag{43}$$

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(iii) *if* $\gamma_+ = \gamma_- = 1$, *then*

$$\begin{aligned}\psi_X(z) =&\delta_+ \frac{\lambda_+}{\alpha_+}\left(\frac{\Phi\left(-\frac{\lambda_+}{\theta_+^{\alpha_+}}, 1, \frac{\alpha_+-1}{\alpha_+}\right)}{\theta_+^{\alpha_+-1}} - \frac{\Phi\left(-\frac{\lambda_+}{(\theta_+-iz)^{\alpha_+}}, 1, \frac{\alpha_+-1}{\alpha_+}\right)}{(\theta_+-iz)^{\alpha_+-1}}\right)\\ &+\delta_+\frac{\theta_+-iz}{\alpha_+}\log\left(\frac{(\theta_+-iz)^{\alpha_+}+\lambda_+}{\theta_+^{\alpha_+}+\lambda_+}\right)+\\ &\delta_-\frac{\lambda_-}{\alpha_-}\left(\frac{\Phi\left(-\frac{\lambda_-}{\theta_-^{\alpha_-}}, 1, \frac{\alpha_--1}{\alpha_-}\right)}{\theta_-^{\alpha_--1}} - \frac{\Phi\left(-\frac{\lambda_-}{(\theta_-+iz)^{\alpha_-}}, 1, \frac{\alpha_--1}{\alpha_-}\right)}{(\theta_-+iz)^{\alpha_--1}}\right)\\ &+\delta_-\frac{\theta_-+iz}{\alpha_-}\log\left(\frac{(\theta_-+iz)^{\alpha_-}+\lambda_-}{\theta_-^{\alpha_-}+\lambda_-}\right)+iz(\delta_+-\delta_-+\mu_1+\mu).\end{aligned} \tag{44}$$

The remaining cases can be derived from the given expressions by combining positive and negative parts as needed. Furthermore, all the above characteristic functions can be analytically continued on $S = \{z \in \mathbb{C}: \mathrm{Im}(z) \in (-\theta_+, \theta_-)\}$.

Proof. We only treat the positive part, the negative one being identical with the obvious parameter modification, and by substituting x with $|x|$. We thus remove the subscripts for ease of notation.

Assume $\gamma = 0$. We notice that in that case $x\,m(x)$ is integrable around zero, and we can compute the Lévy–Khintchine integral without truncation. Interchanging series and integral using Fubini's theorem, we have

$$\begin{aligned}\int_0^\infty \left(e^{izx}-1\right) q(x) x^{-1-\gamma}\,dx &= \int_0^\infty \sum_{k=1}^\infty \frac{(izx)^k}{k!}\frac{e^{-\theta x}}{x} E_\alpha\left(-\lambda\, x^\alpha\right) dx\\ &= \sum_{k=1}^\infty \frac{(iz)^k}{k!}\int_0^\infty e^{-\theta x} x^{k-1} E_\alpha\left(-\lambda\, x^\alpha\right) dx\\ &= \sum_{k=1}^\infty \frac{(iz)^k}{k!}\sum_{j=0}^\infty (-\lambda)^j \frac{1}{\Gamma(1+\alpha j)}\int_0^\infty e^{-\theta x} x^{\alpha j+k-1} dx\\ &= \sum_{k=1}^\infty \frac{\left(\frac{iz}{\theta}\right)^k}{k!}\sum_{j=0}^\infty \left(\frac{-\lambda}{\theta^\alpha}\right)^j \frac{\Gamma(k+\alpha j)}{\Gamma(1+\alpha j)}\\ &= \sum_{j=0}^\infty \left(\frac{-\lambda}{\theta^\alpha}\right)^j \frac{1}{\Gamma(1+\alpha j)}\sum_{k=1} \frac{\left(\frac{iz}{\theta}\right)^k}{k!}\Gamma(k+\alpha j).\end{aligned} \tag{45}$$

For $j = 0$ the summations on k in (45) reduce to a logarithmic series. For $j \geq 1$, recalling the binomial series, it holds that

$$\sum_{k=1} \frac{\left(\frac{iz}{\theta}\right)^k}{k!}\frac{\Gamma(k+\alpha j)}{\Gamma(\alpha j)} = \left(\frac{\theta}{\theta - iz}\right)^{\alpha j} - 1. \tag{46}$$

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Substitute (46) in (45), and under the convergence conditions $|z| < \theta$, $\theta^\alpha > \lambda$, we obtain

$$\sum_{j=1}^{\infty}\left(\frac{-\lambda}{(\theta - iz)^\alpha}\right)^j \frac{1}{\alpha j} - \sum_{j=1}^{\infty}\left(\frac{-\lambda}{\theta^\alpha}\right)^j \frac{1}{\alpha j} + \sum_{k=1} \frac{(iz/\theta)^k}{k}$$
$$= -\frac{1}{\alpha}\left(\log\left(\frac{(\theta - iz)^\alpha + \lambda}{(\theta - iz)^\alpha}\right) - \log\left(\frac{\theta^\alpha + \lambda}{\theta^\alpha}\right) + \log\left(\frac{(\theta - iz)^\alpha}{\theta^\alpha}\right)\right)$$
$$= \frac{1}{\alpha}\log\left(\frac{\theta^\alpha + \lambda}{(\theta - iz)^\alpha + \lambda}\right). \tag{47}$$

Here we consider the principal branch of the complex logarithm and the power function $y \mapsto y^\alpha$ for $\mathrm{Arg}(y) \in (-\pi, \pi]$.

For $\gamma \in (0, 2)$, we observe that since m decays exponentially, it holds $\int_{\{|x|>1\}} xm(x)dx = \mu_1 < \infty$. We can thus use the representation of the characteristic exponent with constant truncation function 1. The same calculations of (45) produce

$$\int_0^\infty \left(e^{izx} - 1 - izx\right) q(x)x^{-1-\gamma}\,dx = \theta^\gamma \sum_{j=0}^{\infty}\left(\frac{-\lambda}{\theta^\alpha}\right)^j \frac{1}{\Gamma(1+\alpha j)} \sum_{k=2} \frac{\left(\frac{iz}{\theta}\right)^k}{k!}\Gamma(k - \gamma + \alpha j). \tag{48}$$

Now first let $\gamma \neq 1$. Using the binomial series again we have

$$\sum_{k=2}^{\infty} \Gamma(k + j\alpha - \gamma)\frac{(iz/\theta)^k}{k!} = \Gamma(j\alpha - \gamma)\left(\left(\frac{\theta - iz}{\theta}\right)^{-\alpha j + \gamma} - iz\frac{j\alpha - \gamma}{\theta} - 1\right), \tag{49}$$

and therefore, whenever $|z| < \theta$ and $\theta^\alpha > \lambda$, (48) becomes, in view of the series expression (29) for the ${}_2R_1$ function,

$$\sum_{j=0}^{\infty} \frac{\Gamma(j\alpha - \gamma)}{\Gamma(j\alpha + 1)}\left((\theta - iz)^\gamma \left(\frac{-\lambda}{(\theta - iz)^\alpha}\right)^j - \theta^\gamma \left(\frac{-\lambda}{\theta^\alpha}\right)^j \left(iz\frac{j\alpha - \gamma}{\theta} + 1\right)\right)$$
$$= (\theta - iz)^\gamma \sum_{j=0}^{\infty} \frac{\Gamma(j\alpha - \gamma)}{\Gamma(j\alpha + 1)}\left(\frac{-\lambda}{(\theta - iz)^\alpha}\right)^j - \theta^\gamma \sum_{j=0}^{\infty} \frac{\Gamma(j\alpha - \gamma)}{\Gamma(j\alpha + 1)}\left(\frac{-\lambda}{\theta^\alpha}\right)^j$$
$$- iz\theta^{\gamma-1} \sum_{j=0}^{\infty} \frac{\Gamma(j\alpha - \gamma + 1)}{\Gamma(j\alpha + 1)}\left(\frac{-\lambda}{\theta^\alpha}\right)^j$$
$$= \Gamma(-\gamma)\left((\theta - iz)^\gamma\ {}_2R_1\left(1, -\gamma, 1, \alpha; \frac{-\lambda}{(\theta - iz)^\alpha}\right) + iz\gamma\ \theta^{\gamma-1}\ {}_2R_1\left(1, 1-\gamma, 1, \alpha; -\frac{\lambda}{\theta^\alpha}\right)\right.$$
$$\left. -\theta^\gamma\ {}_2R_1\left(1, -\gamma, 1, \alpha; -\frac{\lambda}{\theta^\alpha}\right)\right). \tag{50}$$

Finally, for $\gamma = 1$ the term $j = 0$ in (48) becomes yet another log series,

$$\sum_{k=2}^{\infty} \Gamma(k-1)\frac{(iz/\theta)^k}{k!} = \sum_{k=2}^{\infty} \frac{(iz/\theta)^k}{k(k-1)} = \sum_{k=2}^{\infty} \frac{(iz/\theta)^k}{k-1} - \sum_{k=2}^{\infty} \frac{(iz/\theta)^k}{k}$$
$$= \frac{iz}{\theta} + \log\left(1 - \frac{iz}{\theta}\right)\left(1 - \frac{iz}{\theta}\right). \tag{51}$$

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Separating this term and using (49) results in the following expression for (48):

$$\frac{\theta - iz}{\alpha}\sum_{j=1}^{\infty}\frac{1}{j(\alpha j-1)}\left(\frac{-\lambda}{(\theta-iz)^{\alpha}}\right)^{j} - \frac{\theta}{\alpha}\sum_{j=1}^{\infty}\frac{1}{j(\alpha j-1)}\left(\frac{-\lambda}{\theta^{\alpha}}\right)^{j}$$
$$-\frac{iz}{\alpha}\sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{-\lambda}{\theta^{\alpha}}\right)^{j} + iz + \log\left(1-\frac{iz}{\theta}\right)(\theta - iz)\,. \tag{52}$$

Observe that for $y \in \mathbb{C}$, $|y| < 1$ it holds that

$$\sum_{j=1}^{\infty}\frac{y^j}{j(\alpha j-1)} = \alpha\sum_{j=1}^{\infty}\frac{y^j}{(\alpha j-1)} - \sum_{j=1}^{\infty}\frac{y^j}{j} = \sum_{j=0}^{\infty}\frac{y^{j+1}}{(j+\frac{\alpha-1}{\alpha})} + \log(1-y)$$
$$= y\,\Phi\left(y,\,1,\,\frac{\alpha-1}{\alpha}\right) + \log(1-y). \tag{53}$$

Replacing (53) in (52), and recalling (30), the case $\gamma = 1$ specializes to

$$\int_0^{\infty}\left(e^{izx}-1-izx\right)q(x)x^{-\gamma-1}dx =$$
$$\frac{\lambda}{\alpha}\left(\frac{\Phi\left(-\frac{\lambda}{\theta^{\alpha}},\,1,\,\frac{\alpha-1}{\alpha}\right)}{\theta^{\alpha-1}} - \frac{\Phi\left(-\frac{\lambda}{(\theta-iz)^{\alpha}},\,1,\,\frac{\alpha-1}{\alpha}\right)}{(\theta-iz)^{\alpha-1}}\right) + \frac{\theta-iz}{\alpha}\log\left(\frac{(\theta-iz)^{\alpha}+\lambda}{(\theta-iz)^{\alpha}}\right) -$$
$$\frac{\theta}{\alpha}\log\left(\frac{\theta^{\alpha}+\lambda}{\theta^{\alpha}}\right) + iz + \frac{\theta-iz}{\alpha}\log\left(\frac{(\theta-iz)^{\alpha}}{\theta^{\alpha}}\right) + \frac{iz}{\alpha}\log\left(\frac{\theta^{\alpha}+\lambda}{\theta^{\alpha}}\right)$$
$$= \frac{\lambda}{\alpha}\left(\frac{\Phi\left(-\frac{\lambda}{\theta^{\alpha}},\,1,\,\frac{\alpha-1}{\alpha}\right)}{\theta^{\alpha-1}} - \frac{\Phi\left(-\frac{\lambda}{(\theta-iz)^{\alpha}},\,1,\,\frac{\alpha-1}{\alpha}\right)}{(\theta-iz)^{\alpha-1}}\right) + \frac{\theta-iz}{\alpha}\log\left(\frac{(\theta-iz)^{\alpha}+\lambda}{\theta^{\alpha}+\lambda}\right) + iz. \tag{54}$$

Integrating the obtained expressions (47), (50), and (54) in $\sigma(du)$ as in (39), subtracting $iz(\mu_0 - \mu)$ to (47), and adding $iz(\mu_1 + \mu)$ to (54)–(50), $(i)-(iii)$ follow for all $|z| < \theta$ and $\lambda < \theta^{\alpha}$.

We must finally prove the analyticity of Ψ_X on S and that the constraint $\lambda < \theta^{\alpha}$ can be lifted. We know that Ψ_X can be analytically continued on a horizontal strip $\{z \in \mathbb{C} : -a < \mathrm{Im}(z) < b\}$, $a, b \in \mathbb{R} \cup \{\infty\}$, $a \neq b$ (see [41]), if $E[e^{-cX}] < \infty$, for all $c \in (-a, b)$. In turn — by, e.g. [53, Corollary 25.8] — this is equivalent to $\int_{\{|x|>1\}} e^{-cx}m(x)dx < \infty$ for all such c. In our instance we thus have $a = \theta_+,\ b = \theta_-$. Since the expression for Ψ_X has been found on the real line, that (42)–(44) also provide the analytic continuation of Ψ_X on S would then follow from the identity principle of analytic functions, if we can show that such expressions are analytic on S.

Assume first $\gamma = 0$. The function $y \mapsto y^{\alpha}$ is analytic outside the logarithm branch cut $\mathbb{C} \setminus (-\infty, 0]$ yielding the condition $(\theta_+ - iz) \notin (-\infty, 0]$, i.e. $\mathrm{Im}(z) \neq \theta_+$. Hence we obtain, $S^+ := \{z \in \mathbb{C}{:}\mathrm{Im}(z) > -\theta_+\}$. Furthermore, the logarithm branch cut is never crossed by the function $\frac{\theta_+^{\alpha_+}+\lambda_+}{(\theta_+-iz)^{\alpha}+\lambda_+}$, so that $\log\left(\frac{\theta_+^{\alpha_+}+\lambda_+}{(\theta_+-iz)^{\alpha_+}+\lambda_+}\right)$ is analytic on S^+. The same argument applies to $(\theta_- + iz)$, which is analytical on $S^- := \{z \in \mathbb{C}{:}\mathrm{Im}(z) < \theta_-\}$, and hence ψ_X is analytical on S, and so is Ψ_X.

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For $\gamma \in (0, 2)$, $\gamma \neq 1$, we exploit the fact that it is known (e.g. [27]) that ${}_2R_1$ can be analytically continued on $\mathbb{C} \setminus [1, \infty)$. On the other hand, on the regions $S^{\pm}$, the functions $-\lambda_{\pm}/(\theta_{\pm} \pm iz)^{\alpha_{\pm}}$ never attain real positive values, again yielding analiticity of ψ_X on S. Lerch's transcendent can also be analytically continued on $\mathbb{C} \setminus [1, \infty)$, e.g. [23, Lemma 2.2], and then the claim also follows for the case $\gamma = 1$. That the condition $\lambda < \theta^{\alpha}$ can be lifted also follows from the arguments above, since the parameter ranges of λ, θ, α always lie in the domain of analyticity of the continued functions.

Remark 3. For $\gamma_+, \gamma_- \in (0, 1)$, another expression for the characteristic exponent is available. For such a parameter range we still have that, as in the $\gamma = 0$ case, $\int_0^{\infty} \left(e^{izx} - 1\right) q(x) x^{-1-\gamma} dx < \infty$, and then the summation in (48) would yield the expression

$$\begin{aligned}\psi_X(z) =& \Gamma(-\gamma_+)\delta_+ \left((\theta_+ - iz)^{\gamma_+} {}_2R_1\left(1, -\gamma_+, 1, \alpha_+; \frac{-\lambda_+}{(\theta_+ - iz)^{\alpha_+}}\right) \right. \\ & \left. -\theta_+^{\gamma_+} {}_2R_1\left(1, -\gamma_+, 1, \alpha_+; -\frac{\lambda_+}{\theta_+^{\alpha_+}}\right)\right) + \\ & \Gamma(-\gamma_-)\delta_- \left((\theta_- + iz)^{\gamma_-} {}_2R_1\left(1, -\gamma_-, 1, \alpha_-; \frac{-\lambda_-}{(\theta_- + iz)^{\alpha_-}}\right) \right. \\ & \left. -\theta_-^{\gamma_-} {}_2R_1\left(1, -\gamma_-, 1, \alpha_-; -\frac{\lambda_-}{\theta_-^{\alpha_-}}\right)\right) + iz(\mu - \mu_0). \end{aligned} \tag{55}$$

Assume $X \sim \mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu_0)$. Equating (55) and (43), computing the expression in $z = -i$, and noticing that by [53, Chapter 25], in this case it holds that $E[X] = \mu_0 + \mu_1$, one deduces that

$$\begin{aligned} E[X] =& \frac{\delta_+ \Gamma(1-\gamma_+)}{\theta_+^{1-\gamma_+}} {}_2R_1\left(1, 1-\gamma_+, 1, \alpha_+, -\frac{\lambda_+}{\theta_+^{\alpha_+}}\right) \\ & - \frac{\delta_- \Gamma(1-\gamma_-)}{\theta_-^{1-\gamma_-}} {}_2R_1\left(1, 1-\gamma_-, 1, \alpha_-; -\frac{\lambda_-}{\theta_-^{\alpha_-}}\right) \end{aligned} \tag{56}$$

whenever $\theta_+, \theta_- > 0$. We will study cumulants more in detail in Section 4.2.

Example 1. *The* TPL *distribution. Consider a* $\mathrm{RGTS}_{\alpha}^{+}(\lambda, \theta, \delta; \mu_0)$ *r.v. X. From (42) we have the characteristic function*

$$\psi_X(z) = \frac{\delta}{\alpha} \log\left(\frac{\theta^{\alpha} + \lambda}{(\theta - iz)^{\alpha} + \lambda}\right) \tag{57}$$

for some positive constants λ, α, θ. *Letting* $c_1 = \delta/\alpha$, $c_2 = 1/(\theta^{\alpha} + \lambda)$ *we can write*

$$\psi_X(z) = -c_1 \log\left(1 + c_2((\theta - iz)^{\alpha} - \theta^{\alpha})\right), \tag{58}$$

which is the characteristic exponent of a $\mathrm{TPL}(\alpha, c_2, \theta, c_1)$ *r.v. in the parametrization of [61], confirming Remark 2(vi).*

As a consequence of the analyticity of the characteristic functions above we know by the general theory of [41], that under the assumptions above a moment generating function for

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a TGS can be defined, all the moments exist, and so do the θ-exponential moments for $\theta \in (-\theta_-, \theta_+)$.

Remark 4. *Scaling and independent sums.* Let $c > 0$. If X is as in Theorem 1, by inspection on all cases we observe

$$cX \sim \mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, (\lambda_+ c^{-\alpha_+}, \lambda_- c^{-\alpha_-}), \boldsymbol{\theta} c^{-1}, (\delta_+ c^{\gamma_+}, \delta_- c^{\gamma_-}); \mu c^{-1}). \tag{59}$$

Furthermore, if $X \sim \mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$ and $X' \sim \mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}'; \mu')$ are independent and $\theta_\pm > 0$, then $X + X' \sim \mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta} + \boldsymbol{\delta}'; \mu + \mu')$ as it follows directly from the i.d. property.

4.1. The GRGS and RGS cases

When $\boldsymbol{\theta} = 0$, we have the $\mathrm{GRGS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\delta}; \mu)$ class and its $\mathrm{RGS}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \boldsymbol{\delta}; \mu)$ subclass. Distributions in this class are structurally more similar to positive Linnik laws, in that their Lévy measure takes the form of the ratio of a Mittag-Leffler function over a power function.

Before stating the results we show the following technical lemma on some analytical properties of the ${}_2R_1$ functions of interest.

Lemma 1. *For $b, c > 0$, $a \le b < 1$, and $z \in \mathbb{C}$, $\mathrm{Arg}(z) \neq \pi/b$, it holds that*

(i)

$$\lim_{z \to 0} \frac{\Gamma(a)}{z^a} {}_2R_1\left(1, a, 1, b; \frac{-c}{z^b}\right) = c^{-a/b} \frac{\pi}{b\Gamma(1-a)\sin\left(\frac{\pi a}{b}\right)}; \tag{60}$$

(ii)

$$\frac{d^n}{dz^n} \frac{\Gamma(a)}{z^a} {}_2R_1\left(1, a, 1, b; \frac{-c}{z^b}\right) = (-1)^n \frac{\Gamma(a+n)}{z^{a+n}} {}_2R_1\left(1, n+a, 1, b; \frac{-c}{z^b}\right). \tag{61}$$

Proof. Under the given assumptions we can use [28, Corollary 5.2.1] and conclude that for all $w \in \mathbb{C}$ such that $|\mathrm{Arg}(-w) < \pi|$ the following asymptotics when $w \to \infty$ for such function hold true:[1]

$$\begin{aligned} {}_2R_1(1, a, 1; b, w) &= \frac{\Gamma(a-b)}{\Gamma(a)\Gamma(1-b)}(-w^{-1}) + O(w^{-1}) \\ &+ \frac{\Gamma(1-a/b)\Gamma(a/b)}{b\Gamma(a)\Gamma(1-a)}(-w^{-\frac{a}{b}}) + O(-w^{-\frac{a}{b}}). \end{aligned} \tag{62}$$

Now setting $w = -c/z^b$, since $\mathrm{Arg}(z) \neq \pi/b$ then $\mathrm{Arg}(-w) \neq \pi$, and as $z \to 0$, we have $w \to \infty$. Thus (62) implies

$$\Gamma(a) z^{-a} {}_2R_1\left(1, a, 1, b, \frac{-c}{z^b}\right) \sim z^{-a} \frac{\Gamma(1-a/b)\Gamma(a/b)}{b\Gamma(1-a)} \left(\frac{z^b}{c}\right)^{\frac{a}{b}}, \tag{63}$$

and the claim follows combining the above and Euler's reflection formula.

[1]We believe that there is a typo in [28, Theorem 5.2, Equation (5.2)]. A factor μ/ω appears to be missing from the second term, since $A_2 = \omega/\mu$ is apparently not accounted for at the denominator of the outer summation, as it should follow from equation (4.9) of Theorem 4.2 from which equation (5.2) is derived. In our case $\mu = 1$, $\omega = b$.

Regarding (*ii*), first assume $n = 1$. We have, using the chain rule and differentiating term by term the uniformly convergent series

$$\begin{aligned}
&\frac{d}{dz} z^{-a}\Gamma(a)\, {}_2R_1\left(1, a, 1, b; \frac{-c}{z^b}\right) = -a\Gamma(a)z^{-a-1}\, {}_2R_1\left(1, a, 1, b; \frac{-c}{z^b}\right) \\
&- z^{-a}\sum_{j=1}^{\infty}\frac{\Gamma(bj+a)}{\Gamma(bj+1)}\frac{(-c)^j}{z^{bj+1}}bj = -a\Gamma(a)z^{-a-1}\, {}_2R_1\left(1, a, 1, b; \frac{-c}{z^b}\right) \\
&+ z^{-a-1}\left(-\sum_{j=1}^{\infty}\frac{\Gamma(bj+a+1)}{\Gamma(bj+1)}\frac{(-c)^j}{z^{bj}} + a\sum_{j=1}^{\infty}\frac{\Gamma(bj+a)}{\Gamma(bj+1)}\frac{(-c)^j}{z^{bj}}\right) \\
&= -z^{-a-1}\left(a\Gamma(a) + \sum_{j=1}^{\infty}\frac{\Gamma(bj+a+1)}{\Gamma(bj+1)}\frac{(-c)^j}{z^{bj}}\right) \\
&= -z^{-a-1}\Gamma(a+1)\, {}_2R_1\left(1, a+1, 1, b; \frac{-c}{z^b}\right). \qquad (64)
\end{aligned}$$

Using (64) with $a' = a + n$ and appealing to the principle of mathematical induction the proof is complete.

We can now prove the main result regarding the characteristic exponent of GRGS r.v.s.

Theorem 2. *Let X be a $\mathrm{GRGS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\delta}; \mu)$ r.v., with $\alpha_+, \alpha_- \in (0, 1)$, $\gamma_+, \gamma_- \neq 1$, and define, for $z \in \mathbb{R}$,*

$$\Theta_{\pm}(z) = \cos\frac{\alpha_{\pm}\pi}{2}\left(1 - i\tan\frac{\alpha_{\pm}\pi}{2}\mathrm{sgn}(z)\right). \qquad (65)$$

We have, in the notation of Theorem 1:

(i) *if $X \in \mathrm{RGS}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \boldsymbol{\delta}; \mu)$ then*

$$\psi_X(z) = \frac{\delta_+}{\alpha_+}\log\left(\frac{\lambda_+}{|z|^{\alpha_+}\Theta_+(z) + \lambda_+}\right) + \frac{\delta_-}{\alpha_-}\log\left(\frac{\lambda_-}{|z|^{\alpha_-}\Theta_-(z) + \lambda_-}\right) + iz(\mu - \mu_0), \qquad (66)$$

otherwise;

(ii) *if $\max\{\alpha_+ + \gamma_+, \alpha_- + \gamma_-\} \in (0, 1)$ then*

$$\begin{aligned}
\psi_X(z) = {}&\delta_+\left(\Gamma(-\gamma_+)|z|^{\gamma_+}\Theta_+(z)\, {}_2R_1\left(1, -\gamma_+, 1, \alpha_+; \frac{-\lambda_+}{|z|^{\alpha_+}\Theta_+(z)}\right) - \ell(\gamma_+, \alpha_+, \lambda_+)\right) \\
&+\delta_-\left(\Gamma(-\gamma_-)|z|^{\gamma_-}\Theta_-(z)\, {}_2R_1\left(1, -\gamma_-, 1, \alpha_-; \frac{-\lambda_-}{|z|^{\alpha_-}\Theta_-(z)}\right) - \ell(\gamma_-, \alpha_-, \lambda_-)\right) \\
&+ iz(\mu - \mu_0); \qquad (67)
\end{aligned}$$

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(iii) *if* $\min\{\alpha_+ + \gamma_+, \alpha_- + \gamma_-\} \in [1, 3)$ *then*

$$\begin{aligned}\psi_X(z) = \delta_+\Bigg(&\Gamma(-\gamma_+)|z|^{\gamma_+}\Theta_+(z)\,{}_2R_1\left(1, -\gamma_+, 1, \alpha_+; \frac{-\lambda_+}{|z|^{\alpha_+}\Theta_+(z)}\right)\\ &- \ell(\gamma_+, \alpha_+, \lambda_+) - iz\,\ell(\gamma_+ - 1, \alpha_+, \lambda_+)\Bigg)\\ +\delta_-\Bigg(&\Gamma(-\gamma_-)|z|^{\gamma_-}\Theta_-(z)\,{}_2R_1\left(1, -\gamma_-, 1, \alpha_-; \frac{-\lambda_-}{|z|^{\alpha_-}\Theta_-(z)}\right)\\ &- \ell(\gamma_-, \alpha_-, \lambda_-) + iz\,\ell(\gamma_- - 1, \alpha_-, \lambda_-)\Bigg) + iz(\mu + \mu_1);\end{aligned} \tag{68}$$

where

$$\ell(x, y, z) = z^{x/y}\frac{\pi}{\sin\left(-\pi\frac{x}{y}\right)}\frac{1}{y\Gamma(1+x)}. \tag{69}$$

The cases not accounted by (*i*) − (*iii*) *can be obtained by combining the expressions for positive and negative parts corresponding to the relevant inequalities satisfied by* $\alpha_\pm + \gamma_\pm$.

Proof. Let m be the Lévy density of X given by (39) and let θ_+^n, θ_-^n, $n \in \mathbb{N}$, be two sequences of real numbers such that $\theta_+^n, \theta_-^n \to 0$ as $n \to \infty$, set $\boldsymbol{\theta}^n = (\theta_+^n, \theta_-^n)$ and let X_n be $\mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}^n, \boldsymbol{\delta}; 0)$ r.v.s with Lévy densities m^n.

We argue by dominated convergence. The sequence $(e^{izx} - 1 - izx\mathbb{1}_{\{|x|<1\}})m^n(x)$ is dominated in x for all z by the integrable function $(e^{izx} - 1 - izx\mathbb{1}_{\{|x|<1\}})m(x)$. Therefore

$$\int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbb{1}_{\{|x|<1\}})m(x)dx = \lim_{n\to\infty}\int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbb{1}_{\{|x|<1\}})m^n(x)dx. \tag{70}$$

We consider then the decompositions in positive/negative parts $m(x) = m_+(x) + m_-(x)$ and $m^n(x) = m_+^n(x) + m_-^n(x)$ and for brevity only analyze the positive one, the negative one being identical upon the usual sign and parameter modifications.

Assume first $\gamma = 0$. Using Theorem 1(i), in view of (70) and continuity of the logarithm

$$\begin{aligned}\int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbb{1}_{\{|x|<1\}})m_+(x)dx &= \frac{\delta_+}{\alpha_+}\lim_{n\to\infty}\log\left(\frac{(\theta_+^n)^{\alpha_+} + \lambda_+}{(\theta_+^n - iz)^{\alpha_+} + \lambda_+}\right) - i\lim_{n\to\infty} z\mu_0^{n,+}\\ &= \frac{\delta_+}{\alpha_+}\log\left(\frac{\lambda_+}{(-iz)^{\alpha_+} + \lambda_+}\right) - iz\mu_0^+\end{aligned} \tag{71}$$

with $\mu_0^+ = \int_0^1 x\,m_+(x)dx$, $\mu_0^{n,+} = \int_0^1 x\,m_+^n(x)dx$, when $\mu_0^{n,+} \to \mu_0^+$ follows again by dominated convergence. But now, for $\alpha \in (0, 1)$, $w \in \mathbb{R}$ a standard computation [53, p. 84–85] shows,

$$(-iw)^\alpha = |w|^\alpha \cos\frac{\alpha\pi}{2}\left(1 - i\tan\frac{\pi\alpha}{2}\mathrm{sgn}(w)\right) \tag{72}$$

and, after adding $iz\mu$, (66) is proved.

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Next, assume $\alpha_+ + \gamma_+ \in [1, 3)$. Proceeding as in (71) we obtain

$$\int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbb{1}_{|x|<1})m_+(x)dx$$
$$= \delta_+\Gamma(-\gamma_+)|z|^{\gamma_+}\Theta_+(z)\, {}_2R_1\left(1, -\gamma_+, 1, \alpha_+; \frac{-\lambda_+}{|z|^{\alpha_+}\Theta_+(z)}\right)$$
$$- \delta_+\Gamma(-\gamma_+)\lim_{n\to\infty}(\theta_+^n)^{\gamma}\, {}_2R_1\left(1, -\gamma_+, 1, \alpha_+; -\frac{\lambda_+}{(\theta_+^n)^{\alpha_+}}\right)$$
$$-iz\delta_+\Gamma(1-\gamma_+)\lim_{n\to\infty}(\theta_+^n)^{\gamma_+-1}\, {}_2R_1\left(1, 1-\gamma_+, 1, \alpha_+; -\frac{\lambda_+}{(\theta_+^n)^{\alpha_+}}\right) + iz\mu_1^{n,+}, \tag{73}$$

where $\mu_1^+ = \int_{\{x>1\}} m_+(x)dx$, $\mu_1^{n,+} = \int_{\{x>1\}} m_+^n(x)dx$, with $\mu_1^{n,+} \to \mu_1^+$ by dominated convergence, which is ensured by the condition $\alpha_+ + \gamma_+ > 1$ combined with the asymptotic relation (26). Now for the limits above we apply Lemma 1(i) with $b = \alpha_+$, $z = \theta_+^n$, $c = \lambda_+$, and a respectively equal to $-\gamma_+$, and $1 - \gamma_+$, to obtain (68).

The case $\alpha_+ + \gamma_+ \in (0, 1]$ is dealt with similarly, but instead uses expression (55) for the characteristic functions of X_n.

We have been unable to treat the cases $\gamma_+ = \gamma_- = 1$, since we could not derive asymptotic results for Φ in the domain of interest, but we conjecture a similar expression to hold. For complex numbers $z \in \mathbb{C} \setminus \mathbb{R}$ an asymptotic series is given in [16]. The lack of analyticity of the GRGS class compared to the analyticity of GRGTS, $\theta_\pm > 0$, puts these two cases in a similar relationship relationship to the one between the S and CTS classes, in that the GRGTS class is just an exponentially tempered version of the GRGS class. The $\text{RGS}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \boldsymbol{\delta}; \mu)$ family can be of interest for applications, and by analogy with the $\text{BG}(\boldsymbol{\lambda}, \boldsymbol{\delta})$ distribution we could also refer to these distributions as bilateral Linnik $\text{BL}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\delta})$.

Example 2. Linnik distributions. *Continuing Example 1, by letting X be a* $\text{RGS}_\alpha^+(\lambda, \delta; \mu_0)$ *r.v., we have the characteristic exponent*

$$\psi_X(z) = \delta \log\left(\frac{\lambda}{|z|^\alpha\Theta_+(z) + \lambda}\right), \tag{74}$$

which is the characteristic exponent of a $\text{PL}(\alpha, \lambda^{-1}, \delta)$ *r.v.*

Example 3. Subordinated representation of a symmetric bilateral Linnik process. *The Lévy measure for general GS distributions on the real line is provided as a Bochner integral in [33] and does not correspond to a characteristic function of the form (66). Assume* $X \in \text{RGS}_\alpha^s(\lambda, \delta; \mu_0) =: \text{BL}^s(\alpha, \lambda, \delta)$. *In this case we have, observing that $\Theta_\pm$ are complex conjugates,*

$$\psi_X(z) = -\delta \log\left(|z|^{2\alpha}\lambda^{-2} + 2\lambda^{-1}|z|^\alpha + 1\right). \tag{75}$$

By indicating G a $\text{G}(1, \delta)$ *law we see that it is possible to write $\psi_X(z) = \psi_G(\psi_{Y_1+Y_2}(z))$ for independent stable laws* $Y_1 \sim \text{S}_\alpha(2\lambda^{-1}, 0; 0)$ *and* $Y_2 \sim \text{S}_{2\alpha}(\lambda^{-2}, 0; 0)$. *In terms of the associated Lévy processes, the process $X = (X_t)_{t\geq 0}$ is such that $X_t = Y_{1,G_t} + Y_{2,G_t}$ where $G_1 \sim G$ is a gamma subordinator and $Y_{1,t}$, $Y_{2,t}$ independent stable processes with unit time laws Y_1 and Y_2, respectively.*

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4.2. Cumulants

The difference in the analytic structure of the characteristic functions of GRGTS laws depending upon $\boldsymbol{\theta}=0$ or $\boldsymbol{\theta}>0$ highlighted in Theorems 1 and 2 is naturally reflected on cumulants. As we noticed when $\boldsymbol{\theta}>0$, the GRGTS characteristic function is analytical and thus all the moments exist and can be computed by differentiating the characteristic function. However, when $\theta_\pm=0$ a crude analysis of the Lévy measure using (26) yields that, for example, as $x\to\infty$ then $m(x)\sim x^{-1-\alpha_+-\gamma_+}$, so that not all the Lévy moments exist. Because of the equivalence of the finiteness of Lévy and distribution moments (e.g. [53, Chapter 25]), this implies that not all of the $\mathrm{GRGS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{\delta};\mu)$ cumulants will be finite. We will explore rigorously the relation between Lévy and probability density tails in Section 6. Recall that if X is a univariate r.v. such that $E[e^{cX}]<\infty$ for all $c\in(-\epsilon,\epsilon)$ and some $\epsilon>0$, its nth cumulant k_n^X is defined as $k_n^X=\frac{d^n}{ds^n}\log E[e^{sX}]\Big|_{s=0}$. We have the following proposition.

Proposition 2. *Let X be a $\mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{\theta},\boldsymbol{\delta};\mu)$ r.v. with $\theta_+,\theta_->0$, $\gamma\neq1$. Then its cumulants k_n^X, $n\in\mathbb{N}_0$ are given by*

$$k_1^X=\mu_1+\mu; \tag{76}$$

$$k_n^X=\frac{\delta_+\Gamma(n-\gamma_+)}{\theta_+^{n-\gamma_+}}\,{}_2R_1\left(1,n-\gamma_+,1,\alpha_+;-\frac{\lambda_+}{\theta_+^{\alpha_+}}\right)+ (-1)^n\frac{\delta_-\Gamma(n-\gamma_-)}{\theta_-^{n-\gamma_-}}\,{}_2R_1\left(1,n-\gamma_-,1,\alpha_-;-\frac{\lambda_-}{\theta_-^{\alpha_-}}\right),\quad n>1. \tag{77}$$

Let instead Y be a $\mathrm{GRGS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{\delta};\mu)$ r.v. Then Y has finite expectation if and only if $\min\{\alpha_++\gamma_+,\alpha_-+\gamma_-\}>1$ and finite variance if and only if $\min\{\alpha_++\gamma_+,\alpha_-+\gamma_-\}>2$, in which cases

$$E[Y]=\mu_1+\mu, \tag{78}$$

$$\mathrm{Var}[Y]=\delta_+\ell(\gamma_+-2,\alpha_+,\lambda_+)+\delta_-\ell(\gamma_--2,\alpha_-,\lambda_-). \tag{79}$$

Proof. Denote $m(x)=m_+(x)+m_-(x)$ the positive and negative parts of the Lévy density. That $k_1^X=\mu+\mu_1$ for all i.d. distribution is well known (e.g [53, Example 25.12]). In the case $\theta_+,\theta_->0$ we apply that, for analytic distributions, cumulants and Lévy moments coincide when $n>1$ (again [53, Chapter 25]). In our case $k_n^X=\int_{\mathbb{R}}x^nm(x)dx$, and for $n\geq1$, $\gamma\neq1$, and $\theta_+^{\alpha_+}>\lambda_+$ it holds that

$$\int_0^\infty x^n m_+(dx)=\delta_+\int_0^\infty e^{-\theta_+x}x^{n-\gamma_+-1}E_{\alpha_+}\left(-\lambda_+x^{\alpha_+}\right)dx$$
$$=\delta_+\theta_+^{n-\gamma_+}\sum_{j=0}^\infty\left(\frac{-\lambda_+}{\theta_+^{\alpha_+}}\right)^j\frac{\Gamma(n-\gamma_++\alpha_+j)}{\Gamma(1+\alpha_+j)} \tag{80}$$

leading, together with the analogous calculation for m_-, to (76)–(77). The case for arbitrary parameters follows by analytic continuation. Because of the analyticity of ψ_X, for $\gamma\neq1$ the conclusion is also immediate from Lemma 1(*i*,*ii*), since we can differentiate ψ in z and take the limit $z\to0$ to obtain the cumulants.

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Regarding Y, again because of [53, Corollary 25.8], the first statement follows from $\int_{\{|x|>1\}} x^k m_\pm(x)dx < \infty$ if and only if $k < \alpha_\pm + \gamma_\pm$, which once again is a consequence of (26). For the variance it is possible to differentiate twice the characteristic exponent (68) and take the limit $z \to 0$. To this end, since $2 - \gamma_+ < \alpha_+$ we can apply Lemma 1 with $b = \alpha_+$, $a = -\gamma_+$, $c = \lambda_+$, and z replaced by $-iz$, so that by part (i) it holds that

$$\mathrm{Var}[Y^+] = \psi''(0) = \frac{\delta_+ \Gamma(2-\gamma_+)}{(-iz)^{2-\gamma_+}} \, {}_2R_1\left(1, 2-\gamma_+, 1, \alpha_+; -\frac{\lambda_+}{(-iz)^{\alpha_+}}\right) \tag{81}$$

and then, using part (ii),

$$\begin{aligned} k_2^{Y^+} &= \delta_+ \lim_{z\to 0} \frac{\delta_+ \Gamma(2-\gamma_+)}{(-iz)^{2-\gamma_+}} \, {}_2R_1\left(1, 2-\gamma_+, 1, \alpha_+; -\frac{\lambda_+}{(-iz)^{\alpha_+}}\right) \\ &= \delta_+ \frac{\lambda_+^{\frac{\gamma_+-2}{\alpha_+}}}{\alpha_+} \frac{\pi}{\sin\left(\pi \frac{2-\gamma_+}{\alpha_+}\right) \Gamma(\gamma_+ - 1)} \\ &= \delta_+ \ell(\gamma_+ - 2, \alpha_+, \lambda_+). \end{aligned} \tag{82}$$

The same arguments apply to $k_2^{Y^-}$, and the proof is finished.

Therefore, GRGS distributions have the peculiar property of retaining finite variance for some ranges of parameters, but no other higher moment. As mentioned in the introduction this can capture empirical findings on financial data and make this distribution an ideal candidate to model such quantities.

Information about the existence of moments can also be extracted by the Rosińsky measure, which we shall study in Section 5.

Example 4. *Cumulants of an RGTS distribution. When* $X \sim \mathrm{RGTS}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu_0)$, $\theta_+, \theta_- > 0$, *we have*

$$\begin{aligned} k_n^X &= \frac{\delta_+(n-1)!}{\theta_+^n} \, {}_2R_1\left(1, n, 1, \alpha_+; -\frac{\lambda_+}{\theta_+^{\alpha_+}}\right) + \\ &\qquad (-1)^n \frac{\delta_-(n-1)!}{\theta_-^n} \, {}_2R_1\left(1, n, \alpha_-; -\frac{\lambda_-}{\theta_-^{\alpha_-}}\right) \\ &= \frac{\delta_+(n-1)!}{\theta_+^n} \sum_{j=0}^{\infty} (\alpha_+ j + 1)_{n-1} \left(\frac{-\lambda_+}{\theta_+^{\alpha_+}}\right)^j + \\ &\qquad (-1)^n \frac{\delta_-(n-1)!}{\theta_-^n} \sum_{j=0}^{\infty} (\alpha_- j + 1)_{n-1} \left(\frac{-\lambda_-}{\theta_-^{\alpha_-}}\right)^j . \end{aligned} \tag{83}$$

This extends the TPL cumulant analysis of [61, Proposition 2.2]. One can show along the lines of such a result that the TGS cumulants are given by

$$\kappa_n^X = \frac{\delta_+}{\theta_+^n} g_{n-1}\left(\frac{-\lambda_+}{\theta_+^{\gamma_+}}; \alpha_+\right) + (-1)^n \frac{\delta_-}{\theta_-^n} g_{n-1}\left(\frac{-\lambda_-}{\theta_-^{\gamma_-}}; \alpha_-\right), \tag{84}$$

where $g_n(x; c)$ *satisfies the recursion*

$$g_n(x; c) = xc \frac{d}{dx} g_{n-1}(x; c) + n g_{n-1}(x; c) \tag{85}$$

with $c > 0$ *and* $g_0(x; c) = \frac{1}{1-x}$.

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5. Spectral representations, limits and absolute continuity

We analyze more in detail the structure of one-dimensional GRGTS distributions in relation to the theory of [51]. We find the spectral and Rosińsky measures of such laws identify short- and long-time Lévy scaling limits and give conditions for absolute continuity with respect to a stable law, as well as with other GRGTS distributions. In order to keep in line with the standard theory, for the most part of this section we assume $\gamma_+ = \gamma_- > 0$ and we remove the boldface throughout to indicate this. Extensions to asymmetric cases can be easily obtained.

Proposition 3. *A* $\mathrm{GRGTS}_\gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$ *with* $\alpha_+, \alpha_- \in (0, 1)$ *and* $\gamma > 0$ *admits both a spectral density s and a Rosińsky density* r_γ *given, respectively, by*

$$s(x; \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}) = \delta_+ \frac{(x-\theta_+)^{\alpha_+-1}}{\pi} \frac{\sin(\alpha_+\pi)}{\lambda_+^{-1}(x-\theta_+)^{2\alpha_+} + 2(x-\theta_+)^{\alpha_+}\cos(\alpha_+\pi) + \lambda_+} \mathbb{1}_{\{x>\theta_+\}} + \delta_- \frac{(|x|-\theta_-)^{\alpha_--1}}{\pi} \frac{\sin(\alpha_-\pi)}{\lambda_-^{-1}(|x|-\theta_-)^{2\alpha_-} + 2(|x|-\theta_-)^{\alpha_-}\cos(\alpha_-\pi) + \lambda_-} \mathbb{1}_{\{x<-\theta_-\}} \tag{86}$$

and, with $1/0 := \infty$,

$$r_\gamma(x; \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}) = \delta_+ \frac{x^{-\gamma+\alpha_+-1}}{\pi} \frac{(1-\theta_+x)^{\alpha_+-1}\sin(\alpha_+\pi)}{\lambda_+^{-1}(1-\theta_+x)^{2\alpha_+} + 2(x(1-\theta_+x))^{\alpha_+}\cos(\alpha_+\pi) + \lambda_+x^{2\alpha_+}} \mathbb{1}_{\{0<x<\theta_+^{-1}\}} + \delta_- \frac{|x|^{-\gamma+\alpha_--1}}{\pi} \frac{(1-\theta_-|x|)^{\alpha_--1}\sin(\alpha_-\pi)}{\lambda_-^{-1}(1-\theta_-|x|)^{2\alpha_-} + 2(|x|(1-\theta_-|x|))^{\alpha_-}\cos(\alpha_-\pi) + \lambda_-|x|^{2\alpha_-}} \mathbb{1}_{\{-\theta_-^{-1}<x<0\}}. \tag{87}$$

Proof. By, for example, [13, Equation (2.16)], the density in the Bernstein representation of $E_\alpha(-\cdot^\alpha)$ is

$$s^E(x) = \frac{x^{\alpha-1}}{\pi} \frac{\sin(\alpha\pi)}{x^{2\alpha} + 2x^\alpha\cos(\alpha\pi) + 1}, \qquad x > 0, \tag{88}$$

which is the p.d.f. of the ratio of two independent α-stable r.v.s (see [25, p. 9]). After an application of the Laplace transform rules we see that the tempering function $q(x) = q(x, 1)\mathbb{1}_{\{x>0\}} + q(|x|, -1)\mathbb{1}_{\{x<0\}}$ is given by

$$q(x) = \begin{cases} \displaystyle\int_0^\infty e^{-yx}\lambda_+^{-1/\alpha_+} s^E\big((y-\theta_+)\lambda_+^{-1/\alpha_+}\big)dy & x > 0, \\ \displaystyle\int_0^\infty e^{-y|x|}\lambda_-^{-1/\alpha_-} s^E\big((y-\theta_-)\lambda_-^{-1/\alpha_-}\big)dy & x < 0. \end{cases} \tag{89}$$

Substituting (88) in (89) and using this in (16) with σ as in (36) we obtain (86). Moreover, writing explicitly (17), for $B \in \mathcal{B}(\mathbb{R}_+)$ we have, using the integral substitution $y = 1/x$,

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$$
\begin{aligned}
R(B) &= \delta_+ \int_{\theta_+}^{\infty} \mathbb{1}_B\left(\frac{\operatorname{sgn}(x)}{x}\right)\frac{x^\gamma}{\pi}\frac{(x-\theta_+)^{\alpha_+-1}\sin(\pi\alpha_+)}{\lambda_+^{-1}(x-\theta_+)^{2\alpha_+}+2(x-\theta_+)^{\alpha_+}\cos(\alpha_+\pi)+\lambda_+}dx \\
&= \delta_+ \int_0^{1/\theta_+} \mathbb{1}_B(y)\frac{y^{-\gamma-2}}{\pi}\frac{(y^{-1}-\theta_+)^{\alpha_+-1}\sin(\pi\alpha_+)}{\lambda_+^{-1}(y^{-1}-\theta_+)^{2\alpha_+}+2(y^{-1}-\theta_+)^{\alpha_+}\cos(\alpha_+\pi)+\lambda_+}dy \\
&= \delta_+ \int_0^{1/\theta_+} \mathbb{1}_B(y)\frac{y^{-\gamma-\alpha_+-1}}{\pi}\frac{(1-\theta_+y)^{\alpha_+-1}\sin(\pi\alpha_+)}{\lambda_+^{-1}y^{-2\alpha_+}(1-\theta_+y)^{2\alpha_+}+2y^{-\alpha_+}(1-\theta_+y)^{\alpha_+}\cos(\alpha_+\pi)+\lambda_+}dy \\
&= \delta_+ \int_0^{1/\theta_+} \mathbb{1}_B(y)\frac{y^{-\gamma+\alpha_+-1}}{\pi}\frac{(1-\theta_+y)^{\alpha_+-1}\sin(\pi\alpha_+)}{\lambda_+^{-1}(1-\theta_+y)^{2\alpha_+}+2(y(1-\theta_+y))^{\alpha_+}\cos(\alpha_+\pi)+\lambda y^{2\alpha_+}}dy.
\end{aligned}
\tag{90}
$$

The analogous computation holds for the negative part when $B \in \mathcal{B}(\mathbb{R}_-)$, and (87) follows.

For a $\mathrm{GRGTS}_\gamma(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$, $\gamma \neq 0$, we denote $\mathrm{GRGTS}_\gamma(r_\gamma(x; \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}); \mu)$ the parametrization using the Rosińsky density r_γ and a drift μ.

Remark 5. *Moments revisited*. According to [51, Proposition 2.7], finiteness of the moments of a GRGTS law is equivalent to the finiteness of the moments of the measure R. More precisely, the pth moment, $p > 0$, is always finite for $p < \gamma$; it is finite for $p > \gamma$ if and only if $\int_{\{|x|>1\}} |x|^p R(dx) < \infty$, and for $p = \gamma$ it is finite if and only if $\int_{\{|x|>1\}} |x|^\gamma \log |x| R(dx) < \infty$. Using (87) we see that these integrals always converge for any p whenever $\theta_+, \theta > 0$, in accordance with Theorem 1 and Proposition 2. Instead, if $\theta_+ = \theta_- = 0$ we have, with r given by (87),

$$
x^p r_\gamma(x) \sim \begin{cases} \delta_+ \dfrac{x^{p-\gamma+\alpha_+-1}}{\pi} \dfrac{\sin(\alpha_+\pi)}{\lambda_+^{-1} + 2x^{\alpha_+}\cos(\alpha_+\pi) + \lambda_+ x^{2\alpha_+}} = O(x^{p-\gamma-1-\alpha_+}), & x \to \infty \\ \delta_- \dfrac{|x|^{p-\gamma+\alpha_--1}}{\pi} \dfrac{\sin(\alpha_-\pi)}{\lambda_-^{-1} + 2|x|^{\alpha_-}\cos(\alpha_-\pi) + \lambda_-|x|^{2\alpha_-}} = O(|x|^{p-\gamma-1-\alpha_-}), & x \to -\infty. \end{cases}
\tag{91}
$$

Therefore, for $p = \gamma$ it is $x^{-\gamma} r_\gamma(x) \log(x) = O(\log(x) x^{-\alpha_+-1})$ as $x \to \infty$ and $x^{-\gamma} r_\gamma(x) \log(x) = O(\log(-x)(-x)^{-\alpha_--1})$ as $x \to -\infty$, which both converge, and thus the boundary moment is finite. Moreover, when $p > \gamma$, the convergence condition is $\min\{\alpha_+ + \gamma, \alpha_- + \gamma\} > p$, again consistent with Proposition 2. Furthermore, still by [51, Proposition 2.7], the condition for the finiteness of the exponential moments (clearly unavailable when $\theta_+ = \theta_- = 0$) of order $\beta > 0$ is $R(\{x{:}|x| > \beta^{-1}\}) = 0$. From (87) the latter holds if and only if $\beta \leq \theta_0 = \min\{\theta_+, \theta_-\}$. By the standard theory of [41], this implicates the analyticity of the characteristic function of the GRGTS law with positive exponential tempering at least in the strip $B = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \theta_0\}$, in accordance with Theorem 1.

Spectral measures are useful to understand the short- and long-time behavior of generalized geometric tempered stable Lévy processes. For clarity of exposition we further confine our treatment to the case $\alpha_+ = \alpha_-$; how to deal with the case $\alpha_+ \neq \alpha_-$, involving stable limits with different positive and negative stability indices, should be clear.

Proposition 4. *Let $X = (X_t)_{t\geq 0}$ be a $\mathrm{GRGTS}_\gamma(r_\gamma(x; (\alpha, \alpha), \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}), \mu)$ Lévy process with $\gamma \neq 1$ and $\mu = \mu_0$ if $\gamma \in (0, 1)$, $\mu = -\mu_1$ if $\gamma \in (1, 2)$. Define for all $h > 0$ the scaled processes $X^h = (X_{ht})_{t\geq 0}$. We have, as $h \to 0$, that*

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(i) $h^{-1/\gamma}X^h \to Z$ *where* $Z = (Z_t)_{t\geq 0}$ *is a stable Lévy process such that* $Z_1 \sim S_\gamma(\boldsymbol{\delta}; \mu^*)$, *with* $\mu^* = \mu_0^*$ *if* $\gamma \in (0, 1)$ *and* $\mu^* = -\mu_1^*$ *if* $\gamma \in (1, 2)$, *with* μ_1^*, μ_0^* *relative to the Lévy measure of* Z;

and, as $h \to \infty$, *that*

(ii) *if* $\theta_+ = \theta_- = 0$ *and* $\alpha + \gamma \in (0, 1) \cup (1, 2)$ *then* $h^{-(\alpha+\gamma)^{-1}}X_h \to Z$ *where* Z *is a stable Lévy process such that* $Z_1 \sim S_{\alpha+\gamma}((\delta_+^*, \delta_-^*); \mu^*)$ *with*

$$\delta_+^* = \frac{\delta_+}{\Gamma(1-\alpha)\lambda_+}, \qquad \delta_-^* = \frac{\delta_-}{\Gamma(1-\alpha)\lambda_-}; \tag{92}$$

(iii) *if* $\theta_+, \theta_- > 0$ *or* $\alpha + \gamma > 2$, *then* $h^{-1/2}X^h \to B$ *where* $B = (B_t)_{t\geq 0}$ *is a Gaussian Lévy process with triplet* $(0, \mathrm{Var}[X_1], 0)$.

The convergences above are uniform on compact sets in probability, i.e.

$$\lim_{h\to 0,\infty} P\left(\sup_{s\leq t} |X_s^h - Y_s| > \epsilon\right) = 0, \tag{93}$$

for all $\epsilon, t > 0$, *with* $Y = Z, B$, *respectively, in cases* $(i) - (ii)$ *and* (iii).

Proof. Let r_γ be the Rosińsky density (87). For (i) from [51, Theorem 3.1], a sufficient condition for the statement to hold is that

$$\int_{\mathbb{R}} x^\gamma r_\gamma(x; \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta})dx < \infty. \tag{94}$$

When $\theta_+, \theta_- > 0$, the above is trivially verified since in that case r_γ is supported on a bounded set. When $\theta_+ = \theta_- = 0$, then using (91) with $p = \gamma$, one has $x^\gamma r_\gamma(x) \sim O(|x|^{-1-\alpha})$, as $x \to \pm\infty$ so that (94) still holds. Notice that convergence in distribution can be strengthened to (93), since such is the conclusion of [26, Theorem 15.17], used in the proof of [51, Theorem 3.1].

Now to show $(ii) - (iii)$ we begin by proving the Gaussian limit in (iii) under the assumption $\theta_+, \theta_- > 0$. Denote by $\psi_h(z)$ the characteristic exponent of the Lévy process X^h. By the mentioned result [26, Theorem 15.17], for the claim to hold it is sufficient to show convergence in distribution which we verify on the characteristic exponents. Write the decomposition of ψ_h in spectrally positive and negative parts as $\psi_h = \psi_h^+ + \psi_h^-$, Now we can use the integral (48) since $h^{-1} \sim 0$, which yields, after interchanging the summation order,

$$\begin{aligned}
\lim_{h\to\infty} h^{-1/2}\psi_h^+(z) &= \lim_{h\to\infty} h\psi^+(zh^{-1/2}) \\
&= \theta_+^\gamma \delta_+ \lim_{h\to\infty} h \sum_{j=0}^{\infty} \left(\frac{-\lambda_+}{\theta_+^\alpha}\right)^j \frac{1}{\Gamma(1+\alpha j)} \sum_{k=2} \frac{\left(\frac{izh^{-1/2}}{\theta_+}\right)^k}{k!} \Gamma(k-\gamma+\alpha j) \\
&= \delta_+ \lim_{h\to\infty} h\, \theta_+^{\gamma_+ - 2} \sum_{j=0}^{\infty} \left(\frac{-\lambda_+}{\theta_+^\alpha}\right)^j \frac{\Gamma(2-\gamma+\alpha j)}{\Gamma(1+\alpha j)} \left(-\frac{z^2 h^{-1}}{2}\right) + o(1) \\
&= -\frac{z^2}{2}\delta_+\theta_+^{\gamma-2}\Gamma(2-\gamma)\, {}_2R_1\left(1, 2-\gamma, 1; \alpha, -\frac{\lambda_+}{\theta_+^\alpha}\right).
\end{aligned} \tag{95}$$

After carrying out the corresponding computation for ψ_h^-, recalling Proposition 2, we notice that in the final expression the factor $\mathrm{Var}[X_1]$ appears as multiplying the characteristic exponent of the standard Brownian motion, which establishes the claim.

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To prove (*ii*) and (*iii*) when $\theta_+ = \theta_- = 0$, we proceed by setting $\kappa > 0$ and analyze $\psi_h(zh^{-1/k}) = h\psi(zh^{-1/\kappa})$. We must expand (67) and (68) around $h^{-1} \sim 0$; in order to do this we can use [28, Theorem 5.2], providing a series representation for the ${}_2R_1$ function of complex argument outside the unit circle. Under our parameter specification it holds (recall also footnote 1)

$$\begin{aligned}
&\Gamma(-\gamma)\,{}_2R_1\,(1,-\gamma,1,\alpha;w) = \\
&-\sum_{k=0}^{\infty}\frac{\Gamma(-\gamma-\alpha(k+1))}{\Gamma(1-\alpha(k+1))}(w^{-k-1}) \\
&+\frac{(-w^{\gamma/\alpha})}{\alpha}\sum_{k=0}^{\infty}\frac{\Gamma((k-\gamma)/\alpha)\Gamma(1-(k-\gamma)/\alpha)}{k!\Gamma(1+\gamma-k)}(-1)^k(-w)^{-k/\alpha} \\
&=-\sum_{k=0}^{\infty}\frac{\Gamma(-\gamma-\alpha(k+1))}{\Gamma(1-\alpha(k+1))}(w^{-k-1})+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\ell(\gamma-k,\alpha,-w) \qquad (96)
\end{aligned}$$

after using Euler's summation and recalling that $\ell(\cdot,\cdot,\cdot)$ is given by (69). Setting $w = \frac{-\lambda_+}{h^{-\alpha/\kappa}|z|^\alpha\Theta_+(z)}$, we obtain further

$$\begin{aligned}
&\Gamma(-\gamma)h^{-\gamma/\kappa}\,{}_2R_1\left(1,-\gamma,1,\alpha;\frac{-\lambda_+h^{\alpha/\kappa}}{|z|^\alpha\Theta_+(z)}\right) \\
&=-\sum_{k=0}^{\infty}\frac{\Gamma(-\gamma-\alpha(k+1))}{\Gamma(1-\alpha(k+1))}\left(\frac{(-iz)^\alpha}{-\lambda_+}\right)^{k+1}h^{-(\gamma+\alpha(k+1))/\kappa} \\
&\quad+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}h^{(\gamma-k)/\kappa}\ell\left(\gamma-k,\alpha,\frac{\lambda_+}{(-iz)^\alpha}\right). \qquad (97)
\end{aligned}$$

If $\alpha+\gamma<1$, observing that $\ell\left(\gamma-k,\alpha,\frac{\lambda_+}{(-iz)^\alpha}\right) = (-iz)^{-\gamma}\ell(\gamma-k,\alpha,\lambda_+)$ in the positive part of (67) with $\mu = -\mu_0$ we obtain,

$$\begin{aligned}
&\delta_+\left(\Gamma(-\gamma)|z|^\gamma\Theta_+(z)h^{-\gamma/\kappa}\,{}_2R_1\left(1,-\gamma,1,\alpha;\frac{-\lambda_+h^{\alpha/\kappa}}{|z|^\alpha\Theta_+(z)}\right)-\ell(\gamma,\alpha,\lambda_+)\right) \\
&=\delta_+\left(-(-iz)^\gamma\sum_{k=0}^{\infty}\frac{\Gamma(-\gamma-\alpha(k+1))}{\Gamma(1-\alpha(k+1))}\left(\frac{(-iz)^\alpha}{-\lambda_+}\right)^{k+1}h^{-(\gamma+\alpha(k+1))/\kappa}\right. \\
&\qquad\left.+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}h^{-k/\kappa}\ell\left(\gamma-k,\alpha,\frac{\lambda_+}{(-iz)^\alpha}\right)\right). \qquad (98)
\end{aligned}$$

The leading order in (98) corresponds to the term $k=0$ of the first series. We thus have

$$\lim_{h\to\infty} h\,\psi^+(h^{-1/\kappa}) = \delta_+\lim_{h\to\infty} h\frac{\Gamma(-\gamma-\alpha)}{\Gamma(1-\alpha)\lambda_+}(-iz)^{\alpha+\gamma}h^{-(\gamma+\alpha)/\kappa} = \delta_+\frac{\Gamma(-\gamma-\alpha)}{\Gamma(1-\alpha)\lambda_+}(-iz)^{\alpha+\gamma}, \qquad (99)$$

with the last equality holding if and only if $\kappa = \alpha+\gamma$. Now it is well known that (e.g. [53, Lemma 14.11])

$$(-iz)^{\alpha+\gamma}\Gamma(-\alpha-\gamma) = \int_0^\infty \frac{(e^{izx}-1)}{x^{1+\gamma+\alpha}}dx \qquad \alpha+\gamma\in(0,1),\ z\in\mathbb{R}, \qquad (100)$$

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which is the characteristic exponent of a spectrally positive $\alpha+\gamma$ stable r.v. with unit δ_+ and $\mu=\mu_0^*$. Substituting in (99) produces the positive part of the characteristic exponent of Z. Repeating the above for ψ_h^- yields (*iii*) when $\alpha+\gamma\in(0,1)$.

If, instead, $\alpha+\gamma\in(1,2)$ we must use (68) with $\mu=-\mu_1$ and in (97) we have, from the relation $\ell\left(\gamma-1,\alpha,\frac{\lambda_+}{(-iz)^\alpha}\right)=(-iz)^{1-\gamma}\ell(\gamma-1,\alpha,\lambda_+)$, that

$$\begin{aligned}\delta_+\Bigg(\Gamma(-\gamma)|z|^\gamma\Theta_+(z)h^{-\gamma/\kappa}\,{}_2R_1\left(1,-\gamma,1,\alpha;\frac{-\lambda_+h^{\alpha/\kappa}}{|z|^\alpha\Theta_+(z)}\right)-\ell(\gamma,\alpha,\lambda_+)\\ -izh^{-1/\kappa}\ell(\gamma-1,\alpha,\lambda_+)\Bigg)\\ =\delta_+\Bigg(-(-iz)^\gamma\sum_{k=0}^\infty\frac{\Gamma(-\gamma-\alpha(k+1))}{\Gamma(1-\alpha(k+1))}\left(\frac{(-iz)^\alpha}{-\lambda_+}\right)^{k+1}h^{-(\gamma+\alpha(k+1))/\kappa}+\\ \sum_{k=2}^\infty\frac{(-1)^k}{k!}h^{-k/\kappa}\ell\left(\gamma-k,\alpha,\frac{\lambda_+}{(-iz)^\alpha}\right)\Bigg).\end{aligned}\qquad(101)$$

Now, when $\alpha+\gamma\in(0,2)$, again the $k=0$ term in the first series leads, and the statement follows as in (99), this time observing that

$$(-iz)^{\alpha+\gamma}\Gamma(-\alpha-\gamma)=\int_0^\infty\frac{(e^{izx}-1-izx)}{x^{1+\gamma+\alpha}}dx\qquad\alpha+\gamma\in(1,2),\ z\in\mathbb{R},\qquad(102)$$

which is again the characteristic exponent of a spectrally positive $\alpha+\gamma$ stable r.v. and unit δ_+, but this time with $\mu=-\mu_1^*$. This concludes the proof of (*ii*).

Finally, if $\alpha+\gamma>2$ in (97), then the leading order corresponds to the term $k=2$ in the second series, so that

$$\begin{aligned}\lim_{h\to\infty}h\,\psi^+(h^{-1/\kappa})=\delta_+\lim_{h\to\infty}h\frac{h^{-2/\kappa}}{2}(-iz)^\gamma\ell\left(\gamma-2,\alpha,\lambda_+\right)(-iz)^{2-\gamma}\\ =-\frac{z^2}{2}\delta_+\ell\left(\gamma-2,\alpha,\lambda_+\right)\end{aligned}\qquad(103)$$

provided that $\kappa=2$. Together with the analogous computation for ψ_h^-, this completes the proof of (*iii*) and of the proposition.

Observe that in view of Proposition 2 (or Remark 5) the second condition in (*iii*) of Proposition 4 is equivalent to the finiteness of the variance. We see that the familiar short-time stable/long-time Gaussian behavior of CTS laws (e.g. [36]) is reproduced for positive $\theta_\pm$ or when the process is $\mathrm{GRGS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{\delta};\mu)$ but with finite variance, according to the central limit theorem intuition. This last instance is the most interesting since it accounts for persistently heavy tails but has a Gaussian limit, and is consistent with the empirical studies motivating our work.

Instead, case (ii) shows that pure Mittag-Leffler tempering determines a stable limit whose stability index is increased by the Mittag-Leffler factor α, compared to the short-time γ-stable limit. This regime may also be of interest for applications.

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Another useful information that can be derived from the measure R is the range of values of γ for which absolute continuity (in the sense of [53, Chapter 33]) with respect to a stable process holds. The situation is akin to the generalized exponential tempering situation described in [22].

Proposition 5. *Let $X=(X_t)_{t\geq 0}$ be a* $\mathrm{GRGTS}_\gamma(r_\gamma(x;\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{\theta},\boldsymbol{\delta}),\mu)$ *Lévy process. Assume that there exists a second probability measure P' under which X is a $S_\gamma(\boldsymbol{\delta},\mu^*)$ Lévy process, with σ given by (36). Then P' is absolutely continuous with respect to P if and only if*

$$\mu^* = \begin{cases} \mu & \text{if } \gamma \in (0,1), \\ \mu + \int_{\mathbb{R}} x(\log|x|-1) r_\gamma(x;\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{\theta},\boldsymbol{\delta})dx & \text{if } \gamma = 1, \\ \mu + \Gamma(1-\gamma)\int_{\mathbb{R}} x\, r_\gamma(x;\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{\theta},\boldsymbol{\delta})dx & \text{if } \gamma \in (1,2), \end{cases} \tag{104}$$

and $\min\{\alpha_+,\alpha_-\} > \gamma/2$. *Furthermore, in such cases there exists a density Lévy process $Z=(Z_t)_{t\geq 0}$ such that for all $t>0$*

$$\left.\frac{dP'}{dP}\right|_{\mathcal{F}_t} = e^{Z_t}. \tag{105}$$

Proof. Indicating q the GRGTS tempering function in Cartesian coordinates, from [51, Theorem 4.1], we have that a necessary and sufficient condition for the absolute continuity of a TS_γ Lévy process with respect to the given stable one is

$$\int_{\{|x|<1\}} (1-q(x))^2 x^{-\gamma-1} dx < \infty. \tag{106}$$

That is, for $x>0$

$$\int_0^1 \left(1 - e^{-\theta_+ x} E_{\alpha_+}(-\lambda_+ x^{\alpha_+})\right)^2 dx < \infty. \tag{107}$$

We have, when $x\sim 0$ and using (27),

$$e^{-\theta_+ x} \sim 1-\theta_+ x, \qquad E_{\alpha_+}(-\lambda_+ x^{\alpha_+}) \sim 1 - \frac{\lambda_+ x^{\alpha_+}}{\Gamma(\alpha_+ + 1)}, \tag{108}$$

and hence

$$1 - e^{-\theta_+ x} E_{\alpha_+}(-\lambda_+ x^{\alpha_+}) \sim \theta_+ x + \frac{\lambda_+ x^{\alpha_+}}{\Gamma(\alpha_+ + 1)}, \qquad x \sim 0. \tag{109}$$

Since $\alpha_+ < 1$ this implies the following leading order for $x\sim 0$,

$$\left(1 - e^{-\theta_+ x} E_{\alpha_+}(-\lambda_+ x^{\alpha_+})\right)^2 \sim \frac{\lambda_+^2 x^{2\alpha_+}}{\Gamma(\alpha_+ + 1)^2}. \tag{110}$$

Comparing with (106) we see that the condition for convergence is $\alpha_+ > \gamma/2$. The same calculation on $-1<x<0$ in (106) establishes the claim.

When comparing among them two GRGTS Lévy processes, conditions of absolute continuity are somewhat analogous to the TS case (see e.g. [12, Example 9.1]), but additional constraints on the Mittag-Leffler parameter are present. Below, spectral measures do not play a role, so we allow asymmetric stability indices.

Proposition 6. *Let $X=(X_t)_{t\geq 0}$ be a* $\mathrm{GRGTS}_{\boldsymbol{\gamma}}(m_{\boldsymbol{\gamma}}(x;\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{\theta},\boldsymbol{\delta}),\mu)$ *Lévy process. Assume that there exists a second probability measure P' under which X is a*

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GRGTS$_{\boldsymbol{\gamma}'}(m_{\boldsymbol{\gamma}'}(x; \boldsymbol{\alpha}', \boldsymbol{\lambda}', \boldsymbol{\theta}', \boldsymbol{\delta}'), \mu')$ *Lévy process. Then* $P \sim P'$ *if and only if* $\boldsymbol{\gamma} = \boldsymbol{\gamma}'$, $\boldsymbol{\delta} = \boldsymbol{\delta}'$, $\min\{\alpha_\pm, \alpha'_\pm\} > \gamma_\pm/2$, *and* $\mu - \mu' = \int_{\{|x|<1\}} (m_\gamma(dx) - m_{\gamma'}(dx))$. *Furthermore,*

$$\left.\frac{dP'}{dP}\right|_{\mathcal{F}_t} = e^Z \tag{111}$$

where $Z = (Z_t)_{t\geq 0}$ *is the Lévy process with triplet* $(\mu_Z, 0, m_Z(x)dx)$ *given by*

$$\mu_Z = -\int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbb{1}_{\{|x|<1\}}\right) m_Z(x)dx, \tag{112}$$

$$m_Z(x) = m_{\boldsymbol{\gamma}}(l^{-1}(x); \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}), \tag{113}$$

where

$$l(x) = (\theta_+ - \theta'_+)x + \log\left(\frac{E_{\alpha'_+}(-\lambda'_+ x^{\alpha'_+})}{E_{\alpha_+}(-\lambda_+ x^{\alpha_+})}\right)\mathbb{1}_{\{x>0\}} + (\theta'_- - \theta_-)x$$
$$+ \log\left(\frac{E_{\alpha'_-}(-\lambda'_- |x|^{\alpha'_-})}{E_{\alpha_-}(-\lambda_- |x|^{\alpha_-})}\right)\mathbb{1}_{\{x<0\}}. \tag{114}$$

Proof. According to [53, Theorems 33.1, 33.2], $P \sim P'$ if and only if the given relation between μ and μ' holds, and the Hellinger distance between the absolutely continuous Lévy measures $m(x)dx$ and $m'(x)dx$ is finite, that is

$$\int_{\mathbb{R}} \left(\sqrt{m(x)} - \sqrt{m'(x)}\right)^2 dx < \infty. \tag{115}$$

Furthermore, $l(x) = \log(m'(x)/m(x))$, and then (114) is clear by (39), once the first assertion is proved.

Now letting $I(x) = \left(\sqrt{m(x)} - \sqrt{m'(x)}\right)^2$ and using (26), for large x we have

$$I(x) \sim \left(\sqrt{\frac{\delta_+}{\lambda_+\Gamma(1-\alpha_+)x^{\gamma_+ + \alpha_+ + 1}}} e^{-x\theta_+/2} - \sqrt{\frac{\delta'_+}{\lambda'_+\Gamma(1-\alpha'_+)x^{\gamma'_+ + \alpha'_+ + 1}}} e^{-x\theta'_+/2}\right)^2, \tag{116}$$

which is always integrable at $+\infty$, whatever the value of θ_+ by (26). The corresponding convergence holds for large negative x.

In a right neighborhood of 0 we have the condition

$$\int_0^1 \left(\sqrt{\delta_+ \frac{E_{\alpha_+}(-\lambda_+ x^{\alpha_+})}{x^{1+\gamma_+}}} e^{-x\theta_+/2} - \sqrt{\delta'_+ \frac{E_{\alpha'_+}(-\lambda'_+ x^{\alpha'_+})}{x^{1+\gamma'_+}}} e^{-x\theta'_+/2}\right)^2 dx < \infty. \tag{117}$$

We write

$$I(x) = \delta'_+ \frac{e^{-\theta'_+ x} E_{\alpha'_+}(-\lambda'_+ x^{\alpha'_+})}{x^{\alpha'_+ + 1}} \left(x^{\frac{\gamma'_+ - \gamma_+}{2}} \sqrt{\frac{\delta_+ E_{\alpha_+}(-\lambda_+ x^{\alpha_+})}{\delta'_+ E_{\alpha'_+}(-\lambda'_+ x^{\alpha'_+})}} e^{\frac{-\theta_+ - \theta'_+}{2}x} - 1\right)^2, \quad x > 0. \tag{118}$$

Assuming $\gamma'_+ < \gamma_+$ and $\delta_+ \neq \delta_-$, since $\exp(\cdot) \sim E_\alpha(\cdot) \sim 1$ as $x \to 0^+$, we have that $I(x) \sim \delta_+ x^{-1-\alpha'_+-\gamma_++\gamma'_+}$, which diverges. The case $\gamma_+ > \gamma'_+$ is similarly excluded. Therefore, for convergence we must have $\gamma_+ = \gamma'_+$. Once this is further assumed, if we allow $\delta_+ \neq \delta'_+$ then $I(x) \sim (\sqrt{\delta_+} - \sqrt{\delta'_+})^2 x^{-1-\alpha'_+}$, again a divergent integrand. Therefore, for convergence it is necessary that both $\gamma_+ = \gamma'_+$ and $\delta_+ = \delta'_+$. Expanding the right-hand term in (118) in its McLaurin series using (27) we obtain, with $\alpha_* = \min\{\alpha_+, \alpha'_+\}$,

$$
\begin{aligned}
I(x) &\sim \frac{\delta_+}{x^{\gamma_++1}} \left(\left(\frac{1 - \frac{\lambda_+ x^{\alpha_+}}{\Gamma(1+\alpha_+)}}{1 - \frac{\lambda'_+ x^{\alpha'_+}}{\Gamma(1+\alpha'_+)}} \right)^{1/2} - 1 \right)^2 \\
&\sim \frac{\delta_+}{x^{\gamma_++1}} \left(\left(1 - \frac{\lambda_+ x^{\alpha_+}}{2\Gamma(1+\alpha_+)}\right) \left(1 + \frac{\lambda'_+ x^{\alpha'_+}}{2\Gamma(1+\alpha'_+)}\right) - 1 \right)^2 \\
&\sim \frac{\delta_+}{x^{\gamma_++1}} \left(c x^{\alpha_*}\right)^2 = \frac{c^2 \delta_+}{x^{\gamma_++1-2\alpha_*}}
\end{aligned}
\tag{119}
$$

for some $c \neq 0$. The convergence condition is thus $\gamma_+ + 1 - 2\alpha_* < 1$, i.e. $\alpha_* > \gamma_+/2$. Repeating in a left neighborhood of 0 proves the result.

We observe that the condition of equivalence of Proposition 5 is embedded in Proposition 6. Therefore, in order to have mutual equivalence between two GRGTS_γ distributions under different measures, it is necessary that both are equivalent to the same S_γ law.

6. Probability density analysis

An analytic theory of the probability densities of self-decomposable distributions is available. Continuity and differentiability properties are studied in [54, 65, 67, 68], who also discuss unimodality. Tail properties and the relation between Lévy and probability densities are analyzed in [63, 64], among others. Such theory has been already applied in [36] to describe some properties of the CTS probability densities, and it can be successfully exploited in our more general GRGTS context.

As anticipated in Section 2, we begin by observing that there is at least one instance in which the the p.d.f. of a GRGTS distribution is known, i.e. the TPL distribution. For such r.v.s the density f_0 is given in [1], and in our parametrization it reads

$$
f_0(x) = (\lambda + \theta^\alpha)^{\delta/\alpha} e^{-\theta x} x^{\delta-1} E^{\delta/\alpha}_{\alpha,\delta}\left(-\lambda x^\alpha\right), \qquad \lambda, \delta > 0, \ \alpha \in (0, 1), \ x > 0. \tag{120}
$$

Using the classification in [54], we can deduce the following interesting facts on the regularity of p.d.f.s of the GRGTS distribution class.

Proposition 7. *Let X be a* $\text{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$ *r.v. and denote its p.d.f. by f_X. The following hold true:*

(i) *if $\gamma_+\delta_+ > 0$ or $\gamma_-\delta_- > 0$, then $f_X \in C^\infty(\mathbb{R})$;*

(ii) *if $\gamma_+\delta_+ = 0$ and $\gamma_-\delta_- = 0$, then for $n \in \mathbb{N}$, if $n < \delta_+ + \delta_- \leq n+1$ both $f_X \in C^{n-1}(\mathbb{R})$ and $f_X \in C^n(\mathbb{R} \setminus \{\mu_*\})$. Furthermore, f_X is unbounded around $\mu_* = \mu - \mu_0 \in \mathbb{R}$ and continuous on $\mathbb{R} \setminus \{\mu_*\}$ if and only if $\delta_+ + \delta_- \leq 1$;*

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(iii) *f_X is strictly unimodal, i.e. there exists a point x_m such that f_X is increasing on $(-\infty, x_m) \cap \operatorname{supp} f_X$ and decreasing on $(x_m, \infty) \cap \operatorname{supp} f_X$.*

Proof. According to the theory in [54, 65, 68], the regularity/boundedness of the density depends on the values of the integral part of the critical parameter $\lambda = k(0+) + |k(0-)|$, whose value coincides with the maximum differentiability order plus one of the p.d.f. on its support. Recall that k is given by (40).

First, when $\gamma_+\delta_+ > 0$ we have $k(0+) = \infty$ and hence by [54, Theorem 1.2], the smoothness of the density follows. Similarly, if $\delta_-\gamma_- > 0$, noticing $k(0-) = \infty$. This proves *(i)*.

For *(ii)*, observe first that in such case $\mu_0 < \infty$. Assume $\gamma_+ = \gamma_- = 0$ and $\delta_+, \delta_- > 0$; then $k(0+) = \delta_+$, $|k(0-)| = \delta_-$, so that $\lambda = \delta_+ + \delta_-$ and the first claim follows again by [54, Theorem 1.2]. The case in which either δ_+ or δ_- is 0 is analogous. For the second statement, observe that in [54] classification X is now of type III. By [54, Theorem 1.5] then f_X is unbounded around μ_* if and only if the law of X is of subtype III_1 or III_2, which happens if and only if $\delta_+ + \delta_- = \lambda \leq 1$. This shows (*ii*).

To show strict unimodality, denote $k^-(x) = k(-x)$ the canonical density of $-X$. According to [54, Theorem 1.4], for f_X to fail to be strictly unimodal it is necessary and sufficent that both

$$k(0+) + |k(0-)| = k^-(0-) + |k^-(0+)| = \delta_+ + \delta_- = 1 \tag{121}$$

and that for some $\xi, \xi_- > 0$

$$k(\xi) = 1, \qquad k^-(\xi_-) = 1. \tag{122}$$

But in case (i) we know $k(0+) + |k(0-)| = \lambda = \infty$ so that (121) is not satisfied. In cases *(ii)* assume (121) so that both $k(0+), k^-(0+) \leq 1$. Since k and k^- from (40) are strictly decreasing on the positive half line, no strictly positive roots as in (122) can exist. Therefore (121) and (122) cannot hold simultaneously, and *(iii)* follows.

Above, μ_* is a location parameter, which is zero in the case of a driftless Lévy triplet for the GRGTS law when no truncation function is used. By [53, Theorem 24.10], μ^* also coincides with the lower/upper bound of the density support for spectrally positive/negative GRGTS distributions. We have the following result for the left-tail asymptotic behavior of the probability density of a GRGTS^+ law around μ^*.

Proposition 8. *Let X be a $\mathrm{GRGTS}^+_\gamma(\alpha, \lambda, \theta, \delta; \mu)$ r.v. with $\gamma \in [0, 1)$. As $x \downarrow \mu_*$, we have the following leading orders:*

(i) *if $X \in \mathrm{RGTS}^+_\alpha(\lambda, \theta, \delta; \mu)$ then*

$$f_X(x) \sim (\lambda + \theta^\alpha)^{\delta/\alpha} (x - \mu_*)^{\delta - 1}; \tag{123}$$

(ii) *otherwise*

$$\log f_X(x) \sim \delta \frac{\gamma - 1}{\gamma} \Gamma(1 - \gamma)^{\frac{1}{1-\gamma}} (x - \mu_*)^{-\frac{\gamma}{1-\gamma}}. \tag{124}$$

Proof. Part (i) follows by inspection of equation (120), after the location shift $x \mapsto x - \mu_*$, since $E^c_{a,b}(0) = \exp(0) = 1$. Moreover, observe as $x \sim 0$, $k(x) \sim cx^{-\gamma}$, $c > 0$, which is integrable around zero since $\gamma < 1$. This implies that f_X is of type I_6 in the classification in [54], and thus part (ii) follows from an application of [54, Theorem 5.2].

A similar result holds when $X \in \mathrm{GRGTS}^-_\gamma(\alpha, \lambda, \theta, \delta; \mu)$, $\gamma \in [0, 1)$. Proposition 8 is consistent with Proposition 7; in particular, if $\gamma = 0$ the left tail is a power law.

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Next, we study the asymptotics of the tails of the GRGTS p.d.f. at infinity. It is known that the tails of probability densities of an i.d. law correspond to those of the Lévy in a weak sense: for example, finiteness of cumulants is equivalent to finiteness of the truncated Lévy moments. From the results in [63], one obtains the identity of the tail leading orders if the so-called property of convolution equivalence can be shown to hold.

Convolution equivalence can be succinctly defined as follows. For a real measure η we indicate by $\overline{\eta}(x) := \eta((x, \infty))$ its tail function and by $\hat{\eta}(s)$ its moment generating function at $s \in D \subseteq \mathbb{R}$. A r.v. X with law η belongs to the class $L(\beta)$, $\beta \geq 0$, if for all $a, x \in \mathbb{R}$ we have

$$\overline{\eta}(x) > 0, \qquad \bar{\eta}(x+a) \sim e^{-a\beta}\overline{\eta}(x), \tag{125}$$

as $x \to +\infty$. A distribution η is said to be convolution equivalent of order $\beta \geq 0$ if $\hat{\eta}(\beta) < \infty$, $\eta \in L(\beta)$ and

$$\lim_{x\to\infty} \frac{\overline{\eta * \eta}(x)}{\overline{\eta}(x)} = 2\hat{\eta}(\beta), \tag{126}$$

with $*$ indicating the convolution of measures. The convolution equivalence of general TS_γ distributions has been shown in [60], which in particular is shown to be of order θ_+ for $\mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$, distributions with $\gamma > 0$. The proof can be extended with little effort also to $\mathrm{RGTS}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$ distributions as follows.

Proposition 9. *The* $\mathrm{RGTS}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$ *distribution class is convolution equivalent of order* θ_+.

Proof. We drop the parameter subscripts throughout. Let m be the Lévy density of some given $\mathrm{RGTS}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$ law and consider the normalization

$$m_1(dx) := \frac{m(x)}{\overline{m}(1)} \mathbb{1}_{\{x>1\}} dx = \frac{\delta}{\overline{m}(1)x} e^{-\theta x} E_\alpha(-\lambda x^\alpha) \mathbb{1}_{\{x>1\}} dx, \tag{127}$$

which is a probability law. According to [63, Theorem B, (1)–(2)], the stated convolution equivalence holds if and only if so does convolution equivalence of m_1 of same order. Using de l'Hopital's rule, the fundamental theorem of calculus, and (26), for all $a \in \mathbb{R}$ it holds that

$$\begin{aligned}\lim_{x\to\infty} \frac{\overline{m_1}(x+a)}{\overline{m_1}(x)} &= \lim_{x\to\infty} \frac{\displaystyle\int_{x+a}^{\infty} \frac{e^{-y\theta}}{y} E_\alpha(-\lambda y^\alpha) dy}{\displaystyle\int_{x}^{\infty} \frac{e^{-y\theta}}{y} E_\alpha(-\lambda y^\alpha) dy} = \frac{e^{-(x+a)\theta} x}{(x+a)} \frac{E_\alpha(-\lambda(x+a)^\alpha)}{e^{-\theta x} E_\alpha(-\lambda x^\alpha)} \\ &= e^{-\theta a} \lim_{x\to\infty} \left(\frac{x}{x+a}\right) \frac{E_\alpha(-\lambda(x+a)^\alpha)}{E_\alpha(-\lambda x^\alpha)} = e^{-\theta a} \lim_{x\to\infty} \left(\frac{x+a}{x}\right)^{\alpha-1} = e^{-\theta a}\end{aligned} \tag{128}$$

verifying (125) with $\beta = \theta$. If η is the law of X, and $\theta > 0$, that $\hat{\eta}(\theta) = e^{\psi(-i\theta)} < \infty$, with ψ as in Theorem 1, *(ii)* or *(iii)* can be verified directly, and if $\theta = 0$ there is nothing to prove.

To show (126), proceeding as in [60, Equations (2.11)–(2.13)], we arrive at the equality

$$\lim_{x\to\infty} \frac{\overline{m_1 * m_1}(x)}{\overline{m_1}(x)} = \lim_{x\to\infty} \frac{\delta}{\overline{m}(1)} \int_1^{x/2} \left(\frac{x}{(x-z)z}\right) \frac{E_\alpha(-\lambda(x-z)^\alpha) E_\alpha(-\lambda z^\alpha)}{E_\alpha(-\lambda x^\alpha)} dz. \tag{129}$$

Again, referring to [60, Theorem 1], convolution equivalence of m_1 of order θ—and hence, by [63, Theorem B, (1)–(2)], that of order θ of the given RGTS r.v.—would follow by a dominated convergence argument if the last integrand is dominated by an integrable function of z for any

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parameter choice. Now, for $z \in (1, x/2)$ we have $x/(x-z) \leq 2$. Let $k(y) = E_\alpha(-\lambda y^\alpha)$. Since k is c.m., as a consequence of [60, Proposition 1(*ii*)], for all fixed $z \geq 1$ the ratio $k(x-z)/k(x)$ is decreasing in x for all $x > z$ so that $k(x-z)/k(x)\mathbb{1}_{\{1\leq z\leq x/2\}}$ has a maximum in $x = 2z$. Combining with the usual asymptotic estimate (26) this results in

$$\left(\frac{x}{(x-z)z}\right)\frac{E_\alpha(-\lambda(x-z)^\alpha)E_\alpha(-\lambda z^\alpha)}{E_\alpha(-\lambda x^\alpha)}\mathbb{1}_{\{1\leq z\leq x/2\}} < \frac{2}{z}\frac{E_\alpha(-\lambda z^\alpha)^2}{E_\alpha(-\lambda(2z)^\alpha)} \sim \frac{2^{1-\alpha}}{\lambda\Gamma(1-\alpha)}z^{-1-\alpha} \tag{130}$$

as $z \to \infty$. Hence the integrand in (129) is bounded by an integrable function on $[1, \infty)$ for all values of λ and α.

Combining convolution equivalence with density monotonicity, the theory of regular variation can be used to provide the explicit tail asymptotics of GRGTS p.d.f.s.

Proposition 10. *Let X be a* $\mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$ *r.v. and denote by f_X its p.d.f. Then as $x \to \pm\infty$,*

$$f_X(x) \sim \frac{\delta_\pm}{\lambda_\pm\Gamma(1-\alpha_\pm)}\frac{e^{-\theta_\pm x+\psi(-i\theta_\pm)}}{x^{1+\gamma_\pm+\alpha_\pm}}, \tag{131}$$

where ψ is the characteristic exponent given in Theorem 1.

Proof. We assume $x \to \infty$ first and drop the subscript $+$ for notational convenience. From [60, Theorem 1] and Proposition 9 above, we know that the law of X is convolution equivalent of order θ for all $\gamma \in [0, 2)$. Denote by $\overline{F_X}$ the survival function of X and by f_X and m respectively its p.d.f. and Lévy measure. Set X_0 be the $\mathrm{GRGS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\delta}; \mu)$ r.v. with Lévy density $m_0(x) = e^{\theta x}m(x)$ (possibly, $\theta = 0$, and $m_0 = m$), and corresponding survival function $\overline{F_{0,X}}$ having tilted p.d.f. $f_{0,X}(x) = e^{\theta x-\psi(-i\theta)}f_X(x)$, with ψ as in Theorem 1 if $\theta > 0$, replaced by 0, if $\theta = 0$. It is shown in [63, Theorem B, (3)] that convolution equivalence ensures that

$$\overline{F_{0,X}}(x) \sim \overline{m_0}(x)e^{\psi(-i\theta)} = \overline{m_0}(x), \qquad x \to +\infty, \tag{132}$$

where ψ is that of Theorem 2 (but in view of the second equality, this information is irrelevant). Then for all $c > 0$, using (26) we obtain

$$\lim_{x\to\infty}\frac{m_0(cx)}{m_0(x)} = c^{-1-\alpha-\gamma}, \tag{133}$$

that is, m_0 is regularly varying of order $-1-\alpha-\gamma$. Therefore,

$$m_0(x) = x^{-1-\alpha-\gamma}\ell(x) \tag{134}$$

for some slowly varying function ℓ. By [15, VIII.9, Theorem 1(a)], applied with $p = 0$, and (134) we see that

$$\overline{m_0}(x) \sim \frac{x\, m_0(x)}{\gamma+\alpha} = \frac{x^{-\alpha-\gamma}\ell(x)}{\gamma+\alpha}, \qquad x \to +\infty, \tag{135}$$

which replaced in (132) leads to

$$\overline{F_{0,X}}(x) \sim \frac{x^{-\alpha-\gamma}\ell(x)}{\gamma+\alpha}, \qquad x \to +\infty. \tag{136}$$

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Now by the monotone density theorem (MDT) in its version for the tail function (see [5, Theorems 1.7.2, 1.7.2b], and the related discussion of the alternative versions), we have that if $U(x) = \int_x^\infty u(y)dy$ with u ultimately monotone, and $U(x) \sim cx^\rho \ell(x)$, $x \to \infty$, $\ell(x)$ slowly varying, $\rho < 0$, then $u(x) \sim -c\rho x^{\rho-1}\ell(x)$, $x \to \infty$. In the present case, based on (136), and because by Proposition 7*(iii)*, $f_{0,X}$ is ultimately monotone, we can apply the MDT with $U = \overline{F_{0,X}}$, $u = f_{0,X}$, $c = (\gamma + \alpha)^{-1}$, $\rho = -\alpha - \gamma$. Hence

$$f_{0,X}(x) \sim x^{-1-\alpha-\gamma}\ell(x), \qquad x \to \infty. \tag{137}$$

Again, in view of (26) and (39) one has

$$\ell(x) \sim \frac{\delta}{\Gamma(1-\alpha)}, \qquad x \to \infty, \tag{138}$$

which used in (137) proves (131) for $f_{0,X}$.

Letting then $\theta > 0$, recalling the definition of $f_{0,X}$, by what just proved it also follows

$$f_X(x) = f_{0,X}(x)e^{-\theta x + \psi(-i\theta)} \sim \frac{\delta}{\Gamma(1-\alpha)} x^{-1-\alpha-\gamma} e^{-\theta x + \psi(-i\theta)} \tag{139}$$

and we have shown (131) in its generality. The statement for $x \to -\infty$ follows by considering the convolution equivalent GRGTS$_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$ r.v. $-X$ of order θ_- at $x \to +\infty$.

In at least one case it is not hard to verify Proposition 10, namely, for TPL distributions.

Example 5. *Right tail of a* TPL *p.d.f. For a* TPL$(\alpha, \lambda, \theta, \delta)$ *distribution we have the p.d.f. (120) available. Using (25) we can extract the following leading order as $x \to +\infty$,*

$$f_0(x) \sim -\frac{\delta}{\alpha\lambda^{\delta/\alpha+1}} \frac{(\lambda+\theta^\alpha)^{\delta/\alpha}}{\Gamma(-\alpha)} \frac{e^{-\theta x}}{x^{\delta+\alpha}} x^{\delta-1} = \frac{\delta}{\lambda\Gamma(1-\alpha)} \exp\left(\frac{\delta}{\alpha}\log\left(\frac{\lambda+\theta^\alpha}{\lambda}\right)\right) \frac{e^{-\theta x}}{x^{1+\alpha}}, \tag{140}$$

which matches the expression (131) at $+\infty$ with ψ from (42).

7. Conclusions

Motivated by the investigation of heavy-tailed distributions with finite variance in economics and physics, we have proposed the use of positive geometric tempered stable laws to model the radial part of Lévy measure. Such suggestion naturally fits in with the theory of complete monotone tempering of stable distributions. In particular, Propositions 2 and 4 make viable the introduction of processes with Gaussian limit but heavy tails at all time lags in the form of GRGS laws, entailing (very slow) central limit theorem convergence. Also, when $\theta \neq 0$, familiar exponential/semi-heavy tails are obtained, but in principle a faster reversion to Gaussian ought to be observable (because of (27)) compared to, for example, CTS models.

To illustrate such effects, in Figure 1 we plot some comparisons between Lévy densities/tempering functions of S^s/CTS^s laws against those of a GRGTS^s law, $\theta > 0$, and its GRGS^s counterpart. Around 0 the Lévy densities are all unbounded, with S_γ showing the fastest blow-up rate. For large $|x|$, the $|x|^{-\gamma}$ tails of the S_γ density are the heaviest, followed by those of the GRGS$^s_\gamma$ density. As predicted by our results the latter are still heavy, but lightened compared

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TABLE 1. Distribution list. When appearing, the superscripts $s, +, -$ stand, respectively, for the symmetric, spectrally positive, and spectrally negative version of the corresponding distribution.

Symbol	Description	Dimension
$\mathrm{BG}(\boldsymbol{\lambda}, \boldsymbol{\delta})$	bilateral gamma	1
$\mathrm{BL}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\delta})$	bilateral Linnik	1
$\mathrm{CTS}_{\boldsymbol{\alpha}}(\boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$	classical tempered stable	1
$\mathrm{G}(\lambda, \delta)$	gamma	1
$\mathrm{GRGS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\delta}; \mu)$	generalized radially geometric stable	1
$\mathrm{GRGS}_{\gamma}(\sigma, \alpha, \lambda; \mu)$	GRGS, polar parameterization	d
$\mathrm{GRGTS}_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$	generalized radially geometric tempered stable	1
$\mathrm{GRGTS}_{\gamma}(\sigma, \alpha, \lambda, \theta; \mu)$	GRGTS, polar parameterization	d
$\mathrm{GRGTS}_{\gamma}(r_{\gamma}(x; \boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}); \mu)$	GRGTS, Rosińsky parametrization	1
$\mathrm{ML}(\alpha, \lambda)$	Mittag-Leffler	1
$\mathrm{PL}(\alpha, \lambda, \delta)$	positive Linnik	1
$\mathrm{RGS}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \boldsymbol{\delta}; \mu)$	radially geometric stable	1
$\mathrm{RGS}(\sigma, \alpha, \lambda; \mu)$	RGS, polar parameterization	d
$\mathrm{RGTS}_{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\delta}; \mu)$	radially geometric tempered stable	1
$\mathrm{RGTS}(\sigma, \alpha, \lambda, \theta; \mu)$	RGTS, polar parameterization	d
$\mathrm{S}_{\alpha}(\lambda, \beta; \mu)$	stable, classical parameterization	1
$\mathrm{S}_{\boldsymbol{\alpha}}(\boldsymbol{\delta}; \mu)$	stable, Lévy parameterization	1
$\mathrm{S}_{\alpha}(\sigma, \mu)$	stable, polar parameterization	d
$\mathrm{TPL}(\alpha, \lambda, \theta, \delta)$	tempered positive Linnik	1
$\mathrm{TS}_{\gamma}(Q; \mu)$	Rosińsky tempered stable, spectral parameterization	d
$\mathrm{TS}_{\gamma}(R; \mu)$	Rosińsky tempered stable, Rosińsky parameterization	d

to the S_{γ} ones, by a factor $|x|^{-\alpha}$ entailing finite variance in this example (since $\alpha + \gamma > 2$). The CTS^s_{γ} and $\mathrm{GRGTS}^s_{\gamma}$, $\theta \neq 0$, decays are instead both exponential, with the $\mathrm{GRGTS}^s_{\gamma}$ faster again because of the factor $|x|^{-\alpha}$. Such impact eventually becomes negligible, as can be also seen in the considered range. In the bottom panel we visualize the corresponding tempering functions. We notice a stronger tempering around 0 in the $\mathrm{GRGTS}^s_{\gamma}$, $\theta \neq 0$ and GRGS^s_{γ} laws compared with the CTS^s_{γ}, but as $|x|$ becomes larger the $\mathrm{GRGTS}^s_{\gamma}$ tails look like those of an exponential, while those of the pure Mittag-Leffler tempering remain much heavier.

In terms of possible extensions, we notice that one limitation of the radial geometric stable approach is that the tail index of the GRGS densities is always confined to be smaller than 3, which implicates diverging higher moments. This is at odds with some estimates of financial data (e.g. [19]) which instead find evidence of finite skewness. In view of (26), this could be resolved by considering radial functions comprising the three-parameter Mittag-Leffler function, which is known to be c.m. under some parameter constraints (see [21]). We leave this direction of research for further investigations.

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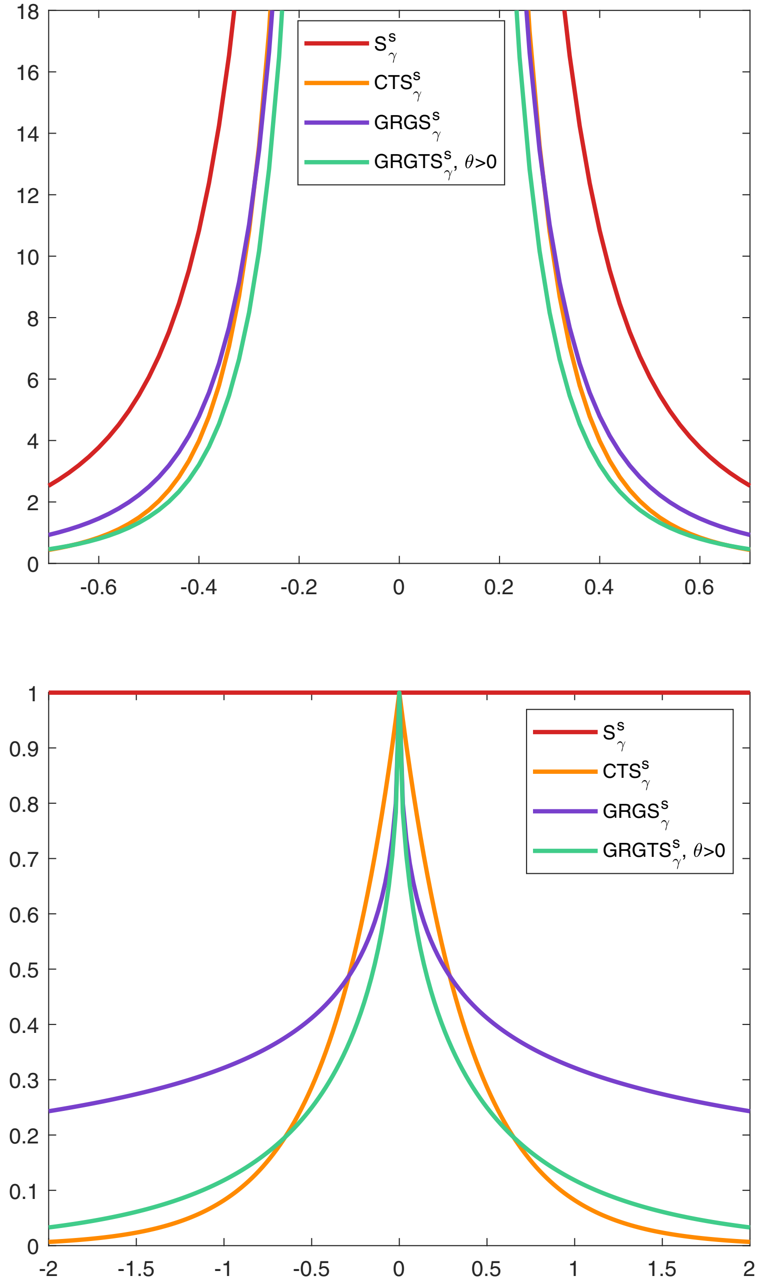

FIGURE 1. Top: S^s_γ, CTS$^s_\gamma$, GRGTS$^s_\gamma$, and GRGS$^s_\gamma$ Lévy densities. Bottom: corresponding tempering functions. The parameters are $\gamma = 1.6$, $\alpha = 0.5$, $\lambda = \theta = \delta = 1$.

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