4
Renormalization

Relativistic quantum field theories generally display infinities at sufficiently high order in a loop expansion. These infinities must first be regulated, meaning that cutoffs are applied to yield finite results that can be manipulated with some mathematical rigor. The results are then renormalized, so that the parameters of the Lagrangian and the cutoffs are eliminated in favor of physical observables such as electric charge and mass. If there are only a finite number of cutoffs as the number of loops increases, the theory is said to be renormalizable and the cutoffs can always be eliminated in favor of a finite number of observables. If the number of required cutoffs increases without bound as the number of loops increases then the theory is said to be nonrenormalizable and one must specify an infinite number of observables to define the theory. The general opinion is that a *fundamental* theory of nature should be renormalizable. This is based on the belief that there are only a finite number of independent parameters in our universe. An *effective* theory only needs to describe nature over a finite range of distances or momenta, and such a theory need not be renormalizable. In this chapter we consider the basic aspects of a renormalizable theory and its implications for finite temperatures. For definiteness we study a scalar field theory; the same principles apply to more complicated theories, such as the gauge theories to be studied in later chapters.

4.1 Renormalizing $\lambda\phi^4$ theory

Recall that the interaction contribution to the partition function is given by

$$\ln Z_{\mathrm{I}} = \ln\left(\frac{\int [d\phi]\mathrm{e}^{S}}{\int [d\phi]\mathrm{e}^{S_0}}\right) \tag{4.1}$$

https://doi.org/10.1017/9781009401968.005 Published online by Cambridge University Press

For the $\lambda\phi^4$ theory the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \lambda\phi^4 \tag{4.2}$$

We found in Chapter 3 that we needed to add a counterterm $-\frac{1}{2}\delta m^2\phi^2$, which is equivalent to saying that $m^2 = m_\mathrm{R}^2 + \delta m^2$, where m_R is the renormalized mass. The cutoff dependence of the self-energy at lowest order could be canceled by a suitable choice of δm^2.

Now we investigate what happens when we scale the field and the coupling constant. Write

$$\phi = \mathcal{Z}_3^{1/2}\phi_\mathrm{R} \tag{4.3}$$

Notice that we can integrate with $[d\phi_\mathrm{R}]$ since $\mathcal{Z}_3$ cancels between the numerator and denominator in (4.1). We also write

$$\lambda = \mathcal{Z}_1\mathcal{Z}_3^{-2}\lambda_\mathrm{R} \tag{4.4}$$

The scaling factors $\mathcal{Z}_1$ and $\mathcal{Z}_3$ are known as the coupling constant and the wavefunction renormalization, respectively. Usually in the literature the symbol Z instead of $\mathcal{Z}$ is used for these, but here we do not want to confuse them with the partition function.

The Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\left[\partial_\mu\phi_\mathrm{R}\partial^\mu\phi_\mathrm{R} - \left(m_\mathrm{R}^2 + \delta m^2\right)\phi_\mathrm{R}^2\right]\mathcal{Z}_3 - \lambda_\mathrm{R}\phi_\mathrm{R}^4\mathcal{Z}_1 \\ &= \mathcal{L}_\mathrm{R} + \tfrac{1}{2}\left[\partial_\mu\phi_\mathrm{R}\partial^\mu\phi_\mathrm{R} - m_\mathrm{R}^2\phi_\mathrm{R}^2\right](\mathcal{Z}_3 - 1) \\ &\quad - \tfrac{1}{2}\mathcal{Z}_3\delta m^2\phi_\mathrm{R}^2 - \lambda_\mathrm{R}\phi_\mathrm{R}^4\left(\mathcal{Z}_1 - 1\right)\end{aligned} \tag{4.5}$$

where

$$\mathcal{L}_\mathrm{R} = \tfrac{1}{2}\partial_\mu\phi_\mathrm{R}\partial^\mu\phi_\mathrm{R} - \tfrac{1}{2}m_\mathrm{R}^2\phi_\mathrm{R}^2 - \lambda_\mathrm{R}\phi_\mathrm{R}^4 \tag{4.6}$$

The Lagrangian is thus expressed as a function of the renormalized field and of the renormalized mass and coupling constant. The latter two have numerical values that must be determined by experiment. All cutoff dependence resides in the unobservable quantities $\mathcal{Z}_1$, $\mathcal{Z}_3$, and δm^2. In a perturbative renormalization scheme they should have power series expansions

$$\begin{aligned}\mathcal{Z}_1 &= 1 + \sum_{n=1}^{\infty} a_n\lambda_\mathrm{R}^n \\ \mathcal{Z}_3 &= 1 + \sum_{n=1}^{\infty} b_n\lambda_\mathrm{R}^n \\ \delta m^2 &= \sum_{n=1}^{\infty} c_n\lambda_\mathrm{R}^n\end{aligned} \tag{4.7}$$

The coefficients a_n, b_n, c_n will depend in general upon the ultraviolet cutoff Λ_c.

All renormalizable field theories can be dealt with in the manner sketched above. The reader is referred to the excellent texts on relativistic quantum field theory listed in the bibliography at the end of the chapter for a full discussion of the renormalization program.

We remark again that whatever regularization and renormalization is necessary and sufficient at zero temperature and chemical potential is also necessary and sufficient at finite temperature and chemical potential. (Recall the discussion in Section 3.4.)

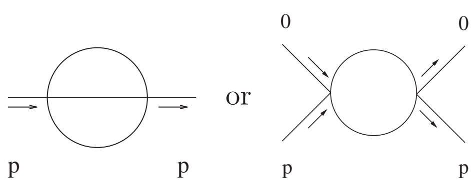
4.2 Renormalization group

For the moment consider the $\lambda\phi^4$ theory at $T = 0$ and with $m_R = 0$. Generalization to $m_R > 0$ and other theories is straightforward. The finite-temperature effects are studied in Section 4.4.

Let $\Gamma^{(n)}$ be a 1PI Green's function of n powers of the field ϕ. The statement that the theory is renormalizable means that

$$\mathcal{Z}_3^{n/2}\left(\lambda, \frac{\Lambda_c}{M}\right)\Gamma^{(n)}(p, \lambda, \Lambda_c) = \Gamma_R^{(n)}(p, \lambda_R, M) \tag{4.8}$$

The unrenormalized Green's function depends on the unrenormalized coupling and on the cutoff Λ_c. The symbol p can represent one momentum or a set of momenta $(p_1, p_2, \ldots)$. Since $\mathcal{Z}_3$ is dimensionless it can only depend on λ and on Λ_c/M. What is M? Green's functions are typically infinite, so we must specify their value at some particular point, for example, $p^2 = M^2$, using one or other of the following diagrams:



We could require $\Gamma_R^{(n)}$ to have its free-field value at $p^2 = M^2$, that is,

$$\Gamma_R^{(n)}(p^2 = M^2, \lambda_R, M) = \Gamma_R^{(n)}(p^2 = M^2, 0, M) \tag{4.9}$$

as is frequently done, but the choice is arbitrary. Physical results should be independent of the renormalization scheme, in particular, independent of the choice of M.

The requirement of renormalizability has consequences. To see them, take the total derivative of the left- and right-hand sides of (4.8) with

https://doi.org/10.1017/9781009401968.005 Published online by Cambridge University Press

respect to M, keeping λ and Λ_c fixed:

$$
\begin{aligned}
M\frac{d}{dM}\left(\mathcal{Z}_3^{n/2}\Gamma^{(n)}\right) &= M\left(\frac{\partial \mathcal{Z}_3^{n/2}}{\partial M}\right)_{\lambda,\Lambda_c}\Gamma^{(n)} \\
M\frac{d}{dM}\Gamma_{\mathrm{R}}^{(n)} &= M\left(\frac{\partial \Gamma_{\mathrm{R}}^{(n)}}{\partial M}\right)_{\lambda_{\mathrm{R}}} + \left(\frac{\partial \Gamma_{\mathrm{R}}^{(n)}}{\partial \lambda_{\mathrm{R}}}\right) M\left(\frac{\partial \lambda_{\mathrm{R}}}{\partial M}\right)_{\lambda,\Lambda_c}
\end{aligned}
\tag{4.10}
$$

Now for sake of convenience of notation define

$$
\begin{aligned}
\gamma_{(n)} &= -\mathcal{Z}_3^{-n/2} M\left(\frac{\partial \mathcal{Z}_3^{n/2}}{\partial M}\right)_{\lambda,\Lambda_c} \\
&= \mathcal{Z}_3^{-n/2}\Lambda_c\left(\frac{\partial \mathcal{Z}_3^{n/2}}{\partial \Lambda_c}\right)_{\lambda,M} \\
&= \frac{1}{2}n\Lambda_c\mathcal{Z}_3^{-1}\left(\frac{\partial \mathcal{Z}_3}{\partial \Lambda_c}\right)_{\lambda,M} \\
&= n\gamma_{(1)}
\end{aligned}
\tag{4.11}
$$

and

$$
\beta_\lambda = M\left(\frac{\partial \lambda_{\mathrm{R}}}{\partial M}\right)_{\lambda,\Lambda_c} = -\Lambda_c\left(\frac{\partial \lambda_{\mathrm{R}}}{\partial \Lambda_c}\right)_{\lambda,M} \tag{4.12}
$$

in the conventional notation. The quantity β_λ must not be confused with the inverse temperature. Putting these all together, we arrive at the renormalization-group equation

$$
\left(M\frac{\partial}{\partial M} + \beta_\lambda\frac{\partial}{\partial \lambda_{\mathrm{R}}} + \gamma_{(n)}\right)\Gamma_{\mathrm{R}}^{(n)} = 0 \tag{4.13}
$$

All the $\Gamma_{\mathrm{R}}^{(n)}$ must satisfy this equation on account of renormalizability. It expresses the invariance of physical observables under changes in M, the renormalization scale.

The renormalized 1PI Green's function has the general functional form

$$
\Gamma_{\mathrm{R}}^{(n)} = p^D z\left(\frac{p}{M}, \lambda_{\mathrm{R}}\right)
$$

where D is the dimension of $\Gamma^{(n)}$ and z is a dimensionless function of the two displayed dimensionless variables. After substitution into (4.13), factoring out p^D, and then defining $x = M/p$, $y = \lambda_{\mathrm{R}}$, we obtain the linear, homogeneous, first-order partial differential equation

$$
\left(x\frac{\partial}{\partial x} + \beta_\lambda(y)\frac{\partial}{\partial y} + \gamma_{(n)}(y)\right) z(x,y) = 0 \tag{4.14}
$$

https://doi.org/10.1017/9781009401968.005 Published online by Cambridge University Press

This equation can be solved by the method of characteristics. The solution is

$$z = f(u(x,y)) \exp\left(\int_x^{x_0} \gamma_{(n)}(x') \frac{dx'}{x'}\right) \tag{4.15}$$

Here x_0 is a reference point, f is an arbitrary function, and $u(x,y) = c$ represents the relationship between x and y when they satisfy the differential equation

$$x\frac{dy}{dx} = \beta_\lambda(y) \tag{4.16}$$

The solution to this equation involves one constant of integration, corresponding to c. What is meant by $\gamma_{(n)}(x)$ is $\gamma_{(n)}(y(x))$ where $y(x)$ is determined from $u(x,y) = c$. Translating this back into the original notation we have the solution to (4.13) as

$$\Gamma_{\mathrm{R}}^{(n)} = G\left(p, \bar{\lambda}\left(\frac{M'}{M}\right)\right) \exp\left(\int_{M/p}^{M'/p} \gamma_{(n)}(x) \frac{dx}{x}\right) \tag{4.17}$$

The function G is arbitrary and undetermined by the renormalization-group equation. The renormalization-group running coupling $\bar{\lambda}$ satisfies the differential equation

$$\chi\frac{d\bar{\lambda}}{d\chi} = \beta_\lambda(\bar{\lambda}) \tag{4.18}$$

where $\chi = M'/M$, subject to the condition

$$\bar{\lambda}(\chi = 1) = \lambda_{\mathrm{R}} \tag{4.19}$$

The exponential in (4.17) is referred to as the anomalous dimension of $\Gamma_{\mathrm{R}}^{(n)}$.

To the lowest nontrivial order, β_λ is computed to be (Exercise 4.1)

$$\beta_\lambda(\bar{\lambda}) = \frac{9}{2\pi^2}\bar{\lambda}^2 \tag{4.20}$$

The differential equation to be solved is

$$\chi\frac{d\bar{\lambda}}{d\chi} = \frac{9}{2\pi^2}\bar{\lambda}^2 \tag{4.21}$$

The solution satisfying (4.19) is

$$\bar{\lambda} = \frac{\lambda_{\mathrm{R}}}{1 - (9/4\pi^2)\lambda_{\mathrm{R}} \ln \chi^2} \tag{4.22}$$

The denominator may be expanded in a power series in $\lambda_{\rm R}$:

$$\bar{\lambda} = \lambda_{\rm R} \sum_{n=0}^{\infty} \left[\frac{9\lambda_{\rm R}}{4\pi^2} \ln\left(\frac{M'^2}{M^2}\right)\right]^n \tag{4.23}$$

This expansion may be arrived at in a completely independent manner, as follows. At each order in perturbation theory compute the logarithmic contribution of the highest power. This is known as the leading-log approximation. One obtains the same result as a consequence of the renormalization group.

The renormalization-group running coupling $\bar{\lambda}$ does not depend on M and $\lambda_{\rm R}$ separately but only on a particular combination of them. In (4.22) define

$$\frac{9}{2\pi^2} \ln \Lambda \equiv \lambda_{\rm R}^{-1} + \frac{9}{2\pi^2} \ln M \tag{4.24}$$

Furthermore, let us choose $M'^2 = p^2$, the only natural scale in the problem. Then

$$\bar{\lambda} = \frac{4\pi^2}{9 \ln\left(\Lambda^2/p^2\right)} \tag{4.25}$$

The effective coupling $\bar{\lambda}$ no longer depends on the coupling $\lambda_{\rm R}$ originally appearing in the Lagrangian! This is often referred to as dimensional transmutation. There is no longer an intrinsic coupling constant, but in its place there is an intrinsic energy scale Λ (not to be confused with the cutoff $\Lambda_{\rm c}$). The effective coupling $\bar{\lambda}$ depends on the momentum p. As $p/\Lambda \to 0$, we have $\bar{\lambda} \to 0$, which is infrared freedom. The coupling effectively goes to zero at large distance so that weak coupling expansions should be quite accurate there. Since to lowest order the beta function β_λ is positive, it follows that $\bar{\lambda}$ must be larger at short distances. In fact, from (4.25), $\bar{\lambda} \to \infty$ as $p/\Lambda \to 1$. This is certainly an artifact of the lowest-order perturbation expansion of β_λ, but nevertheless it indicates that the coupling grows as the distance decreases.

4.3 Regularization schemes

We have regulated the divergences in the scalar field theory by placing an upper limit on the integration over four-momentum in Euclidean space. There are alternative regularization procedures, dimensional regularization being the most commonly used by far. Dimensional regularization is almost indispensable in gauge theories. The idea is to work in $n = 4 - \epsilon$ dimensions where integrals converge and then analytically continue to $\epsilon \to 0$.

https://doi.org/10.1017/9781009401968.005 Published online by Cambridge University Press

Consider the self-energy in scalar field theory. The one-loop expression in Minkowski space is

$$\Pi_1^{\text{vac}} = -12\lambda\kappa^{\epsilon} \int \frac{d^n l}{(2\pi)^n} \frac{i}{l^2 - m^2 + i\epsilon}$$
$$= -\frac{12\lambda\kappa^{\epsilon}}{(2\pi)^n} \frac{\pi^{n/2}\Gamma(1-n/2)}{m^{2-n}} \tag{4.26}$$

This scheme requires the introduction of a mass scale κ to compensate for the deviation from four dimensions and so ensuring that λ remains dimensionless. The Γ function has poles at the negative integers. Using

$$\Gamma(-n+\delta) = \frac{(-1)^n}{n!}\left(\frac{1}{\delta} + \psi(n+1) + \mathcal{O}(\delta)\right), \tag{4.27}$$

with

$$\psi(n+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma_{\text{E}} \tag{4.28}$$

where γ_{E} is Euler's constant, we find that

$$\Pi_1^{\text{vac}} = \frac{3\lambda m^2}{4\pi^2}\left[\frac{2}{\epsilon} + \psi(2) + \ln\left(\frac{\kappa^2}{m^2}\right) + \ln 4\pi + \mathcal{O}(\epsilon)\right] \tag{4.29}$$

This may be compared with the momentum cutoff scheme

$$\Pi_1^{\text{vac}} = \frac{3\lambda}{4\pi^2}\left[\Lambda_{\text{c}}^2 - m^2\ln\left(\frac{\Lambda_{\text{c}}^2}{m^2}\right) + \mathcal{O}\left(\frac{m^4}{\Lambda_{\text{c}}^2}\right)\right] \tag{4.30}$$

In the "minimal subtraction" scheme (MS) none of the constant terms are absorbed into the mass, only the divergent $1/\epsilon$ term. In the "modified minimal subtraction" scheme ($\overline{\text{MS}}$) the finite constant terms are absorbed too. A similar absorption is made for the renormalized coupling.

The arbitrariness in choosing the counterterms is a reflection of the whole regularization and renormalization program in quantum field theory. After expressing physical observables in terms of them, there should be no difference. However, the intrinsic scale Λ does depend on the scheme; for example, there are Λ_{MOM}, Λ_{MS}, $\Lambda_{\overline{\text{MS}}}$, and so on. Their numerical values will in general be different. This is nowhere more apparent than in QCD.

4.4 Application to the partition function

Now we investigate the implications of the renormalization group for the partition function. Let T replace p. As given in (4.1), $\ln Z_{\text{I}}$ is comparable with a Green's function that is zeroth order in the field. It has dimension exactly four and no anomalous dimension. Thus, (4.17) instructs us to replace λ_{R} with $\bar{\lambda}$. If we had an exact expression for $\ln Z$ then the choice of renormalization scale M would indeed be arbitrary. Since we only compute

https://doi.org/10.1017/9781009401968.005 Published online by Cambridge University Press

a finite number of terms in a weak-coupling expansion, we should choose M in an optimal way so as to minimize the contribution of higher-order terms. For the massless self-interacting scalar field we take $M = bT$, where b is a number of order unity, since this is the only energy scale in the problem. We then have

$$\bar{\lambda} = \frac{2\pi^2}{9\ln(\Lambda/bT)} \tag{4.31}$$

As $T/\Lambda \to 0$ the thermodynamics is well approximated by a gas of non-interacting massless bosons. As $T/\Lambda \to 1$ the system becomes strongly coupled and the weak coupling expansion is no longer a reasonable approximation. What really happens at very high temperatures is unknown.

In Chapter 3 the pressure was calculated up to order $\lambda^{3/2}$. It has been calculated up to order λ^2 by Frenkel, Saa, and Taylor [1], and to order $\lambda^{5/2}$ by Parwani and Singh [2]. Using the minimal subtraction scheme,

$$\begin{aligned} P = \frac{\pi^2}{90}T^4 \Bigg\{ 1 &- \frac{5}{24}\left(\frac{9\lambda_{\rm R}}{\pi^2}\right) + \frac{5}{18}\left(\frac{9\lambda_{\rm R}}{\pi^2}\right)^{3/2} \\ &- \frac{5}{36}\left(\frac{9\lambda_{\rm R}}{\pi^2}\right)^{2}\left[\frac{3}{4}\ln\left(\frac{2\pi T}{M}\right) + c_1\right] \\ &+ \frac{5}{36}\left(\frac{9\lambda_{\rm R}}{\pi^2}\right)^{5/2}\left[\ln\left(\frac{9\lambda_{\rm R}}{\pi^2}\right) + \frac{3}{2}\ln\left(\frac{2\pi T}{M}\right) + c_2\right]\Bigg\} \end{aligned} \tag{4.32}$$

is obtained. Here the prime has been dropped from the M in accordance with the notation in Section 4.2. The constants are given by

$$\begin{aligned} c_1 &= \frac{3}{8}\ln(4\pi) + \frac{1}{2}\frac{\zeta'(-3)}{\zeta(-3)} - \frac{\zeta'(-1)}{\zeta(-1)} + \frac{\gamma_{\rm E}}{8} + \frac{59}{60} \approx -0.606\,85 \\ c_2 &= \frac{\zeta'(-1)}{\zeta(-1)} + \frac{\gamma_{\rm E}}{4} - 2\ln 3 - \frac{5}{4} \approx -1.317\,87 \end{aligned} \tag{4.33}$$

If the scale M is held fixed then the perturbative expansion is not reliable at high temperatures on account of the logarithmic terms $\ln(2\pi T/M)$. The renormalization group tells us that we should not choose M constant but proportional to the temperature. If we choose $M = bT$ then the large temperature-dependent logarithms are of order unity. Indeed, if we choose the coefficient b just right then there is no contribution of order $\lambda_{\rm R}^2$ at all! It is compensated by corresponding contributions at higher orders in $\lambda_{\rm R}$. Equivalently, we can eliminate the logarithmic terms $\ln(2\pi T/M)$ by re-expressing the pressure in terms of the renormalization-group running coupling from (4.23),

$$\bar{\lambda} = \lambda_{\rm R}\left[1 + \frac{9\lambda_{\rm R}}{2\pi^2}\ln\left(\frac{bT}{M}\right)\right] + \mathcal{O}(\lambda_{\rm R}^3) \tag{4.34}$$

The result, of course, is the same.

https://doi.org/10.1017/9781009401968.005 Published online by Cambridge University Press

4.5 Exercises

4.1 Derive (4.20) from the definition (4.12). *Hint:* The renormalized coupling λ_{R} can be determined from the expression $-(1/3!)(\delta^2 \ln Z_1/\delta\mathcal{D}_0^2)_{1\mathrm{PI}}$. Use (3.11) and (3.15) to obtain a diagrammatic expansion for λ_{R}. You will find that the order-λ^2 correction is given by a single one-loop diagram. Note that you only need the cutoff (Λ_{c}) dependence to determine β_λ.

4.2 Verify the claim surrounding (4.34).

4.3 Make a plot illustrating the convergence of the expansion of the pressure in (4.32) using $M = T$. Repeat the exercise with $M = \pi T$, $2\pi T$, and $2\pi T \mathrm{e}^{4c_1/3}$.

4.4 Derive a renormalization-group equation for $\Gamma_{\mathrm{R}}^{(n)}$ at finite temperature as well as finite momentum, and then find the solution. Discuss how you might want to choose the optimal value of M when there are two variables, p and T.

References

1. Frenkel, J., Saa, A. V., and Taylor, J. C., *Phys. Rev. D* **46**, 3670 (1992).
2. Parwani, R. R. and Singh, H., *Phys. Rev. D* **51**, 4518 (1995).

Bibliography

Renormalization in quantum field theory

Bjorken, J. D. and Drell, S. D. (1965). *Relativistic Quantum Fields* (McGraw-Hill, New York).

Nash, C. (1978). *Relativistic Quantum Fields* (Academic Press, New York).

Ramond, P. (1989). *Field Theory: A Modern Primer* (Addison-Wesley, Redwood City).

Peskin, M. E., and Schroeder, D. V. (1995). *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading).

Weinberg, S. (1995). *The Quantum Theory of Fields* (Cambridge University Press, Cambridge).

Renormalization group

Collins, J. C. (1984). *Renormalization* (Cambridge University Press, Cambridge).

Politzer, H. D., *Phys. Rep.* **14**, 129 (1974).

Renormalization group and high temperature

Collins, J. C., and Perry., M. J., *Phys. Rev. Lett.* **34**, 1353 (1975).

https://doi.org/10.1017/9781009401968.005 Published online by Cambridge University Press