ON EXTENSIONS OF MONOTONE FUNCTIONS FROM LINEAR SUBLATTICES

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1. Introduction. In this note real valued functions, defined on a linear sublattice S of a linear lattice R and satisfying the two order conditions (M1) and (M2), are studied from the point of view of the existence and uniqueness of extensions to R. The paper is partly expository and supplements and extends § 3 of [4] where S was assumed to be an ℓ-ideal.

A lattice is a set P partially ordered by a binary relation $\leq$ and such that every pair of elements $x, y \in P$ has a greatest lower bound or infimum $x \wedge y$ and a least upper bound or supremum $x \vee y$ in P. A set V that is both a vector space and a lattice is called a vector or linear lattice if the vector and lattice operations are compatible as follows. Writing the (commutative) group operation of V as addition

$$x \leq y \text{ implies } x + a \leq y + a$$

for every $a \in V$. If $V^+ = [x \in V : x \geq 0]$, $x \leq y$ is equivalent to $y - x \in V^+$. Multiplication by scalars satisfies

$$x \leq y, \ \lambda > 0 \text{ implies that } \lambda x \leq \lambda y .$$

It then follows that multiplication by a negative scalar reverses the order. The set V^+ satisfies

$$x, y \in V^+ \Rightarrow x + y \in V^+ ,$$

$$x \in V^+, \ \lambda > 0 \Rightarrow \lambda x \in V^+ ,$$

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conditions that define a cone in a vector space. The set V^+ is called the positive cone of V.

A linear lattice V is called universally continuous if every collection of elements of V^+ has an infimum, sequentially continuous if every countable subset of V^+ has an infimum. (In [1] these are called complete and σ-complete respectively.)

A linear sublattice of a linear lattice R is a linear subspace of R that is also closed under $\wedge$ and $\vee$. Thus if S is a linear sublattice of R and $x, y \in S$, then $x \wedge y$ and $x \vee y$, as defined in R, are in S. If R is universally continuous (sequentially continuous) a linear sublattice of R is called universally continuous (sequentially continuous) if the infimum in R of arbitrary (countable) subsets of S^+ is in S^+.

A linear sublattice S of a linear lattice R is an ℓ-ideal (semi-normal manifold) of R if $x \in S$, $a \in R$, $|a| \le |x|$ implies that $a \in S$. An ℓ-ideal of R is always universally (sequentially) continuous if R is universally (sequentially) continuous.

A function f(x), defined on the positive cone of a linear lattice with values in the non-negative, extended reals is called a monotone function if it satisfies*

(M1) $f(x) \le f(y)$ if $x \le y$ (order-preserving),

(M2) $f(x_i) \uparrow_{i=1}^{\infty} f(x)$ if $x_i \uparrow_{i=1}^{\infty} x$.

In [4] extensions of monotone functions from S^+ to R^+, where S is an ℓ-ideal of R, were studied. We illustrate the application of an elementary part of this theory by an example. Let $(X, \mathcal{Q}, \mu)$ denote an arbitrary measure space with $\mathcal{Q}$ a σ-algebra of subsets of X. Let $(\mathfrak{M}, \le)$ denote the vector lattice of finite real $\mathcal{Q}$-measurable functions under the natural ordering, S the linear sublattice of $\mathfrak{M}$ of bounded functions vanishing outside sets of finite measure. Then $\mathfrak{M}$ is sequentially

* $a_i \uparrow_{i=1}^{\infty}$ means $a_1 \le a_2 \le \dots$; $a_i \uparrow_{i=1}^{\infty} a$ implies in addition that $a = \bigvee_{i=1}^{\infty} a_i$.

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continuous, S an ℓ-ideal of $\mathcal{M}$. If X is σ-finite then to each $x \in \mathcal{M}^+$ corresponds a sequence $\{x_i\} \in S^+$ with $x_i \uparrow_{i=1}^{\infty} x$. It follows from [4], Theorem 3.2, that every monotone function then has a unique monotone extension to $\mathcal{M}^+$. In particular if X is σ-finite any length function, [3], and in particular all length functions corresponding to $\mathcal{L}^p$ norms, $1 \le p \le \infty$, are completely determined on $\mathcal{M}$ by their values on S^+.

Suppose that X is not σ-finite. Then every monotone f on S^+ has unique maximal and minimal extensions f_M and f_m to $\mathcal{M}^+$. If $\mathcal{Q}$ contains a purely infinite set, that is a set E with $\mu(E) = \infty$ such that $\mu(E') = 0$ or ∞ for every measurable subset E' of E then, where f corresponds to $\mathcal{L}^p$ norm, $1 \le p \le \infty$ on S^+, $f_M(\chi_E) = \infty$, $f_m(\chi_E) = 0$ and the maximal and minimal extensions are different. However if $\mathcal{Q}$ contains no purely infinite sets and $1 \le p < \infty$ then for each $x \in \mathcal{M}^+$ with no subsequence $\{x_i\}$ in S^+ increasing to x, x majorizes elements of S^+ on which f assumes arbitrarily large values so that $f_m(x) = \infty = f_M(x)$ and $f_m = f_M$. For $p = \infty$, however, again $f_m \neq f_M$.

If $\mathcal{R}^X$ denotes the universally continuous space of finite real valued functions on X, S as defined above is a sequentially continuous linear sublattice of $\mathcal{R}^X$ but not an ℓ-ideal unless X contains no non-measurable sets of finite outer measure. Thus the theory in [4] does not apply to extensions of monotone functions from S^+ to $\mathcal{R}^{+X}$. However, if f corresponds to the integral on S^+, f can be extended as a <u>mesure abstraite</u> ([2], p. 114) to a monotone function on $\mathcal{R}^X$. A similar extension from the positive cone of the space of continuous functions with compact supports occurs in the general Bourbaki theory [2].

In this note we assume given a vector sublattice S of a sequentially continuous linear lattice R and study the existence

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and uniqueness properties of extensions of monotone functions from S^+ to R^+. Since the case where S is an ℓ-ideal was studied in [4] we consider mainly extensions from S^+ to T^+, where T is the smallest ℓ-ideal containing S. We note that there is a smallest sequentially continuous sublattice S' of R containing S and that each monotone function on S^+ determines a unique monotone extension to S'^+. The smallest ℓ-ideals containing S and S' coincide.

Given a monotone function f defined on S^+, minimal and maximal extensions f_m and f_M, satisfying (M1) are defined as in [4]. As in [4], f_M also satisfies (M2) and thus gives a unique maximal extension of f to T^+, and thus leads to a maximal extension to R^+. In contrast to the ℓ-ideal case, f_m need not be monotone and, in fact, no minimal monotone extension need exist. In order that $f_m = f_M$ on T^+ (which implies a unique monotone extension of f to T^+) it is necessary and sufficient that to each $x \in T^+$ corresponds a pair $s, s' \in S'$ with

$$s \leq x \leq s' , \quad f(s) = f(s') .$$

In order that $f_m = f_M$ for T^+ for every monotone function f on S^+ it is necessary and sufficient that $T = S'$.

2. <u>The sequentially continuous linear sublattice of R generated by S.</u> Let S'^+ be the extension of S^+ obtained by adding to S^+ the collection of all lower envelopes of countable collections of elements of S^+. Thus if $s_i \in S^+$, $i = 1, 2, \ldots$; $\bigwedge_{i=1}^{\infty} s_i \in S'^+$.

It is easy to verify that S'^+ is a cone and $S' = S'^+ - S'^+$ a linear sublattice of R containing S. Suppose that $x_i \in S'^+$, $i = 1, 2, \ldots$. Then $x = \bigwedge_{i=1}^{\infty} x_i$ exists in R.

If $x_i = \Lambda_{j=1}^{\infty} s_{ij}$, $s_{ij} \in S^+$, $i = 1, 2, \ldots$, $x = \Lambda_{i,j} s_{ij} \in S'^+$. Thus S' is sequentially continuous. Since every sequentially continuous linear lattice containing S must contain S'^+ and therefore S', S' is the smallest sequentially continuous linear sublattice of R containing S. We call S' the sequentially continuous sublattice of R generated by S.

We note that if $x \in S'^+$ there exists a sequence $\{s_i\} \in S^+$ with $s_i \downarrow_{i=1}^{\infty} x$. If $x \in S^+$ we take $s_i = x$, $i = 1, 2, \ldots$. If not, there exist $s_i' \in S^+$ with $x = \Lambda_{i=1}^{\infty} s_i'$, and if

$s_i = \Lambda_{j=1}^{i} s_j'$, $s_i \in S^+$, $s_i \downarrow_{i=1}^{\infty} x$. We show that there also exists a sequence $\{s_i'\} \in S^+$ with $s_i' \uparrow_{i=1}^{\infty} x$. We write $x \in S'^-$ if $x \in S'$, $x \leq 0$. If $x \in S'^-$, $-x \in S'^+$ and there exists a sequence $s_i \in S^+$ with $s_i \downarrow_{i=1}^{\infty} -x$. Then $-s_i \uparrow_{i=1}^{\infty} x$. Now suppose that $x \in S'^+$. There then exists $s \in S^+$ with $s \geq x$, $x - s \in S'^-$ and thus a sequence s_i' in S^- with $s_i' \uparrow_{i=1}^{\infty} (x-s)$. Then $s + s_i' \in S$, $i = 1, 2, \ldots$, and

$$(s + s_i') \uparrow_{i=1}^{\infty} s + (x - s) = x .$$

There is no loss of generality in assuming each $s + s_i' \in S^+$ since they could be replaced by $(s + s_i') \vee 0 \in S^+$.

If f is a monotone function on S^+ and $x \in S'^+$ then there is a sequence $\{s_i\}$ in S^+ with $x = \vee_{i=1}^{\infty} s_i$ and if $\bar{f}$ is to be an extension of f to S'^+ satisfying (M2), we must have $\bar{f}(x) = \lim_i f(s_i)$. That the limit does not depend on the actual sequence s_i is shown by the argument of [4, Lemma 3.1]. Thus

Every monotone function f on the positive cone of a linear sublattice S of R has a unique monotone extension $\bar{f}$ to the positive cone of the sequentially continuous linear sublattice of R generated by S given by

$$\bar{f}(x) = \lim_{i\to\infty} f(s_i) ,$$

where $\{s_i\}$ is any sequence of elements of S^+ with $s_i \uparrow_{i=1}^{\infty} s$.

3. The semi-normal manifold of R generated by S.

Let $T = (x \in R\colon |x| \leq S$ for some $s \in S^+)$. Direct verification shows that T is a semi-normal manifold of R containing S and is the smallest one. Since every $x \in S'^+$ is majorized by an element of S^+, $S'^+ \subset T^+$, $S' \subset T$ and S and S' generate the same semi-normal manifold T of R.

In the remainder of this note the notation S', T refers to the sequentially continuous and semi-normal manifolds generated by S.

If f_e is a monotone extension of f from S^+ to T^+, (M1) implies that

$$\sup_{\substack{s \leq x \\ s \in S^+}} f(s) \leq f_e(x) \leq \inf_{\substack{s' \geq x \\ s' \in S^+}} f(s')$$

for every $x \in T^+$. We define functions f_m, f_M on T^+ as follows:

$$f_m(x) = f_M(x) = \lim_{i\to\infty} f(s_i) \text{ if } x \in S'^+ \text{ and } s_i \uparrow_{i=1}^{\infty} x ;$$

$$f_m(x) = \sup_{\substack{s \leq x \\ s \in S^+}} f(s), \quad f_M(x) = \inf_{\substack{s \geq x \\ s \in S^+}} f(s) , \quad x \notin S'^+ .$$

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Clearly $f_m \leq f_M$ and $f_m \leq f_e \leq f_M$ for every monotone extension from S^+ to T^+.

A monotone function f is convex (concave) if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y) \quad (f(\alpha x + \beta y) \geq \alpha f(x) + \beta f(y))$$

for $\alpha + \beta = 1$. $\alpha, \beta \geq 0$; linear if $f(x + y) = f(x) + f(y)$; sublinear if $f(x + y) \leq f(x) + f(y)$; superlinear if $f(x + y) \geq f(x) + f(y)$; homogeneous if $f(\alpha x) = \alpha f(x)$ for $\alpha \geq 0$; and additive if $x \perp y$ implies that $f(x + y) = f(x) + f(y)$.

THEOREM 3.1. *Let S be a linear sublattice of a sequentially continuous linear lattice R, f a monotone function on S. Then (i) f_M is monotone, (ii) f_m satisfies (M1) but is not necessarily monotone, (iii) f_M is sublinear if f is linear or sublinear, (iv) f_m is superlinear if f is linear or superlinear, (v) f_m and f_M are homogeneous if f is homogeneous, (vi) f_M is convex if f is convex, (vii) f_m is concave if f is concave and (viii) f_M or f_m need not be additive if f is additive.*

Proof. Standard arguments show that (M1) holds for f_m and f_M. To prove (i) we note that (M1) implies that if $x, x_i \in T^+$ and $x_i \uparrow_{i=1}^{\infty} x$, then

$$f_M(x_i) \uparrow_{i=1}^{\infty} \quad \text{and} \quad f_M(x) \geq \lim_{i\to\infty} f_M(x_i) .$$

The opposite inequality is trivial if $\lim_{i\to\infty} f_M(x_i) = +\infty$. Assume that this limit is finite. Given $\epsilon > 0$, there exist $s_i \geq x_i$ with

$$f(s_i) \leq f_M(x_i) + \epsilon , \quad i = 1, 2, \ldots .$$

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Since $x \in T^+$, there exists $s \in S^+$ with $s \geq x$. Since $x_i \leq s \wedge s_i \leq s_i$, it follows from (M1) that we can assume each $s_i \leq s$.

Now $x'_n = \bigwedge_{i=n}^{\infty} s_i \in S'^+$ and $x'_n \leq s$, $n = 1, 2, \ldots$.

We note that the fact that S' is sequentially continuous implies that if y, $x'_n \in S'$ and $x'_n \leq y$, $n = 1, 2, \ldots$, then $\bigvee_{n=1}^{\infty} x'_n \in S'$. Thus if $x' = \bigvee_{n=1}^{\infty} x'_n$, $x' \in S'^+$ and

$$x'_n \uparrow_{n=1}^{\infty} x' \quad \text{whence} \quad f_M(x'_n) \uparrow_{n=1}^{\infty} f_M(x'),$$

since (M2) holds for f_M on S'^+. Now $x' \geq x'_n \geq x_n$, $n = 1, 2, \ldots$. Thus $x' \geq x = \bigvee_{n=1}^{\infty} x_n$,

$$f_M(x) \leq f_M(x') = \lim_{n\to\infty} f_M(x'_n) \leq \lim_{n\to\infty} f_M(x_n) + \epsilon .$$

Since ϵ is arbitrary (M2) holds in T^+ and f_M is monotone on T^+.

Proofs of (iii) - (vii) are routine and do not use (M2). The limitations on monotone extensions are illustrated by the following simple example.

Let $\mathcal{R}$ denote the real numbers, let $X = (0, 1)$ and let $R = \mathcal{R}^X$. Then $S = (a\chi_X, -\infty < a < \infty)$ is a sequentially continuous linear sublattice of R and T denotes the bounded functions on X.

Define $f(a\chi_X) = a$ on S^+. Then f is a linear monotone function on S^+ and

$$f_m(x) = \inf_{0<t<1} x(t), \quad f_M(x) = \sup_{0<t<1} x(t)$$

on T^+.

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Let $x_i(t) = 1$, $2^{-i} < t < 1$; $= 0$ elsewhere, $i = 1, 2, \ldots$. Then each $x_i \in T^+$, $f_m(x_i) = 0$, $x_i \uparrow {}_{i=1}^{\infty} \chi_X$ but $f_m(\chi_X) = f(\chi_X) = 1$ showing that f_m is not monotone on T^+. If $x(t) = \chi_{(0,\frac{1}{2})}$, $y(t) = \chi_{[\frac{1}{2},1)}$,

$$f_m(x+y) = 1 \neq f_m(x) + f_m(y) = 0 .$$

Thus linearity need not persist for f_m. Since $x \perp y$ this also gives an example where f is additive on S^+ but f_m is not additive on T^+. The same example shows that linearity or additivity need not persist for f_M.

<u>If</u> f <u>is additive on</u> S^+, $x, y \in T^+$ <u>with</u> $x \perp y$ <u>then</u>

3.1 $$f_m(x+y) \geq f_m(x) + f_m(y) .$$

Given $\epsilon > 0$, there exist $s, s' \in S^+$ with $s \leq x$, $s' \leq y$, $f_m(x) + f_m(y) \leq f(s) + f(s') + \epsilon = f(s+s') + \epsilon \leq f_m(x+y) + \epsilon$, since $s \perp s'$.

<u>If</u> f <u>is additive on</u> S^+ <u>then</u> f_m <u>is additive on</u> T^+ <u>if and only if for each</u> $x, y \in T^+$ <u>with</u> $x \perp y$,

$$\sup_{\substack{s \leq x+y \\ s \in S^+}} f(s) = \sup_{\substack{s' \leq x,\ s'' \leq y \\ s',\ s'' \in S^+}} f(s' + s'') .$$

When S is semi-normal both a maximal and a minimal monotone extension always exist as is shown in [4]. When S is not semi-normal a maximal monotone extension exists but not necessarily a minimal one. For S and R as in the example, let t_i, $i = 1, 2, \ldots$ be distinct points of $(0,1)$. Let

$$f_j(x) = \underset{i=1,2,\ldots,j}{\text{minimum}} \; x(t_i)$$

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in T^+. Then f_j is a monotone extension of f to T^+.

Let $x_j(t) = 0$, $t = t_i$, $i \geq j$; $x_j(t) = 1$ elsewhere in X. Then

$$x_j \uparrow_{j=1}^{\infty} \chi_X .$$

If $\underline{f}$ is a minimal monotone extension of f from S^+ to T^+ then $\underline{f}(x) \leq f_j(x)$ for every $x \in T^+$ and all j. Then

$$\underline{f}(x_j) \leq f_j(x_j) = 0, \quad j = 1, 2, \ldots ; \underline{f}(\chi_X) = 1 ,$$

contradicting the assumption that $\underline{f}$ is monotone on T^+.

When $f_m = f_M$ on T^+ it is clear that all of the additional properties of f except perhaps additivity are preserved. From 3.1 above

If f is additive and sublinear on S^+ and $f_m = f_M$, then $f_m = f_M$ is additive on T^+.

Note that on $(0, 1)$ $S = \mathcal{L}^2$, with the natural order, is a sequentially continuous linear lattice with $f(x) = N^2(x)$ $= \int_0^1 [x(t)]^2 dt$ monotone and additive but superlinear and not sublinear on S^+. Since f has a unique additive extension to $\mathcal{M}^+$ (§ 1) the additional condition of sublinearity is not necessary.

4. *The extensions f_m and f_M.*

THEOREM 4.1. *Let S be a linear sublattice of R, f a monotone function on S^+. Then in order that $f_m = f_M$ on T^+ it is necessary and sufficient that to each $x \in T^+$ and $\epsilon > 0$ corresponds* $s \leq x$, $s' \geq x$, $s, s' \in S^+$ *with*

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(4.1) $$f(s') - f(s) < \epsilon .$$

The proof is routine. We show that if S is sequentially continuous then we can replace (4.1) by

(4.2) $$f(s) = f(s') .$$

If $f_M(x) = \infty$ any s' majorizing x will do. If $f_M(x) < \infty$, there exist $s_i \in S^+$ with $s_i \geq x$, $f(s_i) \leq f_M(x) + 1/i$, $i = 1, 2, \ldots$. By hypothesis $s' = \bigwedge_{i=1}^{\infty} s_i \in S^+$. Since $s' \geq x$, $f(s') \geq f_M(x)$. Since $s' \leq s_i$,

$$f(s') \leq f(s_i) \leq f_M(x) + 1/i, \quad i = 1, 2, \ldots,$$

and $f(s') = f_M(x)$. A similar argument shows that there exists $s \leq x$ with $f(s) = f_m(x)$.

THEOREM 4.2. Let S be a linear sublattice of R. Then in order that $f_m = f_M$ on T^+ for every monotone function f on S^+ it is necessary and sufficient that $T = S'$.

Proof. By definition f_m and f_M always coincide on $S'^+ \subset T^+$ so that the condition is sufficient.

Let $x_0 \in T^+$ and define $f(s) = 0$ if $s \leq x_0$, $f(s) = 1$ otherwise. If $s \leq s'$ and $s' \leq x_0$ then $s \leq s' \leq x_0$ and $f(s) = f(s')$. If $f(s') = 1$, $f(s) \leq f(s')$ trivially. Thus f satisfies (M1) on S^+. If $s_i \uparrow_{i=1}^{\infty} s$ then $f(s_i) = 0$, $i = 1, 2, \ldots$ if $f(s) = 0$. If $f(s) = 1$, then if $f(s_i) = 0$, $i = 1, 2, \ldots$, $s_i \leq x_0$, $\bigvee_{i=1}^{\infty} s_i \leq x_0$ contradicting $\bigvee_{i=1}^{\infty} s_i = s$. Thus (M2) holds and f is monotone on S^+.

If $x_0 \in S'^+$ there exists a sequence $\{s_i\}$ with

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$s_i \uparrow_{i=1}^{\infty} x_0$ and

$$f_M(x_0) = f_m(x_0) = 0 .$$

Always

$$f_m(x_0) = \sup_{\substack{s \leq x_0 \\ s \in S^+}} f(s) = 0 , \quad \inf_{\substack{s' \geq x_0 \\ s' \in S^+}} f(s') = 1 ,$$

showing that the last expression need not coincide with $f_M(x_0)$ in S'^+. If $x_0 \notin S'^+$, $f_M(x_0) = 1 \neq f_m(x_0)$ for the f determined by this x_0. Thus the condition is necessary.

THEOREM 4.3. Let S be a linear sublattice of R. Then in order that every monotone function f on S^+ extend to a monotone function f_m on T^+ it is necessary and sufficient that $x_i \in T^+$, $s \in S^+$, $x_i \uparrow_{i=1}^{\infty} s$ imply the existence of a sequence $s_j \in S^+$, $s_j \uparrow_{j=1}^{\infty} s$ with each s_j majorized by some x_i.

Proof. Sufficiency. For x, x_i in T^+ assume that $x_i \uparrow_{i=1}^{\infty} x$. First assume that $f_m(x) < \infty$. There then exists $s \in S^+$, $s \geq x$, with

$$f_m(x) \leq f(s) + \epsilon , \quad \epsilon > 0 \text{ arbitrary} .$$

Now $x_i' = x_i \wedge s \in T^+$ and $x_i' \uparrow_{i=1}^{\infty} s$. By hypothesis there exists $s_j \uparrow_{j=1}^{\infty} s$ with each s_j majorized by some x_i'. Thus

$$f(s_j) \leq f_m(x_i') \leq \lim_{i \to \infty} f_m(x_i') ,$$

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$$f(s) = \lim_{j\to\infty} f(s_j) \le \lim_{i\to\infty} f_m(x_i')$$

$$f_m(x) \le f(s) + \epsilon \le \lim_{i\to\infty} f_m(x_i') + \epsilon \le \lim_{i\to\infty} f(x_i) + \epsilon .$$

By (M1) $f_m(x) \ge \lim_{i\to\infty} f_m(x_i)$. Since ϵ is arbitrary

$$f_m(x_i)\uparrow_{i=1}^{\infty} f_m(x)$$

and f_m is monotone on T^+. A similar argument applies if $f_m(x) = +\infty$.

Necessity. To prove necessity we show that if the condition is violated we can construct a monotone function on S^+ with f_m not monotone. Suppose that there exists $s_0 \in S^+$ and $x_i \in T^+$, $x_i\uparrow_{i=1}^{\infty} s_0$ and that there exists no sequence $\{s_i\}$, $s_i\uparrow_{i=1}^{\infty} s_0$ with each s_i majorized by some x_i.

Define $f(s) = 0$ if $s \le x_i$ for some i (i.e. if s is majorized by some x_i) or if $s < s_0$ and there exists $s_i\uparrow_{i=1}^{\infty} s$ with each s_i majorized by some x_i. Define $f(s) = 1$ elsewhere in S^+.

We first verify that f is a monotone function on S^+. Assume that $s \le s'$. Then $f(s) \le f(s')$ trivially if $f(s') = 1$. If $f(s') = 0$ and $s' \le x_i$, $s \le s' \le x_i$ and $f(s) = 0$. If $f(s') = 0$ and $s_i\uparrow_{i=1}^{\infty} s$, with each s_i majorized by some x_j, $s_i \wedge s \in S^+$, $i = 1, 2, \ldots$, $s_i \wedge s\uparrow_{i=1}^{\infty} s$, with each $s_i \wedge s$ majorized by some x_j and again $f(s) = 0$. Thus (M1) holds for f on S^+.

Assume that $s_i\uparrow_{i=1}^{\infty} s$. Then $f(s_i)\uparrow_{i=1}^{\infty} f(s)$ trivially if

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$f(s) = 0$. Assume that $s = s_0$. If each $x_i = \bigvee_{j=1}^{\infty} s_{ij}$ with each s_{ij} majorized by some x_i, $s_0 = \bigvee_{i,j} s_{ij}$ and the s_{ij} can be combined to form a sequence increasing to s_0 with each term majorized by some x_i, giving a contradiction. Thus for some i, $f(s_i) = 1$ and, by (M1), $f(s_i) \uparrow_{i=1}^{\infty} f(s_0)$.

Assume $s \neq s_0$, $s \vee s_0 > s_0$. If all $f(s_i) = 0$, $\bigvee_{i=1}^{\infty} s_i = s \leq s_0$ contrary to hypothesis. Thus in this case $\lim_i f(s_i) = 1 = f(s)$. Finally assume $s < s_0$. If for each i there exists s_{ij} with $\bigvee_{j=1}^{\infty} s_{ij} = s_i$ and each s_{ij} is majorized by some x_k, then $f(s) = 0$. If this is false for some i, $f(s) = \lim_i f(s_i) = 1$. Thus (M2) is satisfied and f is monotone on S^+.

Now in T^+, $f_m(s_0) = f(s_0) = 1$, $f_m(x_i) = \sup_{\substack{s \leq x_i \\ s \in S^+}} f(s) = 0$, $i = 1, 2, \ldots$ showing that f_m is not montone on T^+.

We observe that when S is semi-normal the condition of Theorem 4. 3. is trivially satisfied since each x_i is then necessarily in S^+.

Suppose that f is monotone on S^+, f_m monotone on T^+. Let $\dot{f}_m$ and $\dot{f}_M$ denote the minimum and maximum extensions of f_m and f_M respectively from T^+ to R^+. Then $\dot{f}_m$ and $\dot{f}_M$ are monotone on R^+ and in R^+,

$$\dot{f}_m(x) = \sup_{\substack{y \leq x \\ y \in T^+}} f_m(y) = \sup_{\substack{s \leq x \\ s \in S^+}} f(s) ;$$

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$$\dot{f}_M(s) = \lim_{i\to\infty} f(s_i) \text{ if there exist } s_i \in S,\ s_i \uparrow_{i=1}^{\infty} x ;$$

$$= \inf_{\substack{s \geq x \\ s \in S^+}} f(s), \text{ if no such sequence exists and } x \text{ is majorized in } S^+ ;$$

$$= +\infty \text{ if there is no } s \in S^+,\ s \geq x \text{ and no sequence } s_i \uparrow_{i=1}^{\infty} x,\ s_i \in S^+ .$$

For every monotone extension f_e from S^+ to R^+ and all $x \in R^+$,

$$\dot{f}_m(s) \leq f_e(x) \leq \dot{f}_M(x) .$$

Addendum. The referee has pointed out that the third sentence on page 227 in [4] is incorrect without the additional hypothesis that $f(0) = 0$ and that $p_\lambda \in R^+$, $\lambda \in \Lambda$ should be added to the hypotheses of Lemma 4.2.

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