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LONGEST CYCLES IN 2-CONNECTED GRAPHS WITH PRESCRIBED MAXIMUM DEGREE

J. A. BONDY AND R. C. ENTRINGER

1. Introduction. The relationship between the lengths of cycles in a graph and the degrees of its vertices was first studied in a general context by G. A. Dirac. In [**5**], he proved that every 2-connected simple graph on n vertices with minimum degree d contains a cycle of length at least $\min\{2d, n\}$. Dirac's theorem was subsequently strengthened in various directions in [**7**], [**6**], [**13**], [**12**], [**2**], [**1**], [**11**], [**8**], [**14**], [**15**] and [**16**].

Our aim here is to investigate another aspect of this relationship, namely how the lengths of the cycles in a 2-connected graph depend on the maximum degree. Let us denote by $f(n, d)$ the largest integer k such that every 2-connected simple graph on n vertices with maximum degree d contains a cycle of length at least k. We prove in Section 2 that, for $d \geqq 3$ and $n \geqq d + 2$,

(1) $4\log_{d-1}n - 4\log_{d-1}\log_{d-1}n - 20 < f(n, d) < 4\log_{d-1}n + 4.$

Thus, for every $d \geqq 3$,

$$\lim_{n\to\infty} \frac{f(n, d)}{\log_{d-1} n} = 4.$$

In Section 3, we examine the special case of regular graphs. If $g(n, d)$ denotes the largest integer k such that every 2-connected d-regular simple graph on n vertices contains a cycle of length at least k, then it follows from (1) and the above-mentioned theorem of Dirac that, for $d \geqq 3$ and $n \geqq 2d$,

(2) $g(n, d) \geqq \max\{2d, 4\log_{d-1}n - 4\log_{d-1}\log_{d-1}n - 20\}.$

We establish upper bounds on $g(n, d)$ by means of appropriate constructions. In particular, we prove that, for $d \geqq 3$ and $n \geqq \frac{1}{2}(d - 1)(d^2 + 3d + 1)$,

(3) $g(n, d) \leqq 4\{\log_{d-1}n\} + 2d.$

The bounds in (2) and (3) are fairly close to one another both for small

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values of d ($d = O(1)$) and for large values of d ($d = O(n^c)$, where $0 < c \leqq \frac{1}{3}$). However, they are markedly different at intermediate values of d, and the lower bound (2) could, no doubt, be improved in this range. The bound (2) is also rather weak for very large values of d ($d = O(n)$). For example, Jackson [**9**] has proved that, for $d \geqq n/3$,

$$g(n, d) = n$$

and it has been conjectured (see [**3**]) that, for $d \geqq n/k$, where $k \geqq 3$ and n is sufficiently large,

$$g(n, d) \geqq 2n/(k - 1).$$

We conclude the paper with a discussion of some related problems and results.

2. Graphs with prescribed maximum degree. The lower bound in (1) is established by means of a construction based on the following lemma. We first have a definition. If G is a graph whose block-cutvertex tree is a path, and if x and y are two vertices of G belonging to the blocks which correspond to the ends of this path, then G is referred to as an (x, y)*-block-path.*

LEMMA 1. *Let T be a tree with s vertices $u_1, u_2, \ldots, u_s$ of degree one, where $s \geqq 2$. Let T' be a tree isomorphic to T with corresponding vertices $u_1', u_2', \ldots, u_s'$ of degree one and, for $1 \leqq i \leqq s$, let G_i be a (v_i, v_i')-block-path, where $T, T', G_1, G_2, \ldots, G_s$ are pairwise disjoint. Denote by G the graph obtained on identifying u_i with v_i and u_i' with v_i' for all i, $1 \leqq i \leqq s$. Then G is 2-connected and any cycle in G includes vertices of at most two of the graphs G_i.*

Proof. G is clearly 2-connected. Let C be a cycle in G. We may suppose that no G_i entirely contains C. Then, since both T and T' are trees, C must include some (u_i, u_j)-path of T. But if C includes an edge e of T, it also includes the corresponding edge e' of T', because $\{e, e'\}$ is an edge cut of G. Therefore C also includes the (u_i', u_j')-path of T'. It follows that C consists of these two paths together with a (v_i, v_i')-path in G_i and a (v_j, v_j')-path in G_j.

Construction. Let n and d be positive integers with $d \geqq 3$ and $n \geqq d + 2$. Set

$$s = d(d - 1)^t$$

where

$$t = [\log_{d-1}((n(d - 2) + 4)/d^2)]$$

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and let $(a_1, a_2, \ldots, a_s)$ be a sequence of integers, as equal as possible, such that

$$\sum_{i=1}^{s} a_i = \frac{n(d-2) + 4 - d^2(d-1)^t}{d-2}.$$

In order to apply Lemma 1, we now define a tree T and graphs G_i, $1 \leqq i \leqq s$, as follows.

Let T be a tree in which every vertex of degree greater than one has degree d and every vertex of degree one is at distance $t + 1$ from the centre. Observe that T has s vertices of degree one and that, if d_i is the number of vertices of T at distance i from the centre,

$$\nu(T) = \sum_{i=0}^{t+1} d_i = 1 + \sum_{i=1}^{t+1} d(d-1)^{i-1} = \frac{d(d-1)^{t+1} - 2}{d-2}$$

where $\nu(T)$ denotes the number of vertices of T.

For $1 \leqq i \leqq s$, let

$$G_i = \begin{cases} K_1 & \text{if } a_i = 0 \\ K_2 & \text{if } a_i = 1 \\ K_{2,a_i-1} & \text{if } a_i \geqq 2. \end{cases}$$

We denote the ends of T by $u_1, u_2, \ldots, u_s$. If $a_i = 0$, we label the vertex of G_i with v_i and v_i'; if $a_i = 1$, we label one vertex of G_i with v_i and the other with v_i'; if $a_i \geqq 2$, we label one vertex of degree $a_i - 1$ in G_i with v_i and the other with v_i'.

On identifying vertices as in Lemma 1, we obtain a 2-connected graph $G_{n,d}$ with maximum degree d. Now

$$\nu(G_{n,d}) = 2\nu(T) + \sum_{i=1}^{s} (\nu(G_i) - 2)$$
$$= \frac{2d(d-1)^{t+1} - 4}{d-2} + \sum_{i=1}^{s} a_i - s = n$$

and, by Lemma 1, a longest cycle in $G_{n,d}$ has length at most $4t + 8$. Therefore

$$f(n, d) \leqq 4\log_{d-1}(n(d-2) + 4)/d^2 + 8 < 4\log_{d-1}n + 4.$$

Our proof of the lower bound in (1) makes use of the following lemma.

LEMMA 2. *Let G be a 2-connected graph on n vertices with maximum degree d. Then each edge of G lies on a cycle of length at least $2h(n, d) - 1$, where*

$$h(n, d) = \log_{d-1}(n(d-2) + 2)/2.$$

Proof. Let $e = uv$ be any edge of G, and let G' be the graph obtained from G by deleting e, inserting a new vertex x, and joining x to both u

and v. Then G' is also 2-connected and has maximum degree d. In G', let d_i be the number of vertices at distance i from x. Then $d_1 = 2$ and, because G' has maximum degree d, $d_i \leqq 2(d-1)^{i-1}$ for all $i > 1$. Suppose that $d_{r+1} = 0$. Then

$$n = \sum_{i=1}^{r} d_i \leqq \frac{2((d-1)^r - 1)}{d-2}$$

so

$$r \geqq h(n, d).$$

It follows that there is a vertex y in G' whose distance from x is at least $h(n, d)$. Since G' is 2-connected, there are two internally-disjoint (x, y)-paths in G'. Thus x lies on a cycle of length at least $2h(n, d)$ in G', and e lies on a cycle of length at least $2h(n, d) - 1$ in G.

Let G be a 2-connected graph and let C be a cycle of G. For each component G_i of $G - C$, let A_i be the set of vertices of C which are adjacent, in G, to at least one vertex of G_i, and let B_i be the subgraph of G consisting of A_i, G_i and all the edges of G with one end in A_i and the other in G_i. Then the subgraphs B_i are called the *proper bridges* of G (relative to C). The sets A_i are the sets of *vertices of attachment* of the bridges B_i.

THEOREM.

$$f(n, d) > 4\log_{d-1}n - 4\log_{d-1}\log_{d-1}n - 20.$$

Proof. Let G be a 2-connected graph on n vertices with maximum degree d and let C be a longest cycle, of length l, in G. Let G_i, B_i and A_i, $1 \leqq i \leqq r$, be the components of $G - C$, the corresponding proper bridges of G and their sets of vertices of attachment, respectively. For $1 \leqq i \leqq r$, set

$$\nu(G_i) = n_i \quad \text{and} \quad |A_i| = a_i.$$

Then

$$(4) \quad \sum_{i=1}^{r} n_i = n - l$$

and

$$(5) \quad \sum_{i=1}^{r} a_i \leqq l(d-2).$$

Denote by T_i the block-cutvertex tree of G_i, and let G' be the graph one obtains from G on replacing G_i by T_i, $1 \leqq i \leqq r$. Let $\mathscr{P}_i$ denote the set of all paths of length at least two in G' having their ends in A_i and their internal vertices in T_i. Since each such path is determined by its two terminal edges,

$$(6) \quad |\mathscr{P}_i| \leqq \binom{a_i}{2}(d-2)^2.$$

Now each vertex of T_i lies on at least $a_i - 1$ of these paths. Thus, if we define the *weight* $w(P)$ of a path $P \in \mathscr{P}_i$ to be the total number of vertices in the block path of G_i corresponding to the interior of P, we have

$$(7) \quad \sum_{P \in \mathscr{P}_i} w(P) \geqq (a_i - 1)n_i.$$

It follows from (6) and (7) that there is a path $P_i \in \mathscr{P}_i$ with

$$(8) \quad w(P_i) \geqq 2n_i/a_i(d-2)^2.$$

Let $m = \max_i n_i/a_i$. Then $ma_i \geqq n_i$ for all i, so

$$m \sum_{i=1}^{r} a_i \geqq \sum_{i=1}^{r} n_i.$$

Using (4) and (5), we obtain

$$m \geqq (n-l)/l(d-2).$$

We now deduce from (8) the existence of a path $P \in \bigcup_i \mathscr{P}_i$ such that

$$w(P) \geqq 2(n-l)/l(d-2)^3.$$

Suppose that $P \in \mathscr{P}_j$, and that the ends of P are u and v. Then the subgraph H of B_j corresponding to P, together with the edge uv, is 2-connected and

$$\nu(H) \geqq \frac{2(n-l)}{l(d-2)^3} + 2.$$

By Lemma 2, uv lies in a cycle of length at least $2h - 1$, where $h = h(\nu(H), d)$. Therefore B_j contains a (u, v)-path of length at least $2h - 2$. Since C is a longest cycle in G, it follows that $l \geqq 4h - 4$. Thus

$$l \geqq 4 \log_{d-1}\left(\frac{n-l}{l(d-2)^2} + d - 1\right) - 4 > 4 \log_{d-1} n - 4 \log_{d-1} l - 12.$$

But this implies that

$$f(n, d) > 4\log_{d-1} n - 4\log_{d-1}\log_{d-1} n - 20.$$

3. Regular graphs. Here, we describe constructions which yield fairly good upper bounds on $g(n, d)$. The first makes use of Lemma 1.

Let n and d be positive integers with $d \geqq 3$ and $n \geqq d^2 + d + 2$. Set

$$s = t(d-2) + 2$$

where

$$t = \left[\frac{n - 2(d+1)}{d^2 - d}\right]$$

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and let $(a_1, a_2, \ldots, a_s)$ be a sequence of integers, as equal as possible subject to the condition that each a_i be even if d is odd, and such that

$$\sum_{i=1}^{s} a_i = n - 2t.$$

Let T be a tree with s vertices $u_1, u_2, \ldots, u_s$ of degree one and t vertices of degree d. Choose T so that the maximum distance m from the centre of T to a vertex of degree one is as small as possible. Thus

$$m = \{\log_{d-1}(s/d)\} + 1 \leqq \{\log_{d-1}s\}.$$

For $1 \leqq i \leqq s$, let G_i be a (v_i, v_i')-block-path on a_i vertices, where v_i and v_i' have degree $d - 1$ and the remaining vertices have degree d. On identifying vertices as in Lemma 1, we obtain a 2-connected d-regular graph $H_{n,d}$. Now

$$\nu(H_{n,d}) = 2\nu(T) + \sum_{i=1}^{s} (\nu(G_i) - 2) = 2(s + t) + \sum_{i=1}^{s} (a_i - 2) = n$$

and, by Lemma 1, a longest cycle in $H_{n,d}$ has length at most

(9) $4m + 2\max_i a_i - 2.$

Since

$$\max_i a_i \leqq \left\{\frac{n - 2t}{s}\right\} + 1$$

we obtain

$$g(n, d) \leqq 4\{\log_{d-1}s\} + 2\left\{\frac{n - 2t}{s}\right\}.$$

This bound has the disadvantage that the roles of n and d are not expressed explicitly. However, it is amenable to some simplification when $n \geqq \frac{1}{2}(d - 1)(d^2 + 3d + 1)$. In that case, using the fact that

$$t \geqq (n - (d^2 + d + 1))/(d^2 - d)$$

a routine computation yields

$$\left\{\frac{n - 2t}{s}\right\} \leqq d + 3$$

and hence

$$\max_i a_i \leqq d + 3.$$

Also

$$s = t(d - 2) + 2 \leqq (n(d - 2) + 4)/d(d - 1) < n/(d - 1).$$

Substituting these bounds into (9), we obtain

$$g(n, d) \leqq 4\{\log_{d-1}n\} + 2d.$$

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We now briefly describe a construction valid for $d \geqq 3$ and $d + 2 \leqq n \leqq d^2 + d + 2$. It is the natural extension of one due to Lang and Walther [**10**].

Let $(a_1, a_2, \ldots, a_r)$ be a partition of $n - 2$ into integers a_i, where $a_i \geqq d + 1$, a_i is even if d is odd, and $\max_i a_i$ is as small as possible subject to these conditions. For $1 \leqq i \leqq r$, let G_i be a d-regular 2-connected graph on a_i vertices (the graphs G_i being pairwise disjoint), and let $M = \{u_j v_j | 1 \leqq j \leqq d\}$ be a matching in $H = \cup G_i$ which intersects each G_i. Let G be the graph obtained from $H - M$ by adding two new vertices u and v and the edges uu_j, vv_j, $1 \leqq j \leqq d$. Then G is a 2-connected d-regular graph on n vertices with no cycle of length greater than $2 \max_i a_i + 2$.

4. Graphs of higher connectivity. The transition from 2-connected graphs to graphs of higher connectivity has a striking effect on the problems treated above. Bondy and Simonovits [**4**] have proved, for example, that

$$e^{c_1\sqrt{\log_e n}} \leqq f_3(n, 3) \leqq c_2 n^{\log 8/\log 9}$$

where $f_k(n, d)$ is the analogue of $f(n, d)$ for k-connected graphs. They conjecture that

$$f_3(n, 3) > n^c$$

for some $c > 0$. Another conjecture, due to R. Häggkvist (see [**9**]), concerns $g_k(n, d)$, the analogue of $g(n, d)$ for k-connected graphs, and asserts that, for $d \geqq k + 2$ and $n \leqq d(k + 1)$,

$$g_k(n, d) = n.$$

It is perhaps worth pointing out here that Dirac's theorem cannot be improved by considering graphs of connectivity greater than two; if $n \geqq 2d$, then $K_{d,n-d}$ is a d-connected graph, and yet has no cycle of length greater than $2d$.

Added in Proof. Jackson [**9**] has announced that this conjecture is false.

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University of Waterloo,
Waterloo, Ontario;
University of New Mexico,
Albuquerque, New Mexico