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Approximation by Dilated Averages and K-Functionals

Z. Ditzian and A. Prymak

Abstract. For a positive finite measure $d\mu(\mathbf{u})$ on $\mathbb{R}^d$ normalized to satisfy $\int_{\mathbb{R}^d} d\mu(\mathbf{u}) = 1$, the dilated average of $f(\mathbf{x})$ is given by

$$A_t f(\mathbf{x}) = \int_{\mathbb{R}^d} f(\mathbf{x} - t\mathbf{u}) d\mu(\mathbf{u}).$$

It will be shown that under some mild assumptions on $d\mu(\mathbf{u})$ one has the equivalence

$$\|A_t f - f\|_B \approx \inf\{(\|f - g\|_B + t^2 \|P(D)g\|_B) : P(D)g \in B\} \quad \text{for } t > 0,$$

where $\varphi(t) \approx \psi(t)$ means $c^{-1} \leq \varphi(t)/\psi(t) \leq c$, B is a Banach space of functions for which translations are continuous isometries and $P(D)$ is an elliptic differential operator induced by μ. Many applications are given, notable among which is the averaging operator with $d\mu(\mathbf{u}) = \frac{1}{m(S)}\chi_S(\mathbf{u})d\mathbf{u}$, where S is a bounded convex set in $\mathbb{R}^d$ with an interior point, $m(S)$ is the Lebesgue measure of S, and $\chi_S(\mathbf{u})$ is the characteristic function of S. The rate of approximation by averages on the boundary of a convex set under more restrictive conditions is also shown to be equivalent to an appropriate K-functional.

1 Introduction, Set-up, and Main Results

The family of operators A_t for $t > 0$ (or O_n for $n \in \mathbb{N}$) forms an approximation process if $A_t f - f$ tends to zero as $t \to 0+$ (or $O_n f - f$ tends to zero as $n \to \infty$) with respect to some metric. Once the fact that A_t is an approximation process is established, the next important problem is the estimate of $A_t f - f$, or how fast $A_t f$ converges to f. Of course, the rate of convergence of $A_t f$ depends on the properties of f and most times on its smoothness. Peetre K-functionals are one of the ways to measure the smoothness of a function in a given space with respect to some norm. Typically a K-functional representing smoothness is given by

$$K(f, t^\alpha)_B \equiv \inf_{P(D)g \in B} (\|f - g\|_B + t^\alpha \|P(D)g\|_B), \quad t > 0,$$

where, in this paper, "≡" means "by definition", $P(D)$ is a differential operator and $P(D)g \in B$ signifies that g belongs to a class of (very) smooth functions. The estimates discussed in this paper are of the kind described as strong converse inequality of type A introduced in [Di-Iv] yielding $\|A_t f - f\|_B \approx K(f, t^\alpha)_B$, that is

$$C^{-1} K(f, t^\alpha)_B \leq \|A_t f - f\|_B \leq C K(f, t^\alpha)_B.$$

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Among the strong converse inequalities of type A proved in the past, one should mention the celebrated estimate for Bernstein polynomials $B_n f$ given by

$$\|B_n f - f\|_{C[0,1]} \approx \inf(\|f-g\|_{C[0,1]} + \frac{1}{n}\left\|\varphi^2 g''\right\|_{C[0,1]} : g \in C^{(2)}[0,1]),$$

where $\varphi^2(x) = x(1-x)$ (see [To]). Other equivalences of this type were established, in particular using averages of f on a set dilated by t. Results were given for averages on $[-t,t]$ or $[0,t]$ (see [Di-Iv, Sections 6 and 7]), on $[-t,t]\times\cdots\times[-t,t]$ (see [Di-Iv, Section 9]), on $\{\mathbf{x}\in\mathbb{R}^d : |\mathbf{x}| \equiv (x_1+\cdots+x_d)^{1/2} \le t\}$ (see [Di-Ru]) and on $\{\mathbf{x}\in\mathbb{R}^d : |\mathbf{x}| = t\}$. This last mentioned is correct only when $d > 1$ (see [Be-Da-Di, p. 100]).

In this paper we will obtain a general theorem that will include the above results on dilated averages and many more that will be described in Sections 6 and 7. We note that Sections 6 and 7 can be read first as motivation for the need for the results and proofs in Sections 2–5. We mention that results of the above type are given for dilations of averages on any bounded convex set with nonempty interior (see Theorem 6.1). The operator A_t averaging f on the boundary of a convex set is handled under some mild conditions on the curvature of the boundary (see Theorem 7.1).

Equivalence between the K-functionals and $\|A_t f - f\|_B$, when available, establish the latter as a measure of smoothness in its own right.

In this paper we deal with spaces of functions on $\mathbb{R}^d$ for which translations are continuous isometries (see (1.8) and (1.9)).

We now give the set-up and the main results of this paper.

The rate of approximation of the operator $A_t f(\mathbf{x})$ to f, where $A_t f(\mathbf{x})$ is given by

$$(1.1)\qquad A_t f(\mathbf{x}) \equiv \int_{\mathbb{R}^d} f(\mathbf{x}-t\mathbf{u})d\mu(\mathbf{u}), \quad \int_{\mathbb{R}^d} d\mu(\mathbf{u}) = 1, \quad d\mu(\mathbf{u}) \ge 0,$$

will be shown to be equivalent to an appropriate K-functional. That is, $A_t f - f$ will satisfy a direct and a matching converse inequality of type A in the terminology of [Di-Iv] (see Theorem 1.1). Besides $d\mu(\mathbf{u}) \ge 0$ (a condition which is further discussed at the end of this section) and $\int_{\mathbb{R}^d} d\mu(\mathbf{u}) = 1$, the measure μ on $\mathbb{R}^d$ will also satisfy the following conditions.

(i) The center of gravity of $d\mu(\mathbf{u})$ is $(0,\dots,0)$, that is

$$(1.2)\qquad \int_{\mathbb{R}^d} u_j d\mu(\mathbf{u}) = 0, \quad j = 1,\dots,d.$$

(ii) For $|\mathbf{y}| \equiv (y_1+\cdots+y_d)^{1/2} = 1$, $d\mu$ satisfies

$$(1.3)\qquad 0 < c \le \int_{\mathbb{R}^d} (u_1y_1+\cdots+u_dy_d)^2 d\mu(\mathbf{u}) \le C < \infty.$$

This guarantees that $d\mu$ is not supported by a hyperplane (using the left-hand inequality of (1.3)) and that $\int_{\mathbb{R}^d} u_j u_k d\mu(\mathbf{u})$ is finite (using the right-hand inequality of (1.3)).

(iii) The measure $d\mu(\mathbf{u})$ also satisfies

$$\int_{\mathbb{R}^d} |u_j|^r d\mu(\mathbf{u}) \le C_1 \quad \text{for } j = 1, 2, \dots, d \text{ and a given integer } r \ge 3. \tag{1.4}$$

For the converse result we will also need the following condition on $d\mu(\mathbf{u})$.

(iv) For some $\alpha > 0$

$$\|\Delta_{\mathbf{h}} G\|_{L_1(\mathbb{R}^d)} \le C_2 |\mathbf{h}|^\alpha, \quad \Delta_{\mathbf{h}} G(\mathbf{x}) \equiv G(\mathbf{x} + \mathbf{h}) - G(\mathbf{x}), \tag{1.5}$$

where for some integer m

$$\widehat{d\mu}(\mathbf{x}) \equiv \mathfrak{F}(d\mu)(\mathbf{x}) \equiv \int_{\mathbb{R}^d} e^{-i\mathbf{x}\mathbf{u}} d\mu(\mathbf{u}), \quad \big((\widehat{d\mu})(\mathbf{x})\big)^m \equiv \int_{\mathbb{R}^d} e^{-i\mathbf{x}\mathbf{u}} G(\mathbf{u}) d\mathbf{u} \tag{1.6}$$

with $G(\mathbf{u}) \in L_1(\mathbb{R}^d)$. That is, an iteration of our operator will be an operator with $d\mu_m(\mathbf{u}) = G(\mathbf{u})d\mathbf{u}$ and the kernel $G(\mathbf{u})$ is in the class Lipschitz α of $L_1(\mathbb{R}^d)$. In some applications the condition

$$|\mathbf{x}|^{\alpha_1} |\widehat{d\mu}(\mathbf{x})| \le C_3 \quad \text{for some } \alpha_1 > 0, \tag{1.7}$$

which is shown to be equivalent to (1.5), is very useful.

We will deal with functions on $\mathbb{R}^d$ in B, a Banach space of functions for which translations are continuous isometries, that is

$$\|f(\cdot + \mathbf{h})\|_B = \|f(\cdot)\|_B = \|f(-\cdot)\|_B \quad \text{for all } \mathbf{h} \in \mathbb{R}^d \text{ and } f \in B, \tag{1.8}$$

(which means that translations are isometries),

$$\|f(\cdot + \mathbf{h}) - f(\cdot)\|_B = o(1) \quad \text{as } |\mathbf{h}| \to 0, \tag{1.9}$$

(which means that translations are continuous) and for any affine transformation M with $\det M \ne 0$

$$\|L_M f\|_B \equiv \|f(M\cdot)\|_B \le C(M, B) \|f\|_B. \tag{1.10}$$

Many known spaces satisfy (1.8), (1.9), and (1.10), including $B = L_p(\mathbb{R}^d)$ with $1 \le p < \infty$ and $B = UC(\mathbb{R}^d)$ (of uniformly continuous functions on $\mathbb{R}^d$ with the $L_\infty(\mathbb{R}^d)$ norm). The condition (1.10) is needed only when $P(D)$ is not the Laplacian and then only for some matrices M, but in the applications which we encounter, (1.10) is easily satisfied.

Our main result is given in the following Theorem.

Theorem 1.1 *Suppose $f \in B$ with B satisfying* (1.8), (1.9), *and* (1.10) *and $A_t f$ is given by* (1.1) *with μ satisfying* (1.2)–(1.6). *Then*

$$\|f - A_t f\|_B \approx \inf_{P(D)g \in B} (\|f - g\|_B + t^2 \|P(D)g\|_B) \equiv K(f, P(D), t^2)_B, \tag{1.11}$$

where the elliptic operator $P(D)$ is given by

$$P(D)f = P_\mu(D)f = \sum_{j,k} a_{j,k} \frac{\partial^2}{\partial x_j \partial x_k} f \quad \text{with } a_{j,k} = \int_{\mathbb{R}^d} x_j x_k d\mu(\mathbf{x}). \tag{1.12}$$

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In Section 2 we will show that it is sufficient to consider the case $a_{j,k} = \delta_{j,k}$, that is, when $P(D)$ is the Laplacian. In Section 3 we will prove the direct result. Preliminary estimates for the converse result are given in Section 4, and the converse theorem is given in Section 5. In Sections 6 and 7 we give applications of Theorem 1.1 and show that many known results as well as new theorems follow from Theorem 1.1. The applications demonstrate the close relationship between different approximation processes induced by their relation to the K-functionals. We note that (1.11) yields a strong converse inequality of type A (see [Di-Iv]), that is, the information about smoothness is given by the rate of approximation of an average by a single dilation rather than by taking the supremum on a range of dilations. The results shown will be somewhat more general than Theorem 1.1 as the restrictions on $d\mu$ for many of the results, and in particular the direct estimate, will be weaker, and the restriction (1.10) on B is dropped when $P(D)$ of (1.11) is the Laplacian.

We note that while some would consider the condition $d\mu(\mathbf{u}) \geq 0$ onerous, it is satisfied by all applications in Sections 6 and 7 and is used in the proof. Admittedly, $d\mu(\mathbf{u}) \geq 0$ can be replaced in Sections 2 and 3 by $\int |d\mu(\mathbf{u})| \leq M$ (in addition to other conditions in this section). However, in later sections, $d\mu(\mathbf{u}) \geq 0$ is crucial in the proof that A_t is a contraction, which is used to show that $\|A_t^m f\|$ is bounded independently of m. We further remark that strong converse inequalities for operators with nonpositive kernels are very rare (see [Da-Di, Da]), and in the cases where they were established, a very particular kernel is dealt with, not one satisfying some (mild) conditions.

2 Reduction to the Case when $P(D)$ Is the Laplacian

In many applications $P(D)$ given in (1.11) is the Laplacian, but there are some interesting situations in which $P(D)$ is not the Laplacian. In this section we will show that it is sufficient to deal with $P(D) = \Delta \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$. This will be done via several simple lemmas.

Lemma 2.1 *For a bounded positive measure μ satisfying* (1.3), *$P(D)$ given by* (1.12) *is elliptic.*

Proof Being elliptic means $\sum_{j,k} a_{j,k} y_j y_k > 0$ for all $\mathbf{y} \equiv (y_1, \ldots, y_d) \neq 0$, and that is evident from (1.3). ■

We define the measure $\tilde{\mu}$ by

$$\tilde{\mu}(\Omega) \equiv \mu(M^{-1}(\Omega)) \tag{2.1}$$

for some invertible matrix M. It is clear that $\tilde{\mu}$ is a positive measure satisfying $\int_{\mathbb{R}^d} d\tilde{\mu}(\mathbf{u}) = 1$. Other properties of $\tilde{\mu}$ are also inherited from those of μ, as will be shown in the following lemma.

Lemma 2.2 *If the measure μ satisfies conditions* (1.2), (1.3), (1.4), (1.7) *and* (1.5), *with G given by* (1.6), *then the measure $\tilde{\mu}$ given by* (2.1) *also satisfies properties* (1.2), (1.3), (1.4), (1.7), *and* (1.5).

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Proof We recall that $y_k = (M\mathbf{u})_k = \sum_{l=1}^{d} m_{k,l} u_l$. We now write

$$\int_{\mathbb{R}^d} y_j d\tilde{\mu}(\mathbf{y}) = \int_{\mathbb{R}^d} \sum_{l=1}^{d} m_{j,l} u_l d\mu(\mathbf{u}) = \sum_{l=1}^{d} m_{j,l} \int_{\mathbb{R}^d} u_l d\mu(\mathbf{u}) = 0.$$

To show that $\tilde{\mu}$ satisfies (1.3) if μ does, we write for $|\mathbf{z}| = 1$

$$\begin{aligned}
\int_{\mathbb{R}^d} (\mathbf{y} \cdot \mathbf{z})^2 d\tilde{\mu}(\mathbf{y}) &= \int_{\mathbb{R}^d} \Big(\sum_{j=1}^{d} z_j y_j \Big)^2 d\tilde{\mu}(\mathbf{y}) \\
&= \int_{\mathbb{R}^d} \Big(\sum_{j=1}^{d} \sum_{l=1}^{d} m_{j,l} u_l z_j \Big)^2 d\mu(\mathbf{u}) \\
&= \int_{\mathbb{R}^d} \Big(\sum_{l=1}^{d} \Big(\sum_{j=1}^{d} m_{j,l} z_j \Big) u_l \Big)^2 d\mu(\mathbf{u}) \\
&= \int_{\mathbb{R}^d} (M^T \mathbf{z} \cdot \mathbf{u})^2 d\mu(\mathbf{u}),
\end{aligned}$$

and as M is invertible, $|M^T \mathbf{z}| \approx |\mathbf{z}| = 1$ and (1.3) is satisfied (with different constants).

To show that (1.4) with a given r is satisfied by $\tilde{\mu}$ if it is satisfied by μ for the same r, we write

$$\begin{aligned}
\int_{\mathbb{R}^d} |y_j|^r d\tilde{\mu}(\mathbf{y}) &= \int_{\mathbb{R}^d} \Big| \sum_{l=1}^{d} m_{j,l} u_l \Big|^r d\mu(\mathbf{u}) \\
&\le \max_{j,l} |m_{j,l}|^r \int_{\mathbb{R}^d} \Big(\sum_{k=1}^{d} |u_k| \Big)^r d\mu(\mathbf{u}) \\
&\le \max_{j,l} |m_{j,l}|^r d^r \int_{\mathbb{R}^d} \max_{k=1,\dots,d} |u_k|^r d\mu(\mathbf{u}) \\
&\le \max_{j,l} |m_{j,l}|^r d^r \int_{\mathbb{R}^d} \sum_{k=1}^{d} |u_k|^r d\mu(\mathbf{u}) \\
&\le C(M, r, d) \max_{k=1,\dots,d} \int_{\mathbb{R}^d} |u_k|^r d\mu(\mathbf{u}),
\end{aligned}$$

where $C(M, r, d) = \max_{j,l=1,\dots,d} |m_{j,l}|^r d^{r+1}$.

To verify that property (1.5) is satisfied for $\widetilde{G}(\mathbf{x})$, that is induced by $\tilde{\mu}$, we write

$$\begin{aligned}
\mathfrak{F}(d\tilde{\mu})(\mathbf{x}) &= \int_{\mathbb{R}^d} e^{-i\mathbf{x}\mathbf{u}} d\tilde{\mu}(\mathbf{u}) = \int_{\mathbb{R}^d} e^{-i\mathbf{x} \cdot M\mathbf{u}} d\mu(\mathbf{u}) \\
&= \int_{\mathbb{R}^d} e^{-iM^T\mathbf{x} \cdot \mathbf{u}} d\mu(\mathbf{u}) = \mathfrak{F}(d\mu)(M^T \mathbf{x}).
\end{aligned}$$

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Therefore,

$$\big(\mathfrak{F}(d\tilde{\mu})(\mathbf{x})\big)^m = \big(\mathfrak{F}(d\mu)(M^T\mathbf{x})\big)^m = \int_{\mathbb{R}^d} e^{-iM^T\mathbf{x}\cdot\mathbf{y}} G(\mathbf{y})d\mathbf{y}$$
$$= \int_{\mathbb{R}^d} e^{-i\mathbf{x}\cdot M\mathbf{y}} G(\mathbf{y})d\mathbf{y} = \int_{\mathbb{R}^d} e^{-i\mathbf{x}\cdot\mathbf{v}} \frac{G(M^{-1}\mathbf{v})}{|\det M|} d\mathbf{v},$$

which means $\widetilde{G}(\mathbf{v}) = \frac{G(M^{-1}\mathbf{v})}{|\det M|}$. Clearly, $\widetilde{G}(\mathbf{v}) \geq 0$, $\|\widetilde{G}(\cdot)\|_{L_1(\mathbb{R}^d)} = 1$ and

$$\|\widetilde{G}(\cdot + \mathbf{h}) - \widetilde{G}(\cdot)\|_{L_1(\mathbb{R}^d)} \leq \widetilde{C}_2|\mathbf{h}|^\alpha,$$

where $\widetilde{C}_2$ depends on C_2, M, and α. Similarly,

$$|\mathbf{x}|^{\alpha_1}|\mathfrak{F}(d\tilde{\mu})(\mathbf{x})| = |\mathbf{x}|^{\alpha_1}|\mathfrak{F}(d\mu)(M^T\mathbf{x})| \leq C|M^T\mathbf{x}|^{\alpha_1}|\mathfrak{F}(d\mu)(\mathbf{x}^T\mathbf{x})| \leq C_1. \qquad \blacksquare$$

We define $\widetilde{A}_t f(\mathbf{x})$ by

$$\widetilde{A}_t f(\mathbf{x}) \equiv \int_{\mathbb{R}^d} f(\mathbf{x} - t\mathbf{u})d\tilde{\mu}(\mathbf{u}). \tag{2.2}$$

We further note that both $A_t f$ and $\widetilde{A}_t f$ are defined for $f \in B$, where B is a Banach space for which translations are continuous isometries *i.e.*, satisfy (1.8) and (1.9) and both A_t and $\widetilde{A}_t$ are contractions from B into B.

Lemma 2.3 *Suppose B satisfies* (1.8), (1.9), *and* (1.10), *$M\colon \mathbb{R}^d \to \mathbb{R}^d$ is a linear map with* $\det M \neq 0$, *$F(\mathbf{x}) = L_{M^{-1}} f(\mathbf{x}) = f(M^{-1}\mathbf{x})$, and $A_t f$ and $\widetilde{A}_t F$ are given by* (1.1) *and* (2.2) *with μ satisfying* (1.2) *and* (1.3) *and $\tilde{\mu}$ given by* (2.1). *Then we have*

$$\big\|F - \widetilde{A}_t F\big\|_B \approx \|f - A_t f\|_B\,. \tag{2.3}$$

Proof We first observe that

$$\widetilde{A}_t L_{M^{-1}} f(\mathbf{x}) = \int_{\mathbb{R}^d} f(M^{-1}\mathbf{x} - tM^{-1}\mathbf{u})d\tilde{\mu}(\mathbf{u}),$$

and hence

$$L_M \widetilde{A}_t L_{M^{-1}} f(\mathbf{x}) = \int_{\mathbb{R}^d} f(\mathbf{x} - tM^{-1}\mathbf{u})d\tilde{\mu}(\mathbf{u}) = \int_{\mathbb{R}^d} f(\mathbf{x} - t\mathbf{v})d\mu(\mathbf{v}) = A_t f(\mathbf{x}).$$

We now have

$$\|f - A_t f\|_B = \big\|f - L_M \widetilde{A}_t L_{M^{-1}} f\big\|_B = \big\|L_M(F - \widetilde{A}_t F)\big\|_B.$$

Using (1.10) (in fact only for the matrices M and M^{-1}), one has

$$\big\|L_M(F - \widetilde{A}_t F)\big\|_B \approx \|f - A_t f\|_B\,,$$

which completes the proof. $\blacksquare$

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The following (known) lemma is set here to fit with our notations.

Lemma 2.4 *For any positive measure μ satisfying* (1.2) *and* (1.3)*, there exists a matrix M with* $\det M \neq 0$ *and a measure $\tilde{\mu}$ given by* (2.1) *such that*

$$b_{k,l} = \int_{\mathbb{R}^d} y_k y_l d\tilde{\mu}(\mathbf{y}) = \delta_{k,l} = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$

Proof For $a_{i,j} = \int_{\mathbb{R}^d} u_i u_j d\mu(\mathbf{u})$ and the matrix $M = (m_{k,l})_{k,l=1,\dots,d}$,

$$\begin{aligned} b_{k,l} &= \int_{\mathbb{R}^d} y_k y_l d\tilde{\mu}(\mathbf{y}) = \int_{\mathbb{R}^d} \Big(\sum_{i=1}^{d} m_{k,i} u_i\Big)\Big(\sum_{j=1}^{d} m_{l,j} u_j\Big) d\mu(\mathbf{u}) \\ &= \sum_{i,j=1}^{d} m_{k,i} m_{l,j} \int_{\mathbb{R}^d} u_i u_j d\mu(\mathbf{u}) = \sum_{i,j=1}^{d} m_{k,i} m_{l,j} a_{i,j}. \end{aligned}$$

As $(a_{i,j})_{i,j=1,\dots,d}$ is a positive definite matrix, we have an orthogonal matrix M_1 such that for $M = M_1$, the matrix $(b_{k,l})_{k,l=1,\dots,d}$ is a diagonal matrix with positive entries $b_{k,k} = \lambda_k > 0, k = 1, \dots, d$. We can now multiply M_1 by the diagonal $d \times d$ matrix $\widetilde{D}$ with the diagonal entries $d_{k,k} = \lambda_k^{-1/2}$, $k = 1, \dots, d$; and $M = \widetilde{D} M_1$ is the desired matrix. ∎

For the elliptic operator $P(D) = \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}$ and a matrix $M = (m_{i,j})$ with $\det M \neq 0$, we define $P_M(D)$ by

$$P_M(D) = \sum_{k,l=1}^{d} b_{k,l} \frac{\partial^2}{\partial y_k \partial y_l}, \quad b_{k,l} = \sum_{i,j=1}^{d} m_{k,i} m_{l,j} a_{i,j}.$$

Another version of Lemma 2 is that for the elliptic operators $P(D) = \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}$, there exists a matrix M such that $P_M(D) = \frac{\partial^2}{\partial y_1^2} + \dots + \frac{\partial^2}{\partial y_d^2} = \Delta$.

Lemma 2.5 *For a matrix M satisfying* $\det M \neq 0$*, the elliptic operator $P(D) = \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}$ and B satisfying* (1.8), (1.9), *and* (1.10)*, one has*

$$\begin{aligned} K(f, P(D), t^2)_B &\equiv \inf_g (\|f - g\|_B + t^2 \|P(D)g\|_B) \\ &\approx \inf_G (\|L_M f - G\|_B + t^2 \|P_M(D)G\|_B) \equiv K(L_M f, P_M(D), t^2)_B. \end{aligned} \tag{2.4}$$

Proof We can write

$$\begin{aligned} &\inf_G (\|L_M f - G\|_B + t^2 \|P_M(D)G\|_B) \\ &\quad \le C\big(\|L_{M^{-1}}(L_M f - g_1)\|_B + t^2 \|L_M P_M(D) L_{M^{-1}} g_1\|_B\big) \\ &\quad \le C\big(\|f - L_{M^{-1}} g_1\|_B + t^2 \|P(D) L_{M^{-1}} g_1\|_B\big). \end{aligned}$$

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Choosing g_1 close to g for which the infimum of $K(f, P(D), t^2)_B$ is achieved, we obtain

$$K(L_M f, P_M(D), t^2)_B \le CK(f, P(D), t^2)_B,$$

and as M is invertible, (2.4) follows. ■

3 The Direct Estimate

The direct result is given in the following theorem.

Theorem 3.1 *Suppose $A_t f$ given in* (1.1) *with μ satisfying* (1.2), (1.3), *and* (1.4) *for $r = 3$, and suppose further that B is a Banach space of functions satisfying* (1.8), (1.9), *and* (1.10). *Then*

$$\|f - A_t f\|_B \le C \inf_{P(D)g \in B} (\|f - g\|_B + t^2 \|P(D)g\|_B) \equiv CK(f, P(D), t^2)_B, \tag{3.1}$$

where $P(D) = \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}$ and $a_{i,j} = \int_{\mathbb{R}^d} x_i x_j d\mu(\mathbf{x})$.

The infimum in (3.1) is nominally taken on the class of functions g for which $P(D)g \in B$. The proof of Theorem 3.1 will consist of several lemmas.

Lemma 3.2 *Under the conditions of Theorem* 3.1 *we have for $\varphi \in C^{(3)}(\mathbb{R}^d)$*

$$\Big\| A_t \varphi - \varphi - \frac{t^2}{2} P(D)\varphi \Big\|_{C(\mathbb{R}^d)} \le C_1 t^3 \sup_{|\xi|=1} \Big\| \frac{\partial^3 \varphi}{\partial \xi^3} \Big\|_{C(\mathbb{R}^d)}. \tag{3.2}$$

Proof We follow [Di-Iv, Lemma 9.2] and use Taylor's formula on $\varphi(\mathbf{x} + t\mathbf{u})$, writing

$$\varphi(\mathbf{x} + t\mathbf{u}) = \varphi(\mathbf{x}) + t \sum_{j=1}^{d} u_j \frac{\partial \varphi(\mathbf{x})}{\partial x_j} + \frac{t^2}{2} \sum_{k,l=1}^{d} u_k u_l \frac{\partial^2 \varphi(\mathbf{x})}{\partial x_k \partial x_l} + R(t, \mathbf{x}, \mathbf{u})$$

where

$$R(t, \mathbf{x}, \mathbf{u}) = \frac{t^2}{2} \sum_{k,l=1}^{d} u_k u_l \Big(\frac{\partial^2 \varphi(\mathbf{v})}{\partial x_k \partial x_l} - \frac{\partial^2 \varphi(\mathbf{x})}{\partial x_k \partial x_l} \Big)$$

with $\mathbf{v} \in \mathbb{R}^d$ and $\mathbf{v} = \mathbf{v}(t, \mathbf{x}, \mathbf{u})$, a point in the segment $[\mathbf{x}, \mathbf{x} + t\mathbf{u}]$. As A_t is a positive operator on $C(\mathbb{R}^d)$,

$$\Big\| A_t \varphi - \varphi - \frac{t^2}{2} P(D)\varphi \Big\|_{C(\mathbb{R}^d)} \le \|A_t(|R(t, \mathbf{x}, \cdot)|)\|_{C(\mathbb{R}^d)}.$$

We now use (see [Ch-Di])

$$\sup_{|\xi_i|=1} \Big\| \frac{\partial^3 \varphi}{\partial \xi_1 \partial \xi_2 \partial \xi_3} \Big\|_{C(\mathbb{R}^d)} \le \sup_{|\xi|=1} \Big\| \frac{\partial^3 \varphi}{\partial \xi^3} \Big\|_{C(\mathbb{R}^d)}$$

and $|\mathbf{v}-\mathbf{x}| \le t|\mathbf{u}|$ to obtain

$$
\begin{aligned}
|R(t,\mathbf{x},\mathbf{u})| &\le \frac{t^3}{2}\sum_{k,l=1}^{d} |u_k||u_l||\mathbf{u}| \sup_{|\eta|=1}\left\| \frac{\partial}{\partial\eta}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}\varphi\right\|_{C(\mathbb{R}^d)} \\
&\le \frac{t^3}{2}|\mathbf{u}|\sum_{k,l=1}^{d} |u_k u_l| \sup_{|\xi|=1}\left\|\frac{\partial^3}{\partial\xi^3}\varphi\right\|_{C(\mathbb{R}^d)} \\
&\le \frac{t^3}{2} d|\mathbf{u}|^3 \sup_{|\xi|=1}\left\|\frac{\partial^3}{\partial\xi^3}\varphi\right\|_{C(\mathbb{R}^d)}.
\end{aligned}
$$

The use of (1.4) for $r=3$ will complete the proof of (3.2). ■

We observe that the proof of Lemma 3.2 can be simplified for the case $d=1$. We use the averaging operator on the cube in $\mathbb{R}^d$, *i.e.*,

$$
S_t f(\mathbf{x}) = \frac{1}{(2t)^d}\int_{-t}^{t}\cdots\int_{-t}^{t} f(\mathbf{x}+(u_1,\dots,u_d))du_1\dots du_d \tag{3.3}
$$

and

$$
S_t^{k+1} f(\mathbf{x}) = S_t(S_t^k f)(\mathbf{x}) \tag{3.4}
$$

as an intermediary whose approximation properties are known. In fact, we know that, for $S_t f$ and $S_t^m f$ given by (3.3) and (3.4),

$$
\|S_t f - f\|_B \approx \|S_t^m f - f\|_B + t^2\,\|\Delta S_t^m f\|_B \approx K(f,\Delta,t^2)_B \tag{3.5}
$$

for $m \ge 3$ and any B satisfying (1.8) and (1.9) and for $B = C(\mathbb{R}^d)$ (see [Di-Iv, Section 9]). It can be shown that when $f \notin UC(\mathbb{R}^d)$ but $f \in C(\mathbb{R}^d)$, neither $\|S_t f - f\|_{C(\mathbb{R}^d)} \to 0$, nor $K(f,\Delta,t^2)_{C(\mathbb{R}^d)} \to 0$ as $t \to 0+$, and hence (3.5) is not useful for such f. We now investigate the behaviour of $\widetilde{A}_t$ given by (2.2) with M given in Lemma 2.4.

Lemma 3.3 *For $\widetilde{A}_t F$ given in* (2.2) *with M of Lemma* 2.4 *and $S_t F$ of* (3.3)*, we have*

$$
\big\|\widetilde{A}_t F - F\big\|_{C(\mathbb{R}^d)} \le C\big\|S_t F - F\big\|_{C(\mathbb{R}^d)} \le \widetilde{C} K(F,\Delta,t^2)_{C(\mathbb{R}^d)}. \tag{3.6}
$$

Proof Recalling that $\widetilde{A}_t$ is a contraction, we have

$$
\begin{aligned}
\big\|\widetilde{A}_t F - F\big\|_{C(\mathbb{R}^d)} &\le \big\|\widetilde{A}_t F - \widetilde{A}_t S_t^m F\big\|_{C(\mathbb{R}^d)} + \big\|\widetilde{A}_t S_t^m F - S_t^m F\big\|_{C(\mathbb{R}^d)} + \big\|S_t^m F - F\big\|_{C(\mathbb{R}^d)} \\
&\le 2\big\|S_t^m F - F\big\|_{C(\mathbb{R}^d)} + \big\|\widetilde{A}_t S_t^m F - S_t^m F\big\|_{C(\mathbb{R}^d)}.
\end{aligned}
$$

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As $S_t^m F \in C^{(m-1)}(\mathbb{R}^d)$, we can use Lemma 3.2 for $\varphi = S_t^m F$ with $m \geq 5$ and $P(D) = \Delta$ to obtain

$$\Big\| \widetilde{A}_t S_t^m F - S_t^m F - \frac{t^2}{2} \Delta S_t^m F \Big\|_{C(\mathbb{R}^d)} \leq C_1 t^3 \sup_{|\xi|=1} \Big\| \frac{\partial^3}{\partial \xi^3} S_t^m F \Big\|_{C(\mathbb{R}^d)}.$$

We now recall from [Di-Iv, Lemma 9.3] and [Di-II, Lemma 2.1] that for $\psi \in C^{(4)}(\mathbb{R}^d)$

$$\begin{aligned} \Big\| \frac{\partial^3 \psi}{\partial \xi^3} \Big\|_{C(\mathbb{R}^d)} &\leq \sqrt{6} \sup_{h \neq 0} \big\| h^{-2} \Delta^3_{h\xi} \psi \big\|^{1/2}_{C(\mathbb{R}^d)} \cdot \Big\| \frac{\partial^4 \psi}{\partial \xi^4} \Big\|^{1/2}_{C(\mathbb{R}^d)} \\ &\leq C_2 \, \|\Delta \psi\|^{1/2}_{C(\mathbb{R}^d)} \Big\| \frac{\partial^4 \psi}{\partial \xi^4} \Big\|^{1/2}_{C(\mathbb{R}^d)}. \end{aligned} \tag{3.7}$$

Using [Di-Iv, Lemma 9.4] and $\big\| S_t^k F - S_t^l F \big\|_{C(\mathbb{R}^d)} \leq |k-l| \, \|S_t F - F\|_{C(\mathbb{R}^d)}$, we have

$$\begin{aligned} \Big\| \frac{\partial^4}{\partial \xi^4} S_t^m F \Big\|_{C(\mathbb{R}^d)} &= \Big\| \frac{\partial}{\partial \xi} S_t \frac{\partial^3}{\partial \xi^3} S_t^{m-1} F \Big\|_{C(\mathbb{R}^d)} \leq \frac{C_3}{t} \Big\| \frac{\partial^3}{\partial \xi^3} S_t^{m-1} F \Big\|_{C(\mathbb{R}^d)} \\ &\leq \frac{C_3}{t} \Big\| \frac{\partial^3}{\partial \xi^3} S_t^m F \Big\|_{C(\mathbb{R}^d)} + \frac{C_3}{t} \Big\| \frac{\partial^3}{\partial \xi^3} S_t^3 (S_t^{m-3} F - S_t^{m-4} F) \Big\|_{C(\mathbb{R}^d)} \\ &\leq \frac{C_3}{t} \Big\| \frac{\partial^3}{\partial \xi^3} S_t^m F \Big\|_{C(\mathbb{R}^d)} + \frac{C_4}{t^4} \|S_t F - F\|_{C(\mathbb{R}^d)} \\ &\leq C_5 \max\Big(t^{-1} \Big\| \frac{\partial^3}{\partial \xi^3} S_t^m F \Big\|_{C(\mathbb{R}^d)}, t^{-4} \|S_t F - F\|_{C(\mathbb{R}^d)} \Big). \end{aligned}$$

Combining all of the above, we have

$$\big\| \widetilde{A}_t F - F \big\|_{C(\mathbb{R}^d)} \leq C_6 \big(\|S_t F - F\|_{C(\mathbb{R}^d)} + t^2 \, \|\Delta S_t^m F\|_{C(\mathbb{R}^d)} \big).$$

Following (3.5), we have

$$t^2 \, \|\Delta S_t^m F\|_{C(\mathbb{R}^d)} \leq \widetilde{C}_1 \, \|S_t^m F - F\|_{C(\mathbb{R}^d)} \leq m \widetilde{C}_1 \, \|S_t F - F\|_{C(\mathbb{R}^d)},$$

and, therefore, the first inequality of (3.6). The second inequality of (3.6) follows from (3.5). ■

We now extend the result to any Banach space B satisfying (1.8) and (1.9).

Lemma 3.4 *For $F \in B$ satisfying* (1.8) *and* (1.9), *$\widetilde{A}_t$ of Lemma* 2.3, *and $S_t F$ of* (3.3)

$$\big\| \widetilde{A}_t F - F \big\|_B \leq C \, \|S_t F - F\|_B \leq \widetilde{C} K(F, \Delta, t^2)_B. \tag{3.8}$$

Proof Using (3.5), we have to show only the first inequality. For $F \in B$ satisfying (1.8) and (1.9) we take $G \in B'$ (the dual to B) such that $\|G\|_{B'} = 1$. Clearly, the convolution of F and G $\psi = F * G \in UC(\mathbb{R}^d)$, and we may choose G so that

$$|\widetilde{A}_t\psi(0) - \psi(0)| = |(\widetilde{A}_t F - F) * G(0)| \geq \left\|\widetilde{A}_t F - F\right\|_B - \varepsilon.$$

Using (3.6), we have

$$|\widetilde{A}_t\psi(0) - \psi(0)| \leq \left\|\widetilde{A}_t\psi - \psi\right\|_{C(\mathbb{R}^d)} \leq C\,\|S_t\psi - \psi\|_{C(\mathbb{R}^d)} \leq C\,\|S_t F - F\|_B\,,$$

and as $\varepsilon > 0$ is arbitrary, (3.8) follows. ■

Proof of Theorem 3.1 Using (2.3) and (2.4), we obtain (3.1) as a consequence of (3.8), where $\widetilde{A}_t$ is given by (2.2) with $\tilde{\mu}$ of (2.1) and M of Lemma 2.4. ■

4 Some Preliminary Results Needed for the Converse Theorem

Following [Di-Iv, Section 4], we need some inequalities of the Bernstein type to prove the converse inequality. The estimates needed will be proved in this section using condition (1.5), that is the Lipschitz condition on G, the k-th convolution iterate of $d\mu$. Alternatively (see Lemma 4.4), we have condition (1.7) on the asymptotic behaviour of the Fourier transform of $d\mu$. We give both conditions as each one happens to be more easily checked in some applications.

The immediate implications of (1.5) are summarized in the following lemma.

Lemma 4.1 *Suppose*

$$G \equiv G_1 \in L_1(\mathbb{R}^d),\, \|\Delta_{\mathbf{h}} G\|_{L_1(\mathbb{R}^d)} \equiv \|G(\cdot + \mathbf{h}) - G(\cdot)\|_{L_1(\mathbb{R}^d)} \leq \widetilde{C}|\mathbf{h}|^\alpha$$

for some $\alpha > 0$ and $G_m \equiv G_{m-1} * G_1$. Then

$$\left\|\Delta^l_{\mathbf{h}} G_m\right\|_{L_1(\mathbb{R}^d)} \leq \widetilde{C}_1|\mathbf{h}|^{\alpha m} \quad \text{for } l > \alpha m, \tag{4.1}$$

$$\frac{\partial^s}{\partial\xi^s} G_m \in L_1(\mathbb{R}^d) \quad \text{for } s < \alpha m, \tag{4.2}$$

$$\left\|\Delta^l_{\mathbf{h}} G_m\right\|_{L_\infty(\mathbb{R}^d)} \leq \widetilde{C}_2|\mathbf{h}|^{\alpha m - d} \quad \text{for } l > \alpha m - d > 0, \tag{4.3}$$

and

$$|\mathbf{x}|^s|\mathfrak{F}(G_m)(\mathbf{x})| \equiv |\mathbf{x}|^s|\widehat{G}_m(\mathbf{x})| \leq \widetilde{C}_3 \quad \text{for } s < \alpha m, \tag{4.4}$$

where $\Delta^l_{\mathbf{h}}\psi(\mathbf{x}) = \Delta^{l-1}_{\mathbf{h}}\Delta_{\mathbf{h}}\psi(\mathbf{x})$, and $\mathfrak{F}(G_m)(\mathbf{x}) = (\mathfrak{F}(G)(\mathbf{x}))^m$ is the Fourier transform of G_m.

Proof We write $\|\Delta^m_{\mathbf{h}} G_m\|_{L_1(\mathbb{R}^d)} = \|\Delta_{\mathbf{h}} G * \cdots * \Delta_{\mathbf{h}} G\|_{L_1(\mathbb{R}^d)} \leq \widetilde{C}^m|\mathbf{h}|^{m\alpha}$, and using the Marchaud inequality, we obtain (4.1). The Ul'yanov inequality (see for instance [Di-Pr, Theorem 2.3]) now implies (4.3). The existence of the derivative and (4.2) is routine. From (4.2) it follows that $\Delta^s G_{2m} \in L_1(\mathbb{R}^d)$ for $s < \alpha m$, and hence, $|\mathbf{x}|^{2s}|\mathfrak{F}(G_{2m})(\mathbf{x})| \leq C\,\|\Delta^s G_{2m}\|_{L_1(\mathbb{R}^d)}$, which implies (4.4). ■

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Lemma 4.2 *Suppose $G(\mathbf{u}) \ge 0$, $\int_{\mathbb{R}^d} G(\mathbf{u})d\mathbf{u} = 1$, $\|\Delta_{\mathbf{h}} G\|_{L_1(\mathbb{R}^d)} \le C|\mathbf{h}|^\alpha$ for some $\alpha > 0$, and $\int_{\mathbb{R}^d} |\mathbf{u}|^{d+1} G(\mathbf{u})d\mathbf{u} \le C$. Then for any $\varepsilon > 0$ and $l, r = 1, 2, \dots$, there exists $k = k(\varepsilon, l, r)$ such that*

$$\|\Delta^l G_k\|_{L_1(\mathbb{R}^d)} \le \varepsilon \quad \text{and} \quad \left\|\frac{\partial^r}{\partial \xi^r} G_k\right\|_{L_1(\mathbb{R}^d)} \le \varepsilon \tag{4.5}$$

where Δ is the Laplacian and $G_k = G_{k-1} * G$.

Proof Using iterations and the Kolmogorov-type inequality given by

$$\left\|\frac{\partial^r}{\partial \xi^r} G_k\right\|_{L_1(\mathbb{R}^d)} \le K \|\Delta^l G_k\|_{L_1(\mathbb{R}^d)}^{\frac{r}{2l}} \|G_k\|_{L_1(\mathbb{R}^d)}^{1-\frac{r}{2l}} \quad \text{for } r < 2l$$

(see [Di-I, Theorem 6.2] with $B = L_1(\mathbb{R}^d)$), we have to prove only the first inequality of (4.5) for some fixed l. (We choose $l = d + 3$.) It was shown in the proof of [St-We, Lemma 3.17, p. 26] that

$$\|\psi\|_{L_1(\mathbb{R}^d)} \le C \sum_{|\beta| \le d+1} \|D^\beta \widehat{\psi}\|_{L_1(\mathbb{R}^d)}, \tag{4.6}$$

where $D^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\beta_d}$ and $|\beta| = \beta_1 + \dots + \beta_d$.

We apply (4.6) to $\psi = \Delta^l G_k$ with k to be chosen later. Therefore, we have to show

$$\|D^\beta\{|\mathbf{x}|^{2l} \widehat{G}(\mathbf{x})^k\}\|_{L_1(\mathbb{R}^d)} \le \varepsilon_1 \quad \text{for } |\beta| \le d + 1 \text{ and } k \ge k_0. \tag{4.7}$$

Under the conditions of our lemma and using (4.4) of Lemma 4.1, we have

$$|\widehat{G}(\mathbf{x})| \le 1, \ |D^\gamma \widehat{G}(\mathbf{x})| \le C, \ |\mathbf{x}|^{\alpha_1} |\widehat{G}(\mathbf{x})| \le C \quad \text{for } \alpha_1 < \alpha, \text{ and } |\gamma| \le d + 1. \tag{4.8}$$

We set $\mathbb{R}^d = D_1 \cup D_2 \cup D_3$, where $D_1 = \{\mathbf{x} : |\mathbf{x}| \le r\}$, $D_2 = \{\mathbf{x} : r < |\mathbf{x}| < R\}$, and $D_3 = \{\mathbf{x} : |\mathbf{x}| \ge R\}$, and we estimate $\|D^\beta\{|\mathbf{x}|^{2l}\widehat{G}(\mathbf{x})^k\}\|_{L_1(D_j)}$ to obtain (4.7). We deal first with the estimate on D_3. We choose R (using (4.8)) such that $|\widehat{G}(\mathbf{x})| \le \frac{1}{2}$ for $|\mathbf{x}| \ge R$. As $|\widehat{G}(\mathbf{x})| \le C|\mathbf{x}|^{-\alpha_1}$ for $\alpha_1 < \alpha$, $|\widehat{G}(\mathbf{x})|^m \le C^m |\mathbf{x}|^{-d-2l-1}$ for $m = [\frac{d+2l+2}{\alpha}]$, and as $|D^\gamma \widehat{G}(\mathbf{x})| \le C$ for $|\gamma| \le d + 1$, we have

$$\begin{aligned} |D^\gamma\{\widehat{G}(\mathbf{x})^k\}| &\le Ck^{|\gamma|} |\widehat{G}(\mathbf{x})|^{k-|\gamma|} \le \widetilde{C} k^{|\gamma|} |\widehat{G}(\mathbf{x})|^{k-m-|\gamma|} |\mathbf{x}|^{-d-2l-1} \\ &\le C' k^{|\gamma|} \frac{1}{2^k} |\mathbf{x}|^{-d-2l-1}. \end{aligned}$$

Since $|D^\gamma |\mathbf{x}|^{2l}| \le C \max(|\mathbf{x}|^{2l}, 1)$ for all γ, we conclude that

$$\|D^\beta\{|\mathbf{x}|^{2l}\widehat{G}(\mathbf{x})^k\}\|_{L_1(D_3)} \le C_2 k^{|\beta|} \frac{1}{2^k} \le C_2 \frac{k^{d+1}}{2^k},$$

which is small enough provided that $k \geq k_3$.

We choose l such that $l = d + 3$ (as it is sufficient to prove (4.5) for some fixed l) and we estimate $\|D^\beta\{|\mathbf{x}|^{2l}\widehat{G}(\mathbf{x})^k\}\|_{L_1(D_1)}$ on $D_1 = \{\mathbf{x} : |\mathbf{x}| \leq k^{-\lambda}, \lambda = \frac{d+1}{2(d+2)}\}$, that is, $r = k^{-\lambda}$ with $\lambda < \frac{1}{2}$. Using $|D^\gamma|\mathbf{x}|^{2l}| \leq C|\mathbf{x}|^{2l-d-1}$ for $|\gamma| \leq d+1$, we have for $|\beta| \leq d+1$

$$\|D^\beta\{|\mathbf{x}|^{2l}\widehat{G}(\mathbf{x})^k\}\|_{L_1(D_1)} \leq C_1 k^{d+1}(k^{-\lambda})^{2l-d-1}(k^{-\lambda})^d = C_1 k^{d+1}k^{-\lambda(2d+4)}k^{-\lambda} = Ck^{-\lambda},$$

and $\|D^\beta\{|\mathbf{x}|^{2l}\widehat{G}(\mathbf{x})^k\}\|_{L_1(D_1)}$ is sufficiently small for $k \geq k_1 \geq k_3$.

Using (4.3) for m satisfying $\alpha m - d > 1$, and recalling $\int_{\mathbb{R}^d} G_m(\mathbf{u})d\mathbf{u} = 1$, there exists a point $\mathbf{u}_0$ such that $G_m(\mathbf{u}_0) = a > 0$. Using (4.3) again (with $\alpha m - d > 1$), we have

$$|G_m(\mathbf{u}+\mathbf{h}) - G_m(\mathbf{u})| \leq A|\mathbf{h}| \quad \text{for all } \mathbf{u}, \mathbf{h} \in \mathbb{R}^d,$$

and therefore,

$$G_m(\mathbf{u}) \geq \frac{a}{2} \quad \text{for } \Big\{\mathbf{u} : |\mathbf{u} - \mathbf{u}_0| \leq \frac{a}{2A}\Big\}.$$

We now define $\psi_{a,R}(\mathbf{u}) \equiv \psi(\mathbf{u})$ with R of $D_3(R)$ by

$$\psi(\mathbf{u}) = \psi_{a,R}(\mathbf{u}) = \begin{cases} \frac{a}{2}, & |\mathbf{u} - \mathbf{u}_0| \leq \min\big(\frac{a}{2A}, \frac{\pi}{R}\big), \\ 0, & \text{otherwise.} \end{cases}$$

This implies $\int_{\mathbb{R}^d} \psi_{a,R}(\mathbf{u})d\mathbf{u} = b$ with $0 < b < 1$ and b depending only on m, A, a, and R. To estimate the Fourier transform of G_m, we write $\widehat{G}_m(\mathbf{x}) = (G_m - \psi_{a,R})\widehat{\ }(\mathbf{x}) + \widehat{\psi}_{a,R}(\mathbf{x})$, and as $G_m(\mathbf{u}) - \psi_{a,R}(\mathbf{u}) \geq 0$, $|(G_m - \psi_{a,R})\widehat{\ }(\mathbf{x})| \leq 1 - b$. The Fourier transform of $\psi_{a,R}$ is given by

$$\begin{aligned}\widehat{\psi}_{a,R}(\mathbf{x}) &= e^{-i\mathbf{u}_0\mathbf{x}}\frac{a}{2}\int_{|\mathbf{v}|\leq\min(\frac{a}{2A},\frac{\pi}{R})} e^{-i\mathbf{x}\mathbf{v}}d\mathbf{v} = e^{-i\mathbf{u}_0\mathbf{x}}\frac{a}{2}\int_{|\mathbf{v}|\leq\min(\frac{a}{2A},\frac{\pi}{R})} \cos\mathbf{x}\mathbf{v}d\mathbf{v} \\ &= e^{-i\mathbf{u}_0\mathbf{x}}\frac{a}{2}\int_{|\mathbf{v}|\leq\min(\frac{a}{2A},\frac{\pi}{R})} d\mathbf{v} - e^{-i\mathbf{u}_0\mathbf{x}}a\int_{|\mathbf{v}|\leq\min(\frac{a}{2A},\frac{\pi}{R})} \sin^2\frac{\mathbf{x}\mathbf{v}}{2}d\mathbf{v}.\end{aligned}$$

Hence, using $\sin^2\frac{\xi}{2} \geq (\frac{\xi}{\pi})^2$ for $|\xi| \leq \pi$, we have for $|\mathbf{x}| \leq R$

$$\begin{aligned}|\widehat{\psi}_{a,R}(\mathbf{x})| &= b - a\int_{|\mathbf{v}|\leq\min(\frac{a}{2A},\frac{\pi}{R})} \sin^2\frac{\mathbf{x}\mathbf{v}}{2}d\mathbf{v} \\ &\leq b - \frac{a}{\pi^2}\int_{|\mathbf{v}|\leq\min(\frac{a}{2A},\frac{\pi}{R})} (\mathbf{x}\mathbf{v})^2 d\mathbf{v} \\ &\leq b - \frac{a}{\pi^2}|\mathbf{x}|^2\int_{|\mathbf{v}|\leq\min(\frac{a}{2A},\frac{\pi}{R})} \Big(\frac{\mathbf{x}\mathbf{v}}{|\mathbf{x}||\mathbf{v}|}\Big)^2|\mathbf{v}|^2 d\mathbf{v}.\end{aligned}$$

Since

$$\int_{|\mathbf{v}|\leq\min(\frac{a}{2A},\frac{\pi}{R})} \Big(\frac{\mathbf{x}\mathbf{v}}{|\mathbf{x}||\mathbf{v}|}\Big)^2|\mathbf{v}|^2 d\mathbf{v} \geq c > 0,$$

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where $c \equiv c(a, A, R, d)$ does not depend on $\mathbf{x}$, we have

$$|\widehat{\psi}_{a,R}(\mathbf{x})| \le b - \frac{a}{\pi^2} c|\mathbf{x}|^2 = b - c_1|\mathbf{x}|^2,$$

and hence $|\widehat{G}_m(\mathbf{x})| \le 1 - c|\mathbf{x}|^2$ for $|\mathbf{x}| \le R$. To estimate $\|D^\beta\{|\mathbf{x}|^{2l}\widehat{G}(\mathbf{x})^k\}\|_{L_1(D_2)}$, we use the fact that for $2l - d - 1 > 0$ and $|\gamma| < |\beta| \le d + 1$, the function $|\mathbf{x}|^{2l-|\gamma|}(1 - c|\mathbf{x}|^2)^{k_1}$ attains maximum at $|\mathbf{x}|^2 \approx \frac{1}{k_1}$ with $k_1 = k/m$, and is decreasing for $|\mathbf{x}|^2 \ge \frac{c}{k}$. Using (4.8) and the estimate above, we have for $|\beta| \le d + 1$

$$\|D^\beta\{|\mathbf{x}|^{2l}\widehat{G}(\mathbf{x})^k\}\|_{L_1(D_2)} \le \widetilde{C}(k^{-\lambda})^{2l-d-1}(1 - Ak^{-\lambda})^k k^{d+1},$$

and as $\lambda < \frac{1}{2}$, the last expression is sufficiently small when $k \ge k_2 \ge k_1 \ge k_3$.

Therefore, for $l = d+3$, we proved $\left\|\Delta^l G_k\right\|_{L_1(\mathbb{R}^d)} < \varepsilon$ provided that k is sufficiently large. ∎

As a corollary of Lemma 4.2, we have the following useful result.

Lemma 4.3 *Suppose μ is a positive measure satisfying $\int_{\mathbb{R}^d} d\mu(\mathbf{u}) = 1$, as well as* (1.2), (1.3), (1.4) *with $r = d + 1$, and* (1.5). *Then for $\varepsilon > 0$ and each r and l, there exists k such that*

$$\left\| \frac{\partial^r}{\partial \xi^r} t^{-d} G_k\left(\frac{\mathbf{x}}{t}\right) \right\|_{L_1(\mathbb{R}^d)} \le \frac{\varepsilon}{t^r} \quad \text{and} \quad \left\| \Delta^{2l} t^{-d} G_k\left(\frac{\mathbf{x}}{t}\right) \right\|_{L_1(\mathbb{R}^d)} \le \frac{\varepsilon}{t^{2l}}, \tag{4.9}$$

where $\widehat{G}_k(\mathbf{x}) = (\widehat{d\mu}(\mathbf{x}))^{mk}$ with m of (1.6). *Moreover, for A_t given by* (1.1), *$A_t^k f = A_t(A_t^{k-1} f)$ and $\varepsilon > 0$, there exists k such that*

$$\left\| \frac{\partial^r}{\partial \xi^r} A_t^k f \right\|_B \le \frac{\varepsilon}{t^r} \|f\|_B \tag{4.10}$$

for any B satisfying (1.8) *and* (1.9).

Proof The proof of (4.9) is essentially just a change of variable in (4.5). The inequality (4.10) for $B = C(\mathbb{R}^d)$ follows immediately from (4.5). To derive (4.10) for all Banach spaces B satisfying (1.8) and (1.9), we follow an often used technique (see [Ch-Di, Di-Iv, Di-II] and Lemma 3.4). ∎

We also have an alternative condition to (1.5) which sometimes is more accessible.

Lemma 4.4 *Suppose a positive measure $d\mu$ satisfies* (1.2), (1.3), *and* (1.4) *for $r = d + 1$. Then*

(i) $|\mathbf{x}|^{\alpha_1}|\widehat{d\mu}(\mathbf{x})| \le A_1$ *for some $\alpha_1 > 0$, and*

(ii) $\|\Delta_{\mathbf{h}} G\|_{L_1(\mathbb{R}^d)} \le A|\mathbf{h}|^\alpha$ *for some $\alpha > 0$ where $\widehat{G}(\mathbf{x}) = (\widehat{d\mu}(\mathbf{x}))^m$ for some integer m, are equivalent.*

Proof Using (4.4), (ii) implies (i). Using (1.4), $|D^\beta \widehat{d\mu}(\mathbf{x})| \le C$ for $|\beta| \le d + 1$. We can now follow the proof of Lemma 4.2 (with $l = d + 3$) and use I to obtain $|D^\beta(\widehat{d\mu}(\mathbf{x}))^m| \le C$ for $|\beta| \le d + 1$, and $\|D^\beta\{|\mathbf{x}|^{2l}(\widehat{d\mu}(\mathbf{x}))^m\}\|_{L_1(\mathbb{R}^d)} \le C_1$ for all β satisfying $|\beta| \le d + 1$ and some integer m and therefore, $G \in L_1(\mathbb{R}^d)$, $\widehat{G}(\mathbf{x}) = (\widehat{d\mu}(\mathbf{x}))^m$ and $\|G(\cdot + \mathbf{h}) - G(\cdot)\|_{L_1(\mathbb{R}^d)} \le C_2|\mathbf{h}|$. ∎

Remark 4.5 For $L_2(\mathbb{R}^d)$, the proof of

$$\Big\| \frac{\partial^r}{\partial \xi^r} A_t^k f \Big\|_{L_2(\mathbb{R}^d)} \le \frac{\varepsilon}{t^r} \|f\|_{L_2(\mathbb{R}^d)}$$

is simpler and does not use (1.4) as we have to show only that the multiplier $|\mathbf{x}|^{2l}(\widehat{d\mu}(\mathbf{x}))^k$ is small provided that k is big enough.

5 The Converse Result

In Section 4 we proved the main ingredients for the converse result. We now state and prove the converse theorem which, together with the direct result given in Section 3, will complete the proof of the equivalence theorem *i.e.,* Theorem 1.1.

Theorem 5.1 *Under the conditions of Theorem* 1.1,

$$\inf_{P(D)g\in B} (\|f-g\|_B + t^2 \|P(D)g\|_B) \le C \|f - A_t f\|_B . \tag{5.1}$$

Proof We first prove our result for $\widetilde{A}_t F$ for which $F \in C(\mathbb{R}^d)$, $F(\mathbf{x}) = f(M^{-1}\mathbf{x})$ and $P_M(D) = \Delta$. Using Lemma 3.2, we write

$$\Big\| \widetilde{A}_t^{l+1} F - \widetilde{A}_t^l F - \frac{t^2}{2} \widetilde{A}_t^l F \Big\|_{C(\mathbb{R}^d)} \le C_1 t^3 \sup_{|\xi|=1} \Big\| \frac{\partial^3}{\partial \xi^3} \widetilde{A}_t^l F \Big\|_{C(\mathbb{R}^d)}$$

with $l \ge l_0$ and l_0 sufficiently large so that $\frac{\partial^4}{\partial \xi^4} \widetilde{A}_t^l F \in C(\mathbb{R}^d)$, and this is possible by combining (4.2), (1.5), and (1.6). Moreover, we will select l to be bigger later in the proof when the need arises. We now use (3.7) (implied by [Di-Iv, Lemma 9.3] and [Di-II, Lemma 2.1]) with $\psi = \widetilde{A}_t^l F$ to obtain

$$\Big\| \frac{\partial^3}{\partial \xi^3} \widetilde{A}_t^l F \Big\|_{C(\mathbb{R}^d)} \le C_2 \| \Delta \widetilde{A}_t^l F \|_{C(\mathbb{R}^d)}^{1/2} \cdot \Big\| \frac{\partial^4}{\partial \xi^4} \widetilde{A}_t^l F \Big\|_{C(\mathbb{R}^d)}^{1/2}.$$

We choose $l_1 < l$ large enough so that $\widetilde{A}_t^{l_1} F \in C^{(3)}(\mathbb{R}^d)$, which is possible following Lemma 4.1 and (1.5), and for a given $\varepsilon_1 > 0$, we choose $l - l_1$ large enough so that

$$\Big\| \frac{\partial}{\partial \xi} \widetilde{A}_t^{l-l_1} \Psi \Big\|_{C(\mathbb{R}^d)} \le \frac{\varepsilon_1}{t} \|\Psi\|_{C(\mathbb{R}^d)}$$

which is possible following (4.10) of Lemma 4.3. Following [Di-Iv, Section 9], we can now write

$$\begin{aligned}
\Big\| \frac{\partial^4}{\partial \xi^4} \widetilde{A}_t^l F \Big\|_{C(\mathbb{R}^d)} &\le \frac{\varepsilon_1}{t} \Big\| \frac{\partial^3}{\partial \xi^3} \widetilde{A}_t^{l_1} F \Big\|_{C(\mathbb{R}^d)} \\
&\le \frac{\varepsilon_1}{t} \Big\| \frac{\partial^3}{\partial \xi^3} \widetilde{A}_t^l F \Big\|_{C(\mathbb{R}^d)} + \frac{\varepsilon_1}{t} \Big\| \frac{\partial^3}{\partial \xi^3} (\widetilde{A}_t^l - \widetilde{A}_t^{l_1}) F \Big\|_{C(\mathbb{R}^d)} \\
&\le \frac{\varepsilon_1}{t} \Big\| \frac{\partial^3}{\partial \xi^3} \widetilde{A}_t^l F \Big\|_{C(\mathbb{R}^d)} + \frac{\varepsilon_1}{t^4} C_2 (l - l_1) \big\| \widetilde{A}_t F - F \big\|_{C(\mathbb{R}^d)} \\
&\le 2 \max\Big(\frac{\varepsilon_1}{t} \Big\| \frac{\partial^3}{\partial \xi^3} \widetilde{A}_t^l F \Big\|_{C(\mathbb{R}^d)}, \frac{\varepsilon_1 C_2 (l - l_1)}{t^4} \big\| \widetilde{A}_t F - F \big\|_{C(\mathbb{R}^d)} \Big).
\end{aligned}$$

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Combining the above, we have

$$C_1 t^3 \Big\| \frac{\partial^3}{\partial \xi^3} \widetilde{A}_t^l F \Big\|_{C(\mathbb{R}^d)} m \le C_3 \big(t^2 \big\| \Delta \widetilde{A}_t^l F \big\|_{C(\mathbb{R}^d)} \big)^{1/2} \Big[\max\Big(\varepsilon_1 t^3 \Big\| \frac{\partial^3}{\partial \xi^3} \widetilde{A}_t^l F \Big\|_{C(\mathbb{R}^d)}, \varepsilon_1 (l - l_1) \big\| \widetilde{A}_t F - F \big\|_{C(\mathbb{R}^d)} \Big) \Big]^{1/2}.$$

Therefore, we have

$$\big\| \widetilde{A}_t^{l+1} F - \widetilde{A}_t^l F - \frac{t^2}{2} \Delta \widetilde{A}_t^l F \big\|_{C(\mathbb{R}^d)} \le C_4 \max\Big(\sqrt{\varepsilon_1} t^2 \big\| \Delta \widetilde{A}_t F \big\|_{C(\mathbb{R}^d)}, \big\| \widetilde{A}_t F - F \big\|_{C(\mathbb{R}^d)} \Big)$$

and

$$t^2 \big\| \Delta \widetilde{A}_t^l F \big\|_{C(\mathbb{R}^d)} \le C_5 \big\| \widetilde{A}_t F - F \big\|_{C(\mathbb{R}^d)}, \tag{5.2}$$

which implies (5.1) with $g = \widetilde{A}_t^l F$ for the operator $\widetilde{A}_t$ and $B = C(\mathbb{R}^d)$.

The technique used in Lemma 3.4 (and elsewhere) now extends (5.2) to all B satisfying (1.8) and (1.9). The considerations in Section 2 and in particular (2.3) of Lemma 2.3 and (2.4) of Lemma 2.5 transfer (5.1) from $\widetilde{A}_t F$ to $A_t f$ with Δ becoming $P(D)$. ■

6 Applications

In this section we will show that the equivalence Theorem 1.1 is applicable to many averaging operators.

Theorem 6.1 *Suppose $S \subset \mathbb{R}^d$ is a bounded convex set with an interior point and center of gravity at $(0, \dots, 0)$, and suppose $G(\mathbf{u}) = \frac{1}{m(S)} \chi_S(\mathbf{u})$, where $m(S)$ is the Lebesgue measure of S and $\chi_S(\mathbf{u})$ is the characteristic function of S. Then*

$$\|A_t f - f\|_B \approx \inf_{P(D) g \in B} (\|f - g\|_B + t^2 \|P(D) g\|_B), \tag{6.1}$$

where

$$A_t f(x) = \frac{1}{m(S)} \int_{\mathbb{R}^d} f(\mathbf{x} + t\mathbf{u}) \chi_S(\mathbf{u}) d\mathbf{u},$$

$P(D)$ is given by (1.12) *for $d\mu(\mathbf{u}) = G(\mathbf{u}) d\mathbf{u}$, and B is a Banach space satisfying* (1.8), (1.9), *and* (1.10).

Proof The boundedness of S implies $m(S) < \infty$, and, setting $d\mu(\mathbf{u}) = G(\mathbf{u}) d\mathbf{u}$, it implies (1.4) for all r and the right-hand inequality of (1.3). The left-hand inequality of (1.3) follows from the fact that S has an interior point. We prove condition (1.5) for $\alpha = 1$, where m of (1.6) equals 1 *i.e.,* that

$$\int_{\mathbb{R}^d} |G(\mathbf{u} + \mathbf{h}) - G(\mathbf{u})| d\mathbf{u} \le M |\mathbf{h}|. \tag{6.2}$$

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To establish (6.2), we define for any $\mathbf{h} \in \mathbb{R}^d$

$$S_{\mathbf{h}} = \{\mathbf{v} : \mathbf{v} = \mathbf{u} + a\mathbf{h}, 0 \le a \le 1, \mathbf{u} \in S\}$$

and ${}_{\mathbf{h}}S = \{\mathbf{v} : \mathbf{v}, \mathbf{v} - \mathbf{h} \in S\}$. We let A be the orthogonal projection of S on the hyperplane perpendicular to $\mathbf{h}$. Clearly,

$$m(S_{\mathbf{h}}) \le m(S) + |\mathbf{h}| m(A) \le m(S) + |\mathbf{h}| m(\partial S)$$

and

$$m({}_{\mathbf{h}}S) \ge m(S) - |\mathbf{h}| m(A) \ge m(S) - |\mathbf{h}| m(\partial S),$$

where $m(A)$ and $m(\partial S)$ are the Lebesgue measures of A and of the boundary of S denoted by ∂S. For any $\mathbf{h} \in \mathbb{R}^d$, we now have

$$\frac{1}{m(S)} \int_{\mathbb{R}^d} |\chi_S(\mathbf{x}+\mathbf{h}) - \chi_S(\mathbf{x})| d\mathbf{x} = \frac{1}{m(S)} \int_{S_{\mathbf{h}} \setminus {}_{\mathbf{h}}S} d\mathbf{x} \le 2|\mathbf{h}| \frac{m(\partial S)}{m(S)},$$

which is (6.2). Satisfying all the conditions of Theorem 1.1, we have (6.1). ■

Remark 6.2 In the application given in Theorem 6.1, it is clear that $P(D)$ does not have to be the Laplacian. Any affine transformation of the simplex, cube, or ball in $\mathbb{R}^d$ satisfies the condition when the center of gravity is moved to $(0, \dots, 0)$. For the ball and the cube, (6.1) was proved in [Di-Ru] and [Di-Iv, Section 9], respectively. Other applications are half of (or other part of) the ball, a cone $0 \le \mathbf{x} \cdot \mathbf{v} \le \cos\theta$ for fixed $\mathbf{v}$ and θ, cut by a hyperplane not passing $(0, \dots, 0)$ and containing a positive multiple of $\mathbf{v}$, or by the ball $|\mathbf{x}| \le a$ (an ice cream-type cone), all properly centered. In fact, one can use sets which are not convex (but with $(0, \dots, 0)$ as their center of gravity), like for instance the set between two balls, cubes or simplices, one inside the other.

We now give an application for which (1.5) is satisfied with some α, $0 < \alpha < 1$.

Theorem 6.3 *Suppose* $\mathbf{u} \in \mathbb{R}^d$, $0 < \alpha < 1$, $d\mu(\mathbf{u}) = G(\mathbf{u}) d\mathbf{u}$,

$$G(\mathbf{u}) = \begin{cases} M|\mathbf{u}|^{\alpha-d} & |\mathbf{u}| \le 1 \\ 0 & \textit{otherwise} \end{cases}, \qquad \int_{\mathbb{R}^d} G(\mathbf{u}) d\mathbf{u} = 1$$

and B *satisfies* (1.8) *and* (1.9), *and suppose also that*

$$A_t f(\mathbf{x}) = \int_{\mathbb{R}^d} f(\mathbf{x} + t\mathbf{u}) G(\mathbf{u}) d\mathbf{u}.$$

Then (6.1) *is satisfied with* $P(D) = \Delta$.

Proof We have to verify the assumption of Theorem 1.1 for $d\mu(\mathbf{u}) = G(\mathbf{u}) d\mathbf{u}$, of which all but (1.5) (or equivalently (1.7)) are immediate. To verify (1.5), we note that for $|\mathbf{x}| \ge 1 + |\mathbf{h}|$ both $G(\mathbf{x})$ and $G(\mathbf{x} + \mathbf{h})$ equal zero, and write

$$\int_{\mathbb{R}^d} |G(\mathbf{x}) - G(\mathbf{x}+\mathbf{h})| d\mathbf{x} = \left\{ \int_{D_1} + \int_{D_2} + \int_{D_3} \right\} |G(\mathbf{x}) - G(\mathbf{x}+\mathbf{h})| d\mathbf{x} \equiv I_1 + I_2 + I_3,$$

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where $D_1 = \{\mathbf{x} : |\mathbf{x}| \le 2|\mathbf{h}|\}$, $D_2 = \{\mathbf{x} : 2|\mathbf{h}| < |\mathbf{x}| < 1 - |\mathbf{h}|\}$ and $D_3 = \{\mathbf{x} : 1 - |\mathbf{h}| \le |\mathbf{x}| \le 1 + |\mathbf{h}|\}$. We now have

$$I_1 \le \int_{D_1} \big(|G(\mathbf{x})| + |G(\mathbf{x}+\mathbf{h})|\big)\,d\mathbf{x} \le \int_{|\mathbf{x}|\le 2|\mathbf{h}|} |G(\mathbf{x})|d\mathbf{x} + \int_{|\mathbf{x}|\le 3|\mathbf{h}|} |G(\mathbf{x})|d\mathbf{x}$$
$$\le M\Big\{\int_{|\mathbf{x}|\le 2|\mathbf{h}|} |\mathbf{x}|^{\alpha-d}d\mathbf{x} + \int_{|\mathbf{x}|\le 3|\mathbf{h}|} |\mathbf{x}|^{\alpha-d}d\mathbf{x}\Big\} \le M_1|\mathbf{h}|^{\alpha}.$$

To estimate I_2, we write $G(\mathbf{x}+\mathbf{h}) - G(\mathbf{x}) = |\mathbf{h}|\frac{\partial}{\partial\eta}G(\xi)$, where η ($|\eta| = 1$) is in the direction of $\mathbf{h}$ and ξ is a point between $\mathbf{x}$ and $\mathbf{x}+\mathbf{h}$. Therefore in D_2, $|G(\mathbf{x})-G(\mathbf{x}+\mathbf{h})| \le M_2|\mathbf{h}||\mathbf{x}|^{\alpha-d-1}$ and

$$I_2 \le M_2|\mathbf{h}| \int_{2|\mathbf{h}|\le|\mathbf{x}|\le 1-|\mathbf{h}|} |\mathbf{x}|^{\alpha-d-1}d\mathbf{x} \le M_3|\mathbf{h}|^{\alpha}.$$

For $\mathbf{x} \in D_3$, the estimates $|G(\mathbf{x})|, |G(\mathbf{x}+\mathbf{h})| \le M(1-|\mathbf{h}|)^{\alpha-d} \le M_4$ hold, and hence

$$I_3 \le \int_{D_3} (|G(\mathbf{x})| + |G(\mathbf{x}+\mathbf{h})|)d\mathbf{x} \le 2M_4 m(D_3) \le M_5|\mathbf{h}|. \qquad \blacksquare$$

As in both above applications $d\mu(\mathbf{u})$ had compact support, we put forward an example for which $d\mu(\mathbf{u})$ has all of $\mathbb{R}^d$ as its support.

Theorem 6.4 *Suppose* $\mathbf{u} \in \mathbb{R}^d$, $\beta > 0$, $d\mu(\mathbf{u}) = G(\mathbf{u})d\mathbf{u}$,

$$G(\mathbf{u}) = Me^{-|\mathbf{u}|^{\beta}}, \qquad \int_{\mathbb{R}^d} G(\mathbf{u})d\mathbf{u} = 1$$

and B satisfies (1.8) *and* (1.9). *Then* (6.1) *is satisfied with* $P(D) = \Delta$ *and* $A_t f(\mathbf{x}) = \int_{\mathbb{R}^d} f(\mathbf{x}+t\mathbf{u})G(\mathbf{u})d\mathbf{u}$.

The proof of Theorem 6.4 consists of verifying (1.2)–(1.5) which are essentially straightforward.

7 Applications when $d\mu(\mathbf{u})$ is Singular

In this section we give applications of Theorem 1.1 when $d\mu(\mathbf{u})$ is supported by a set of Lebesgue measure zero in $\mathbb{R}^d$ and hence is singular. It is well known that in $\mathbb{R}$ (or T) for the operator

$$A_h f(x) = \frac{1}{2}f(x-h) + \frac{1}{2}f(x+h),$$

one has

$$\sup_{0<h\le t} \|A_h f - f\|_B \equiv \omega^2(f,t)_B \approx \inf_{g''\in B}\big(\|f-g\|_B + t^2\,\|g''\|_B\big), \tag{7.1}$$

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and simple examples can be given to show that the supremum cannot be replaced by $\|A_h f - f\|_B$ for a single $h \approx t$. Similarly, for $\mathbf{x} \in \mathbb{R}^d$ and

$$A_h f(\mathbf{x}) = \frac{1}{2d} \sum_{j=1}^{d} [f(\mathbf{x} + h\mathbf{e}_j) + f(\mathbf{x} - h\mathbf{e}_j)],$$

where $\{\mathbf{e}_j\}_{j=1}^d$ is a set of orthogonal unit vectors in $\mathbb{R}^d$, one has

$$(7.1)' \qquad \sup_{0<h\leq t} \|A_h f - f\|_B \approx \inf_{\Delta g \in B} \big(\|f - g\|_B + t^2 \|\Delta g\|_B \big),$$

and the supremum on the left of $(7.1)'$ cannot be dropped. However, it was shown in [Be-Da-Di] that averages on the circle or sphere ($d > 1$) do yield a strong converse inequality of type A. In this section we will show that the result in [Be-Da-Di] is in fact part of much wider phenomena.

Theorem 7.1 *Suppose that $E \subset \mathbb{R}^d$, $d > 1$, is a boundary of a bounded convex set S, $E = \partial S$, and that the Gaussian curvature of E is different from 0 everywhere. Suppose further that $d\mu(\mathbf{u})$ is the $d-1$ Lebesgue measure of E normalized to satisfy $\int_E d\mu(\mathbf{u}) = 1$ and that the center of gravity of E with respect to $d\mu(\mathbf{u})$ is $(0, \dots, 0)$. Then*

$$\|A_t f - f\|_B \approx \inf_{P(D)g \in B} \big(\|f - g\|_B + t^2 \|P(D)g\|_B \big),$$

where A_t is given by (1.1) i.e.,

$$A_t f(\mathbf{x}) = \int_E f(\mathbf{x} + t\mathbf{u}) d\mu(\mathbf{u}) \equiv \int_{\mathbb{R}^d} f(\mathbf{x} + t\mathbf{u}) d\mu(\mathbf{u}),$$

B satisfies (1.8), (1.9), *and* (1.10), *and $P(D)$ is given by* (1.12).

We recall that the Gaussian curvature at a given point $\mathbf{u} \in E$ (see [St, p. 348]) can be described as follows. For the point $\mathbf{u} \in E$ moved to the origin, the tangent plane to E at $\mathbf{u}$ (now the origin) is spanned by the orthogonal vectors $x_1, \dots, x_{d-1}$, and in the neighbourhood of $\mathbf{u}$ the surface E is described by $x_d = \varphi(x_1, \dots, x_{d-1})$ (where $x_d \perp x_j$ $j < d$). The Gaussian curvature at $\mathbf{u}$ is now given by the determinant of the matrix $\frac{\partial\varphi(x_1,\dots,x_{d-1})}{\partial x_l \partial x_k}$ at the origin. In short, we require that any plane containing $\mathbf{u} \in E$ and the perpendicular to the tangent plane of the surface E at $\mathbf{u}$ intersects E with a curve which, written as a function of x representing the perpendicular vector to the tangent plane, has a second derivative different from 0 at $\mathbf{u}$. It is assumed that the boundary is smooth enough, that is, for any point $\mathbf{u}$, φ defined locally, belongs to C^2_{loc}.

Proof The conditions (1.2), (1.3), and (1.4) follow easily. We use [St, p. 348, (27)] to obtain $|\widehat{d\mu}(\mathbf{x})| \leq M|\mathbf{x}|^{(1-d)/2}$, which implies (1.7) for $d > 1$. ■

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Remark 7.2 (i) It was not necessary to have the Gaussian curvature different from 0. In fact, we need (1.7) and not the stronger condition $|\widehat{d\mu}(\mathbf{x})| \leq M|\mathbf{x}|^{(1-d)/2}$. For example, the measure given by the uniform weight on the cylindrical body

$$E = \{\mathbf{x} = (x_1, \dots, x_d) : x_1^2 + \cdots + x_{d-1}^2 = 1, |x_d| \leq 1\}$$

for $d \geq 3$ easily satisfies (1.7) (with $\alpha_1 = \frac{1}{2}$ when $d = 3$) and hence the conditions of Theorem 1.1.

(ii) The uniform weight on part of a sphere or a boundary of a bounded convex set of Gaussian curvature different from zero, centered at the origin, also satisfies the conditions of Theorem 1.1.

(iii) Sums of measures that satisfy the conditions of Theorem 1.1 properly centered will also satisfy the conditions of Theorem 1.1.

(iv) A guide to m-dimensional bounded manifolds ($m < d$) on which a normalized uniform weight will satisfy the conditions of Theorem 1.1 is given in [St, p. 351, Theorem 2].

(v) We conjecture that the measure given by the normalized uniform weight on the boundary of a cube or a simplex properly centered will also yield (1.11). Unfortunately, the methods of this paper cannot imply it, as neither (1.7) nor (1.5) is valid.

(vi) In this section, the support of the measure is of very smooth character. It is interesting to ask about implications such as (1.7) for other situations. For instance, is (1.7) satisfied when we have a boundary of a convex set E which is less smooth, perhaps with the added condition that the measure of E between two parallel hyperplanes, the distance between which is δ, is smaller than $c\delta^\lambda$ with positive λ?

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Department of Math. and Stat. Sciences, University of Alberta, Edmonton, AB
e-mail: zditzian@math.ualberta.ca

Department of Mathematics, University of Manitoba, Winnipeg, MB
e-mail: prymak@gmail.com

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