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BOOLEAN NEAR-RINGS AND WEAK COMMUTATIVITY

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Abstract

It is shown that every boolean right near-ring R is weakly commutative, that is, that $xyz = xzy$ for each $x, y, z \in R$. In addition, an elementary proof is given of a theorem due to S. Ligh which states that a d.g. boolean near-ring is a boolean ring. Finally, a characterization theorem is given for a boolean near-ring to be isomorphic to a particular collection of functions which form a boolean near-ring with respect to the customary operations of addition and composition of mappings.

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1. Introduction

In a recent paper [5], Murty proved that a boolean (right) near-ring N is weakly commutative, that is, $xyz = xzy$ for each $x, y, z \in N$, if it is zero-symmetric. It will be shown here that the condition of zero-symmetry can be removed. It should be noted that this identity was introduced in 1962 by Subrahmanyam [8], in a paper on abelian boolean near-rings under the title of "Boolean semirings". Later, Ligh [4] gave a structure theorem for boolean near-rings satisfying the above identity, which he called β-near-rings. In view of Theorem 1, this apparent restriction can be deleted from these two papers. Next, in the light of weak commutativity, we will re-examine one of Ligh's theorems by presenting an elementary proof of his observation that every d.g. boolean near-ring is a boolean ring. Finally, a characterization of a particular class of boolean (right) near-rings $(N, \oplus, \cdot)$ is given, where N

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is a subcollection of linear mappings from an ordinary boolean ring R into R, namely, $f(x) = ax + b$ for each $x \in R$, and where "$\oplus$" and "$\cdot$" denote, respectively, ordinary addition and composition of mappings.

It will be assumed throughout this paper that the near-rings N are right distributive and "boolean" if $x^2 = x$ for each $x \in N$.

2. Weak commutativity

The following lemma is a specialization to boolean near-rings of a result by Reddy and Murty [7] on strongly regular near-rings.

LEMMA 1. *If N is a boolean near-ring, then $xy = xyx$ for each $x, y \in N$.*

THEOREM 1. *If N is a boolean near-ring, then $abc = acb$ for each $a, b, c \in N$.*

PROOF. Let $a, b, c \in N$. Then $b(a - ac)c = b(ac - ac) = b0$. Thus, $[(a-ac)b(a-ac)]c = (a-ac)b0$ implies $(a-ac)bc = (a-ac)b0$ by Lemma 1. Since $cbc = cb$, we obtain, from the preceding equation that $abc - acb = ab0 - acb0$. Next, $c(a-ac)c = c(ac-ac) = c0$ and thus, by Lemma 1, $c(a-ac) = c0$ and this gives $bc(a-ac) = bc0$. Again, by Lemma 1, $bc(a-ac) = bc(a-ac)bc$ and thus, by the last two equations, $bc(a-ac)bc = bc0$. Now pre-multiplying both sides by $a - ac$ we obtain $[(a-ac)bc(a-ac)bc] = (a-ac)bc0$ and this gives, by idempotency, that $(a-ac)bc = (a-ac)bc0$. Again, using $cbc = cb$, we obtain from the preceding equation that $abc - acb = abc0 - acb0$. But we have seen earlier that $abc - acb = ab0 - acb0$ and so $abc0 - acb0 = ab0 - acb0$ and this gives that $abc0 = ab0$ for each $a, b, c \in N$. Hence, by using the result just obtained, we have that $ab0 = aab0 = aa0 = a0$ for each $a, b \in N$. Returning to the equation $abc - acb = ab0 - acb0$ we conclude that $abc - acb = ab0 - acb0 = a0 - a0 = 0$. Therefore $abc = acb$ for each $a, b, c \in N$.

S. Ligh [4] proved that a boolean (right) near-ring N containing a left multiplicative identity is a boolean ring. With Theorem 1 at our disposal, we give a modification of his result as follows.

THEOREM 2. *Let N denote a boolean near-ring such that, if each of $x, y \in N$, then there exists an $e \in N$ such that $ex = x$ and $ey = y$. Then N is a boolean ring.*

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PROOF. Let $x \in N$. Consider x and $x+x$. By assumption, there exists an idempotent $e \in N$ such that $ex = x$ and $e(x+x) = x+x$. Thus, $x+x = ex+ex = (e+e)x = (e+e)^2x = [e(e+e)+e(e+e)]x = e(e+e)x + e(e+e)x = e(ex+ex)+e(ex+ex) = e(x+x)+e(x+x) = (x+x)+(x+x)$. Hence, it follows that $x+x=0$. Therefore $(N,+)$ is an abelian group. Now, let $x, y \in N$. Then according to our assumption, there exists an idempotent $f \in N$ such that $fx = x$ and $fy = y$. By Theorem 1, $xy = (fx)y = fxy = fyx = (fy)x = yx$. With the multiplication being commutative, it follows that N is a boolean ring.

3. A theorem of S. Ligh

Without using transfinite methods, a proof is offered of the following result from [3].

THEOREM 3 (Ligh). *Every d.g. boolean near-ring N is a boolean ring.*

PROOF. Let N denote a d.g. boolean near-ring and suppose S is a multiplicative semigroup whose elements s generate $(N,+)$ and satisfy $s(x+y) = sx+sy$ for each $x, y \in N$.

It is easy to see that, for each $s, s_1, s_2 \in S$ and $x \in N$, $s0 = 0$, $x0 = 0$, $s+s=0$, and $s_1s_2 = s_2s_1$. Hence $s(x+x) = sx+sx = (s+s)x = 0x = 0$. Next, for $x, y \in N$, let $y = s_1+s_2+\cdots+s_n$, where each $s_i \in S$. Then $y(x+x) = (s_1+s_2+\cdots+s_n)(x+x) = s_1(x+x)+s_2(x+x)+\cdots+s_n(x+x) = 0$. Thus, by Lemma 1, $x+x = (x+x)x = (x+x)x(x+x) = (x+x)0 = 0$, that is, each non-zero element in $(N,+)$ is of order 2. Hence $(N,+)$ is an abelian group. Consequently, N is a ring since $(N,+)$ being abelian implies by an elementary result of Frölich [1] that N is left distributive. Therefore N is a boolean ring.

4. A special class of boolean near-rings

To motivate the last theorem, we will begin with an example of a boolean near-ring which belongs to a more general class of near-rings previously investigated under the name of abstract affine near-rings by Gonshor [2] and discussed by Pilz in [6]. Let R denote a boolean ring. Let each of A and B denote a subring of R such that $A \cap B = \{0\}$ and suppose $ab = 0$ for each $a \in A$ and $b \in B$. Take N to be the set of all mappings: $f\colon R \to R$ such that, for each $x \in R$, $f(x) = ax+b$, where $a \in A$ and $b \in B$. Then $(N, \oplus, \cdot)$ is a boolean

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near-ring where "$\oplus$" and "$\cdot$" denote, respectively, ordinary addition and composition of mappings. Finally, $(N, \oplus, \cdot)$ is boolean since, for each $x \in R$, $(f \cdot f)(x) = f[f(x)] = a(ax+b)+b = ax+ab+b = ax+0+b = ax+b = f(x)$. It is this class of boolean near-rings which we will characterize in the following manner.

Let A denote a boolean ring and let B denote an additive abelian group. Consider the group direct sum $A \oplus B$ of A and B. Define a multiplication in $A \oplus B$ by $(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1)$. It can be verified directly that $A \oplus B$ forms a boolean right near-ring with commutative addition and satisfies the identity $(x-y)0 = xy-yx$. We will denote this boolean near-ring by $N(A, B)$.

THEOREM 4. *Let N denote a boolean near-ring in which the addition is commutative and suppose, for each $x, y \in N$, that*

$$(*) \qquad (x-y)0 = xy - yx.$$

Then there exist a boolean ring A and an abelian group B such that $N \cong N(A, B)$.

PROOF. let $A = \{a \in N | a0 = 0\}$ and let $B = \{b \in N | b0 = b\}$. Clearly, A and B are additive subgroups of N. For each $a_1, a_2 \in A$, we have by $(*)$ that $a_1 a_2 - a_2 a_1 = (a_1 - a_2)0 = a_1 0 - a_2 0 = 0 - 0 = 0$ and thus $a_1 a_2 = a_2 a_1$. Also, A is closed with respect to multiplication since $(a_1 a_2)0 = a_1(a_2 0) = a_1 0 = 0$ for each $a_1, a_2 \in A$ and thus $a_1 a_2 \in A$. Hence A is a boolean ring. Furthermore, $A \cap B = \{0\}$ and, from the definitions of A and B along with Theorem 1, $ab = ab0 = a0b = a0 = 0$, for each $a \in A$ and $b \in B$.

Let $\phi \colon N \to N(A, b)$ denote a mapping defined by $\phi(x) = (x - x0, x0)$ for each $x \in N$. It is easy to see that ϕ is additive. To see that ϕ is also multiplicative, let $x_1, x_2 \in N$. Using the identity $(*)$, we obtain $x_1(x_2 - x_2 0) - (x_2 - x_2 0)x_1 = [x_1 - (x_2 - x_2 0)]0 = x_1 0 - (x_2 - x_2 0)0 = x_1 0 - x_2 0 + x_2 0 = x_1 0$. Thus $x_1(x_2 - x_2 0) - x_2 x_1 + x_2 0 = x_1 0$ and rearranging we obtain $x_1(x_2 - x_2 0) = x_2 x_1 + (x_1 - x_2)0 = x_2 x_1 + x_1 x_2 - x_2 x_1 = x_1 x_2$. Also, by Theorem 1, $x_1 x_2 0 = x_1 0 x_2 = x_1 0$. Thus, $\phi(x_1)\phi(x_2) = (x_1 - x_1 0, x_1 0)(x_2 - x_2 0, x_2 0) = ((x_1 - x_1 0)(x_2 - x_2 0), x_1 0) = (x_1(x_2 - x_2 0) - x_1 0(x_2 - x_2 0), x_1 0) = (x_1 x_2 - x_1 0, x_1 0) = (x_1 x_2 - x_1 x_2 0, x_1 x_2 0) = \phi(x_1 x_2)$. Hence, ϕ is a homomorphism. That ϕ is injective is trivial.

Now, for each $(a, b) \in N(A, B)$, let $c = a + b$. Then $c0 = (a+b)0 = a0 + b0 = 0 + b = b$ and $c - c0 = a + b - b = a$. Thus $\phi(c) = (c - c0, c0) = (a, b)$. This shows that ϕ is surjective. Therefore ϕ is an isomorphism and consequently $N \cong N(A, B)$.

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