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A GENERALIZATION OF "CONCORDANCE OF PL-HOMEOMORPHISMS OF $S^p \times S^q$"

J. P. E. HODGSON

0. Introduction. Let M^m be a closed PL manifold of dimension m. Then a concordance between two PL-homeomorphisms h_0, $h_1: M \to M$ is a PL-homeomorphism $H: M \times I \to M \times I$ such that $H|M \times 0 = h_0$ and $H|M \times 1 = h$. Concordance is an equivalence relation and in his paper [**2**], M. Kato classifies PL-homeomorphisms of $S^p \times S^q$ up to concordance. To do this he treats first the problem of classifying those homeomorphisms that induce the identity in homology, and then describes the automorphisms of the cohomology ring that can arise from homeomorphisms of $S^p \times S^q$. In this paper we show that for sufficiently connected PL-manifolds that embed in codimension 1, one can extend Kato's classification of the homeomorphisms that induce the identity in homology. More specifically, we consider the case of an $\left[\frac{m}{3}\right] + 1$-connected closed PL manifold M of dimension m that embeds in S^{m+1}. In this case the complementary domains of such an embedding collapse to complexes of relatively low dimension, which allows one to use Wall's concept of thickening and the results of the authors paper [**1**] to mimic Kato's arguments and obtain the classification.

1. Notation and statement of the theorem. Let M^m be a closed PL-manifold of dimension m which is $\left[\frac{m}{3}\right] + 1$-connected and let $j: M \subset S^{m+1}$ be an embedding. Let Q_1, Q_2 be the closures of the complementary domains.

LEMMA 0. *Q_1 (respectively Q_2) is $\left[\frac{m}{3}\right] + 1$-connected and is of the homotopy type of a CW-complex K_1 (respectively, K_2) of dimension* $\leqq \left[\frac{2m}{3}\right] + 1$.

Proof. Since $Q_1 \, U_M \, Q_2 = S^{m+1}$, by the Mayer Victoris sequence

$$H_i(Q_1) \cong H_i(Q_2) = 0 \quad \text{for } 0 < i \leqq \left[\frac{m}{3}\right] + 1.$$

Thus by Lefshetz duality

$$H^{m+1-i}(Q_1, M) = 0 \quad \text{for } 0 < i \leqq \left[\frac{m}{3}\right] + 1,$$

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and by Poincaré duality we have

$$H^{m-i}(M) = 0 \quad \text{for } 0 < i \leqq \left[\frac{m}{3}\right] + 1.$$

Further, $H^m(M) \to H^{m+1}(Q_1, M)$ is surjective. Hence by the exact sequence of the pair (Q_1, M),

$$H^i(Q_1) = 0 \quad \text{for } i \geqq \left[\frac{2m}{3}\right] + 1;$$

similarly for Q_2.

Thus by the handle decomposition of Q_1 (respectively, Q_2) $Q_1 \sim K_1$ (respectively, $Q_2 \sim K_2$) with K_1 (respectively, K_2) a CW-complex of dimension $\leqq \left[\frac{2m}{3}\right] + 1$. Further, the Q_i are the trivial $(m + 1)$-thickenings of the K_i, since $Q_i \subseteq R^{m+1}$.

Let $S\mathscr{P}(M)$ be the Δ-set (q.v. [**3**]) of PL-homeomorphisms of M inducing the identity on the homology of M. Then a K simplex of $S\mathscr{P}(M)$ is a PL-homeomorphism

$$f: \Delta^k \times M \to \Delta^k \times M$$

such that:

(1) Restriction to a face Δ^l of Δ^k induces a PL-homeomorphism of $\Delta^l \times M$ and f_* is the identity on homology.

(2) For some fixed disc $D_*^m \subset M, f|\Delta^k \times D_*^m = \mathrm{Id}$; if $\partial M \neq 0$ we assume $D_*^m \cap \partial M = D^{m-1}$.

We shall require all our homeomorphisms to satisfy condition (2). It is not difficult to see that it can be carried through all our arguments; therefore we shall omit mention of it in our proofs.

The main result is the following:

THEOREM 1. *$\pi_0(S\mathscr{P}(M))$ is in (1-1)-correspondence with $\mathscr{T}^{m+2}(SK_1) \oplus \mathscr{T}^{m+2}(SK_2)$ for $m > 5$. Thickenings are taken in the PL-category.*

2. Proof of the main theorem. We mimic the proof in [**2**]. Recall Definition 5.2 of [**1**]: $\mathscr{H}(M)$ = the quasi-homeotopy group of homeomorphisms h of M, h the identity on D_*^m and homotopic to the identity, modulo concordance.

Let us write $\tilde{P}(M)$ for the Δ-group of PL-homeomorphisms of M homotopic to the identity and satisfying condition (2) of the definitions of $S\mathscr{P}(M)$.

Then we have maps:

$$\alpha: \tilde{P}\ (Q_1) \to \mathscr{P}(M),$$
$$\beta: \tilde{P}\ (Q_2) \to \mathscr{P}(M),$$

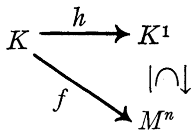
where $\mathscr{P}(M)$ is the Δ-set of all PL-homeomorphisms of M (defined as for $S\mathscr{P}(M)$ but without the homology condition).

PROPOSITION 2. *$\alpha(\tilde{P}(Q_1)) \cap \beta(\tilde{P}(Q_2)) = T\mathscr{P}(M)$, the sub-$\Delta$-set of elements in $\mathscr{P}(M)$ concordant to the identity.*

Proof. Let $h:M \to M$ be a PL-homeomorphism concordant to the identity. Then there exists H: $M \times I \to M \times I$ with $H|M \times 0 = h$, $H|M \times 1 = \text{Id}$. Now $M \times I$ can be regarded as a collar on ∂Q_1 in Q_1, so that a homeomorphism F: $Q_1 \to Q_1$ is given by $F|\partial Q_1 \times I = H$, and $F|Q_1 - (\partial Q_1 \times I) = \text{Id}$; similarly for Q_2. Thus $T\mathscr{P}(M) \subset \alpha(\tilde{P}\ (Q_1)) \cap \beta\ (\tilde{P}(Q_2))$.

Conversely, suppose that $h:M \to M$ is given so that $h = \alpha g_1 = \beta g_2$, where $g_i:Q_i \to Q_i$ is an element of $\tilde{P}\ (Q_i)$. Define a homeomorphism of S^{m+1} by $F|Q_1 = g_1$ and $F|Q_2 = g_2$. By the Alexander trick [**6**, Lemma 16] there exists an isotopy of F to the identity; let $G:S^{m+1} \times I \to S^{m+1} \times I$ be the isotopy. The proof will follow if we can show how to deform G to a concordance G' such that G' restricts to a concordance of M. To do this we recall a theorem of Stallings [4].

THEOREM. *Let $f:K^k \to M^n$ be an r-connected map of K, where K is a finite connected CW-complex and M^n is a compact PL-manifold of dimension n_1. Suppose $k \leqq n - 3$, $r \geqq (2k - n + 1)$. Then there exists a PL-subspace $K^1 \subseteq M^n$ of dimension k and a homotopy equivalence $h:K \to K^1$ such that*



homotopy commutes.

So by considering the homotopy equivalences $K_i \to Q_i$ of Lemma 1, and applying the above theorem we can suppose that K_i is a subcomplex of Q_i. Now consider $G|K_i \times I \to S^{m+1} \times I$. By the induced relative thickening theorem of [**1**], there is a concordance H of $S^{m+1} \times I$, fixed on $S^{m+1} \times S^0$, such that $H|S^{m+1} \times I \times 0 = G$ and $H|S^{m+1} \times I \times 1 = G'$ is such that $G'\ (Q_1 \times I) = Q_1 \times I$. G' then restricts to $\partial Q_1 = M$ to give the required concordance of h to the identity.

PROPOSITION 3. *The homomorphisms*

$$\alpha_0: \mathscr{H}(Q_1) \to \pi_0\ (\mathscr{P}(M)),$$
$$\beta_0: \mathscr{H}(Q_2) \to \pi_0\ (\mathscr{P}(M)),$$

are monomorphisms.

Proof. Let f represent an element of $\tilde{P}(Q_1)$ that maps to an element of $T\mathscr{P}\ (M)$ under α. Then by Proposition 2, there is an element $g \in \tilde{P}\ (Q_2)$ such that $\beta g = \alpha f$. But in the proof of Proposition 3 we saw that such an element f is concordant to the identity (by $G'|Q_1 \times I$), so α_0 is a monomorphism; likewise β_0.

Proof of Theorem 1. Suppose a PL-homeomorphism h: $M \to M$ induces the identity in homology. Then by Van Kampen's Theorem, $Q_1 \cup_h Q_2$ is simply-connected. Hence by the Mayer-Vietoris Theorem, $Q_1 \cup_h Q_2$ is a homotopy sphere and for $m \geqq 5$ a PL-sphere [**7**]; that is, there is a homeomorphism

$$F: Q_1 \cup_h Q_2 = \Sigma \to S^{m+1}.$$

Let $i: Q_1 \to \Sigma$, $j: Q_2 \to \Sigma$ be the natural embeddings such that $j^{-1}i = h$, and consider the maps

$$g_1 = [F \circ i]^{-1}: Q_1 \to Q_1 \text{ and } g_2 = F \circ j: Q_2 \to Q_2.$$

Then $g_1 \in \tilde{P}\ (Q_1)$ and $g_2 \in \tilde{P}\ (Q_2)$. These maps g_i are defined up to concordance class because we can use the argument of Proposition 3 to insure that $(F \circ i)$ $(Q_i) = Q_1$.

Now by Propositions 2 and 3 there is a map

$$\lambda: \mathscr{H}(Q_1) \oplus \mathscr{H}(Q_2) \to \pi_0\ (\mathscr{P}(M)),$$

and clearly $\lambda([g_1],[g_2]) = [h]$. Thus λ is an isomorphism onto $\pi_0\ (S\mathscr{P}(M))$. To complete the proof we remark that $\mathscr{H}(Q_i) = \mathscr{T}^{m+2}(SK_i)$, $i = 1, 2$, by [**1**, Theorem 2.7]

3. Calculations. We observe that there are two problems raised by this result: one is the calculation of the homotopy types of the (dual) complexes K_1, K_2, and the other is the calculation of the automorphisms of $H_*(M)$ that may be induced by PL-homeomorphisms of M. As an example of the kind of calculations involved, we consider the special case where M is the connected sum of k copies of $S^p \times S^q$, $3 < (p + q)/3 + 1 < p \leqq q$.

M embeds in S^{p+q+1}, and we may choose an embedding such that the complementary domains Q_1 and Q_2 have the homotopy types of $\bigvee_{i=1}^k S^p$ and $\bigvee_{i=1}^k S^q$, respectively. Then we have from Theorem 2 and [**5**], Proposition 10.1 and Corollary 6.3]:

THEOREM 5. *If* $M = \#_{i=1}^k (S^p \times S^q)$, p, g, *as above, then* $\pi_0(S\mathscr{P}(M))$ *is in* (1-1)-*correspondence with*

$$\bigoplus_{i=1}^{k} \pi_p(PL(q+1)) \oplus \bigoplus_{j=1}^{\binom{k}{2}} \pi_{p+1}(S^{q+1}) \oplus \bigoplus_{i=1}^{k} \pi_q(PL(p+1)) \oplus \bigoplus_{j=1}^{\binom{k}{2}} \pi_{q+1}(S^{p+1}),$$

where p, q *are as above.*

We must now determine which elements of Aut $H_*(M)$ may be induced by homeomorphisms of M. For this purpose it is easier to study cohomology. Let us define $\mathrm{Aut}_0\, H^*(M)$ as the subgroup of $\mathrm{Aut}\, H^*(M)$ consisting of automorphisms of $H^*(M)$ that can be induced by homotopy equivalences of M.

We have the following results:

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THEOREM 6. *Let M be the connected sum of k copies of $S^p \times S^q$ with $3 < (p + q)/3 + 1 < p < q$. Then any element of* $\mathrm{Aut}_0\, H^*(M)$ *can be induced by a homeomorphism and there is a (split) exact sequence*:

$$0 \to \mathrm{GL}(k, Z) \to \mathrm{Aut}_0 H^*(M) \xrightarrow{\theta} Z_2 \to 0,$$

where θ is induced by orientation.

THEOREM 7. *Let M be the connected sum of k copies of $S^p \times S^p$ with $p > 3$. Then any element of* $\mathrm{Aut}_0\, H^*(M)$ *can be induced by a homeomorphism and there is a split exact sequence.*

$$0 \to G(k, p) \to \mathrm{Aut}_0 H^*(M) \xrightarrow{\theta} Z_2 \to 0,$$

where θ is determined by orientation and $G(k, p) = S_p(k, Z)$ for p odd and $O(2k, k, Z)$ for p even.

We shall give the proof of Theorem 6. The proof of Theorem 7 is similar and we will omit it.

Proof of Theorem 6. We show first that any element of $\mathrm{Aut}_0\, H^*(M)$ can be obtained from a homeomorphism. Suppose that $f: M \to M$ is a homotopy equivalence giving an element $a \in \mathrm{Aut}_0\, H^*(M)$. Let us assume further that f preserves orientation, so that $\theta(a) = O$. Now by a homotopy of f, we can suppose that for some disc D in M with

$$M - D \sim \bigvee_{i=1}^{k} S^p \vee \bigvee_{i=1}^{k} S^q,$$

$f(D) = D$. The restricted map $f\,|\overline{M - D} \to \overline{M - D}$ is a homotopy equivalence. We have thus a map

$$g: \bigvee_{i=1}^{k} S^p \vee \bigvee_{i=1}^{k} S^q \to \overline{M - D}$$

obtained by taking the composition of f and the inclusion of $\bigvee_{i=1}^{k} S^p v \bigvee_{i=1}^{k} S^q$ in $M - D$. By the induced thickening theorem of [**5**], g is homotopic to a map

$$g': \bigvee_{i=1}^{k} S^p \vee \bigvee_{i=1}^{k} S^q \to N \subset \overline{M - D},$$

where N is a thickening of $\bigvee_{i=1}^{k} S^p \vee \bigvee_{i=1}^{k} S^q$. But the regular neighborhood theorem and the fact that $f|\overline{M - D}$ is a homotopy equivalence imply that N is homeomorphic to $\overline{M - D}$. Thus $f|\overline{M - D}$ is homotopic to an embedding f', and by the h-cobordism theorem we can collar the embedding to extend f' to a homeomorphism of $\overline{M - D}$. Using the Alexander trick then extends the homeomorphism to the required homeomorphism of M.

The exactness of the sequence follows by straight-forward algebra. For an orientation preserving homotopy equivalence $f: M \to M$ with the obvious basis for $H^p\,(M, Z)$, f induces $F^*: H^p(M) \to H^p(M)$; this gives an element of $\mathrm{GL}(k, Z)$. Poincaré duality then determines the automorphism of $H^p(M, Z)$.

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University of Pennsylvania,
Philadelphia, Pennsylvania