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TUBES, COHOMOLOGY WITH GROWTH CONDITIONS AND AN APPLICATION TO THE THETA CORRESPONDENCE

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Introduction. In this paper we continue our effort [**11**], [**12**], [**13**], [**14**] to interpret geometrically the harmonic forms on certain locally symmetric spaces constructed by using the theta correspondence. The point of this paper is to prove an integral formula, Theorem 2.1, which will allow us to generalize the results obtained in the above papers to the finite volume case (the previous papers treated only the compact case). We then apply our integral formula to certain finite volume quotients of symmetric spaces of orthogonal groups. The main result obtained is Theorem 4.2 which is described below. We let (,) denote the bilinear form associated to a quadratic form with integer coefficients of signature (p, q). We assume that the fundamental group $\Gamma \subset SO(p, q)$ of our locally symmetric space is the subgroup of the integral isometries of (,) congruent to the identity matrix modulo some integer N. We assume that N is chosen large enough so that Γ is neat (the multiplicative subgroup of $\mathbf{C}^*$ generated by the eigenvalues of the elements of Γ has no torsion), Borel [**2**], 17.1 and that every element in Γ has spinor norm 1, Millson-Raghunathan [**15**], Proposition 4.1. These conditions are needed to ensure that our cycles C_x (see below) are orientable. The methods we will use apply also to unitary and quaternion unitary locally symmetric spaces, see [**13**].

Let G denote $O(p, q)$ and G' denote the non-trivial 2-fold cover of $Sp_n(\mathbf{R})$. Let $V = \mathbf{R}^{p+q}$ be the standard representation space of G. Let $S(V^n)$ denote the Schwartz space of the direct sum of n copies of V. Then G operates on $S(V^n)$ in the obvious way and we may consider the continuous cohomology groups $H^*_{ct}(G, S(V^n))$. It is a remarkable fact that G' also acts on $S(V^n)$ and this action of G' commutes with that of G. The corresponding action of G' is by the oscillator or Weil representation and will be denoted ω. A convenient reference for our purposes is [**14**]. Hence G' also acts via ω on $H^*_{ct}(G, S(V^n))$. By the van Est theorem we may represent elements of the previous cohomology group by closed differential forms $\phi(z)$ on D, the symmetric space of G, with values in $S(V^n)$ which

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are invariant under the action of Γ, the fundamental group of the locally symmetric space. Since G' acts on such ϕ we may define a function $\phi(g', g)$ on $G' \times G$ by

$$\phi(g', g) = \omega(g')\phi(z)$$

where z is the image of g in D. We let K' denote a maximal compact subgroup of G' and we assume that ϕ transforms by a character under $\omega|K'$. We may then consider ϕ to be a section of a homogeneous line bundle on G'/K' as a function of g'. This quotient is of course the Siegel space $\mathfrak{h}_n$. In Section 4 we construct a continuous cohomology class $\phi \in H^{nq}_{ct}(SO(p, q), S(V^n))$ for each n, p, and q which is an eigenclass of $\omega|K'$. We will henceforth specialize our consideration to these ϕ, though there are other continuous cohomology classes.

We now recall, see example [**14**], that given a lattice L in V there is a continuous linear functional Θ_L on $S(V^n)$, the theta distribution, which is a sum of Dirac delta functions, one at each point of the lattice $L^n \subset V^n$. We assume $(\ ,\ )$ takes integral values on L. Then Θ_L is invariant under suitable arithmetic subgroups of G' and G. Fix a lattice L as above and an element $x_0 \in L^n$. We will let Θ denote the sum of Dirac delta functions located at the points in L^n congruent to x_0 modulo NL^n. We may still find arithmetic groups Γ and Γ' which leave Θ invariant. We let $M' = \Gamma' \backslash G'/K'$ and $M = \Gamma \backslash G/K$ and we define:

$$\theta_\phi(g', g) = \Theta(\omega(g')\phi(g)).$$

Then θ_ϕ is a section of a line bundle $\mathscr{L}$ on M' in the g' variable and a closed differential form on M in the g variable. We may accordingly use θ_ϕ as the kernel of an integral transform Λ_ϕ from cuspidal sections of $\mathscr{L}$ to $H^*(M, \mathbf{C})$; it is well known that $\theta_\phi(g', g)$ has moderate growth on M', see [**8**]. The line bundle $\mathscr{L}$ has a holomorphic structure and we restrict Λ_ϕ to the holomorphic cuspidal sections of $\mathscr{L}$. Our goal is to interpret geometrically the image of Λ_ϕ.

The basic fact underlying our program is that the lattice $\mathscr{L}'$ in the symmetric n by n matrices which parametrizes the Fourier coefficients of the holomorphic sections of $\mathscr{L}$ also parametrizes certain reducible cycles in M. This lattice is the following. Let Γ'_∞ be the subgroup of matrices in Γ' of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Then Γ'_∞ is a lattice in the space of symmetric n by n matrices. We let L' be the dual lattice to Γ'_∞ for the trace pairing.

We now describe the cycles. We will restrict ourselves to cycles of positive type, see [**14**], I.5. Let $x = (x_1, x_2, \ldots, x_n)$ in V^n be such that each x_j has rational coordinates relative to the standard basis for $V = R^{p+q}$ and such that the matrix $((x_i, x_j))$ is positive definite. We wish to construct a singular cycle $\iota : C_x \to M$ such that C_x is an orientable finite volume locally symmetric space and ι is a proper map with totally geodesic image. Then C_x will be a locally finite cycle. We recall that D may be

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realized as the subset of the Grassmannian of q-planes in V consisting of those planes P such that $(,)|P$ is negative definite. We define D_x to be the subset of those planes which lie in the orthogonal complement $u^\perp$ of u, where u is the span of x. We observe that D_x is the fixed-point set in D of the involution r_x in $O(p, q)$ defined by:

$$r_x(x) = -x \quad \text{for } x \in u$$
$$r_x(x) = -x \quad \text{for } x \in u^\perp.$$

In particular D_x is totally geodesic. However, there is a remarkable amount of extra structure, see Section 5. We let Γ_x be the centralizer of r_x in Γ and C_x be the finite volume locally symmetric space $C_x = \Gamma_x \backslash D_x$. That C_x has finite volume follows from the standard (but hard) result from reduction theory, see [**2**], that if H is a semi-simple algebraic group defined over $\mathbf{Q}$ then $H(\mathbf{Z})\backslash H(\mathbf{R})$ has finite volume. If Γ is neat then Γ_x fixes x pointwise, see [**9**], Lemma 7.1 and consequently Γ_x can be considered as an element of $SO(p - n, q)$. But since every element in Γ_x has spinor norm 1 we find that Γ_x is contained in $SO_0(p - n, q)$, the connected component of the identity, and consequently preserves the orientation of D_x. Hence, C_x is orientable. The quotient mapping $p:D \to M$ restricted to D_x factors through a map $\iota:C_x \to M$. The map is proper by [**1**], Lemma 2.7 which shows that if $H \subset G$ is an inclusion of reductive algebraic groups over $\mathbf{Q}$ and $\Gamma \subset G(\mathbf{Q})$ is a discrete subgroup of $G(\mathbf{R})$ then the map

$$\Gamma \cap H(\mathbf{R})\backslash H(\mathbf{R}) \to \Gamma\backslash G(\mathbf{R})$$

is proper. We call cycles in locally symmetric spaces which are locally the fixed point set of an involution *special cycles*.

We now construct the cycles promised above as a sum of the cycles C_x just constructed. Let $\beta = (\beta_{ij})$ be a positive definite symmetric n by n matrix which is an element of L'. Given $x = (x_1, x_2, \ldots, x_n)$ in V^n we say x has length β if $(x_i, x_j) = \beta_{ij}$. Let $\mathscr{C}_\beta$ be a set of Γ-orbit representatives for the vectors in L^n of length 2β. Then $\mathscr{C}_\beta$ is finite, [**2**], Theorem 9.11, and we define the reducible cycle $\mathscr{C}_\beta$ by:

$$C_\beta = \sum_{x \in \mathscr{C}_\beta} C_x.$$

In order to prevent C_β from being trivially zero we consider only those x as above congruent to x_0 modulo NL where x_0 was chosen previously in the definition of the modified theta distribution. We let $\mathscr{C}'_\beta$ denote the set of Γ-orbit representatives of such x and we redefine C_β according to:

$$C_\beta = \sum_{x \in \mathscr{C}'_\beta} C_x.$$

We recall that a smooth closed form ω on M is said to be Poincaré dual to a locally finite cycle C_β if for any closed compactly supported form η on M we have:

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$$\int_M \eta \wedge \omega = \int_{C_\beta} \eta.$$

All such forms lie in the same cohomology class. We now state our main theorem (Theorem 4.2) relating the cycles C_β to the image of Λ_ϕ.

THEOREM. *The image of Λ_ϕ is the span of the Poincaré duals of the cycles C_β if $n < (p + q)/4$.*

We now give an indication of why Theorem 2.1 is a key step in the proof of the above theorem. The main ingredient in the proof of the above theorem is a formula for $\alpha_\beta(\theta_\phi(\eta))$, the β-th Fourier coefficient of the section of $\mathscr{L}$ defined by

$$\theta_\phi(\eta) = \int_M \eta \wedge \theta_\phi,$$

here η is a compactly-supported closed form on M. Note that $\alpha_\beta(\theta_\phi(\eta))$ is a function of v where $\tau \in \mathfrak{h}_n$ satisfies $\tau = u + iv$ with u and v symmetric n by n matrices and v positive definite. Our formula is then:

$$\alpha_\beta(\theta_\phi(\eta))(v) = e^{-2\pi \mathrm{tr} \beta v} \int_{C_\beta} \eta.$$

Theorem 2.1 plays a critical role in the proof of this formula. Indeed by definition we have:

$$\alpha_\beta(\theta_\phi(\eta))(v) = \frac{1}{\mathrm{vol}\ \mathscr{D}(v)} \int_{\mathscr{D}(v)} \theta_\phi(\eta)(u + iv) e^{-2\pi i \mathrm{tr} \beta u} du.$$

Here $\mathscr{D}(v)$ is a fundamental domain for the subgroup Γ'_∞ acting on the subset of $\mathfrak{h}_n$ defined by $\mathrm{Im}\, \tau = v$. It is a formal and well-known result that such integrals "unfold" to a sum of integrals (indexed by $\mathscr{C}'_\beta$) of the type considered on the left-hand side of the formula presented in Theorem 2.1. In the case discussed above the degree of ϕ is equal to the codimension of C_β and the right-hand side of Theorem 2.1 is the right-hand side of the above formula.

The third section of our paper is a digression intended to show how the integral formula of Section 2 follows from a comparison of the cohomology of complexes of sufficiently rapidly decreasing forms with that of the cohomology of the complex of forms with compact support. Our results in this section are analogous to those of Borel [**3**].

For example in the case of a cusp of a finite volume quotient of the upper half plane Borel's results imply that the cohomology of the complex of forms that decrease rapidly (along with their derivatives) along the cusp is the same as the cohomology of the complex of forms which are compactly supported on the cusp. His results also imply that the cohomology of the complex of forms that increase slowly (along with their derivatives) is the same as the de Rham cohomology. In our analogue of this example we divide the upper half plane by a hyperbolic element and get a "tube" rather than a cusp. We now discuss what our results in the third section imply for this case.

Let $\mathbf{H}$ denote the upper half plane and Γ_1 be the infinite cyclic group generated by a primitive hyperbolic element γ_1. Let A_1 be the one parameter group generated by γ_1. We let ξ and η be the fixed-points of γ_1 on the boundary of $\mathbf{H}$ and $\tilde{c}$ be the oriented geodesic joining ξ to η. Let E be the "tube" given by $E = \Gamma_1\backslash\mathbf{H}$ and $p:\mathbf{H} \rightarrow E$ be the projection. We have a fibering $\pi:\mathbf{H} \rightarrow \tilde{c}$ by geodesics normal to $\tilde{c}$. The fibering π induces a fibering, also denoted π, of E over c, the image of $\tilde{c}$ under p. We note that c is a closed geodesic. Let r be the function on E defined so that $r(x)$ is the distance from x to c. We then consider the complex of forms $A^*_{-1}(E)$ consisting of those forms η satisfying:

$$\|\eta(x)\| \leqq e^{-r(x)}p_1(r(x))$$
$$\|d\eta(x)\| \leqq e^{-r(x)}p_2(r(x))$$

for polynomials p_1 and p_2 in one variable. By analogy with the results in [**3**], we might expect that the cohomology of $\mathscr{A}^*_{-1}(E)$ would be the cohomology of E with compact supports. However, this is not the case. The first cohomology of $\mathscr{A}^*_{-1}(E)$ is $\mathbf{R}^2$ with basis $\pi^*\mu$ and $*\pi^*\mu$ in the notation of [**10**]. It is proved in Section 3 that if one takes complexes of forms which are sufficiently rapidly decreasing then one obtains the cohomology of E with compact supports and if one allows slow increase then one obtains the absolute cohomology of E. However there is a gap between the two types of growth conditions. We note that if (in the notation of Theorem 2.1) we take $\Phi = \pi^*\mu$ and $\eta = *\pi^*\mu$ then

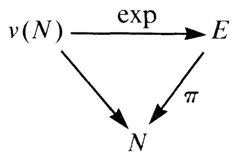
$$\int_E \eta \wedge \Phi \neq \int_c j^*\eta \wedge \pi_*\Phi.$$

Hence the integral formula of Theorem 2.1 will not hold unless Φ is sufficiently rapidly decreasing.

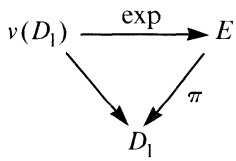
We conclude by observing that there is considerable overlap between our results and those of [**18**] and [**19**]. In [**18**], the authors found the exhaustion argument of Lemma 2.3 independently. However, they do not use our homology argument but instead use an interesting uniqueness theorem, Theorem 2.1 of [**18**] and 4.9 of [**19**]. Using their uniqueness theorem Tong and Wang obtain a canonical piece of the dual form to the cycles C_β via an integral transform of the type described above.

1. Geometric preliminaries on tubes. Let N be an n-dimensional connected complete orientable submanifold of an orientable connected complete m-dimensional Riemannian manifold E with sectional curvature bounded below by $-\rho^2$ (with $\rho > 0$). We will assume E is without boundary or else that E has boundary N, so N is a hypersurface in the case

E has boundary. In this latter case, the assumption that E is complete means that E is complete as a metric space. We assume that N has finite volume in the induced metric and that the Riemannian exponential map exp of the normal bundle $\nu(N)$ in E is a diffeomorphism onto E. Thus there is induced a vector bundle structure $\pi:E \to N$ so that the following diagram commutes.



As a consequence of our assumption above we see that the inclusion of N into E is a homotopy equivalence and hence the universal cover $\tilde{N}$ of N embeds into the universal cover D of E. We denote the image of $\tilde{N}$ by D_1. We again have an induced vector bundle structure:



We let Γ denote the group of deck transformations of the covering $p:D \to E$. Then Γ takes D_1 into itself and $p(D_1) = N$. We assume D_1 is diffeomorphic to $\mathbf{R}^n$. In fact this is not necessary but it is convenient to have global coordinates.

We now introduce coordinates on D. We choose global coordinates $(x_1, x_2, \ldots, x_n)$ on D_1. Let $E_1, E_2, \ldots, E_k$ be an orthonormal frame field for $\nu(D_1)$. We assume that this frame may be chosen so that the functions

$$f_i(T) = \|\nabla_T E_i\|, \quad 1 \leqq i \leqq k,$$

are uniformly bounded for $T \in S(D_1)$, the tangent sphere bundle of D_1. Then we associate to

$$(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_k) \in R^{n+k}$$

the point

$$\exp_x(y_1E_1(x) + \ldots + y_kE_k(x))$$

where x is the point in D_1 with coordinates $(x_1, x_2, \ldots, x_n)$. Such coordinates are often called Fermi coordinates, see [7], page 205. We obtain an atlas on E by composing $(x_1, \ldots, x_n, y_1, \ldots, y_k)$ with local cross-sections of the covering $p:D \to E$. We continue to denote these coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_k)$. We call such coordinates Fermi coordinates in E. We let $\mathcal{D}_1$ denote the interior of a fundamental domain for the action of Γ on D_1 and put $\mathscr{D} = \pi^{-1}(\mathscr{D}_1)$. Hence $\mathscr{D}$ will be the interior of a fundamental domain for the action of Γ on D. We will often identify integrals over E (respectively N) with integrals over $\mathscr{D}$ (respectively $\mathscr{D}_1$). We let $S(r)$ denote the subset of elements of E consisting of those points having distance r from N. If X is a Riemannian manifold then vol X will denote the volume of X and vol_X will denote the Riemannian volume element. Let r be the function on E defined by

$$r = (y_1^2 + \ldots + y_k^2)^{1/2}.$$

Hence, if ξ is a point in E, then $r(\xi)$ is the distance from ξ to N.

LEMMA 1.1. $\mathrm{vol}\ S(r) \leqq Cr^{k-1}e^{(m-1)\rho r}$.

Proof. We estimate the integral expressing vol $S(r)$ by using [7]. There exists a smooth function $A(x, y)$ on E such that

$$\mathrm{vol}_E = A(x, y)dx_1 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_k.$$

We introduce polar Fermi coordinates $(r, u_1(y), \ldots, u_{k-1}(y))$. We obtain:

$$\mathrm{vol}_E = A(x, ru)dx_1 \wedge \ldots \wedge dx_n \wedge dr \wedge du_1 \wedge \ldots \wedge du_{k-1}$$

and

$$\mathrm{vol}_{S(r)} = A(x, ru)dx_1 \wedge \ldots \wedge dx_n \wedge du_1 \wedge \ldots \wedge du_{k-1}.$$

By [7], Lemma 6.2 we have:

$$A(x, ru) \leqq C_1 r^{k-1}e^{(m-1)\rho r}A(x, u).$$

Hence:

$$\mathrm{vol}\ S(r) \leqq C_1 r^{k-1}e^{(m-1)\rho r} \int_{\mathscr{D}_1 \times S^{k-1}} A(x, u)dx_1 \wedge \ldots \wedge dx_n \wedge du_1 \wedge \ldots \wedge du_{k-1}$$

$$\leqq C_1\ \mathrm{vol}\ S(1) r^{k-1}e^{(m-1)\rho r}.$$

With this the lemma is proved.

In Section 3 we will need a lower bound for the lengths of the coordinate differentials in a coordinate patch. We have the following lemma. Let U be a standard coordinate patch. By this we mean that U is the inverse image under π of a small ball in N. The coordinates on U are induced by a local section of p.

LEMMA 1.2. *If N is totally geodesic then there exist constants C_1 and C_2 such that for any $x \in U$ we have*:

(a) $C_1 e^{-\rho r(x)} \leqq \|dx_i|_x\| \leqq C_2$ *for* $i = 1, 2, \ldots, n$

(b) $C_1 e^{-\rho r(x)} \leqq \|dy_j|_x\| \leqq C_2$ *for* $j = 1, 2, \ldots, k$.

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This lemma follows from standard techniques in Riemannian geometry (the Rauch comparison theorems). It is proved in the appendix for the convenience of the reader.

LEMMA 1.3. *If η is a bounded form on N then $\pi^*\eta$ is a bounded form on E.*

Proof. From Lemma A2 of the appendix it follows that:

$$||\pi^*\eta|_x|| \leqq ||\eta|_{\pi(x)}|| \quad \text{for all } x \in E.$$

With this the lemma is proved.

2. An integral formula. In what follows s and t will be positive integers satisfying $s \geqq k$ and $t = m - s$. For $x \in E$, let $r(x)$ be the geodesic distance from x to N. Let $j:N \to E$ denote the inclusion and C a generic positive constant.

THEOREM 2.1. *Let Φ be a differential s-form on E satisfying:*

(i) *Φ is closed*

(ii) $||\Phi(x)|| \leqq e^{-m\rho r}p(r)$ *for some polynomial p.*

Then, if η is any closed, bounded t-form on E we have:

$$\int_E \eta \wedge \Phi = \int_N j^*(\eta) \wedge \pi_*(\Phi).$$

Notation. π_* denotes the operation on forms on E of "integration over the fiber", the adjoint of π^* for the pairing:

$$[\eta, \phi] = \int_E \eta \wedge \phi.$$

See [**4**], page 61 for details.

Remark 2.1. We observe that Φ and $\eta \wedge \Phi$ are integrable over E. Since $||\eta \wedge \Phi|| \leqq C||\Phi||$ it is sufficient to prove the former. Using Lemma 1.1 we have:

$$\int_E ||\Phi|| = \int_{\mathscr{D}_1} \int_{R^k} ||\Phi||A(x, y)dxdy$$
$$\leqq C \int_0^\infty e^{-m\rho r}p(r) \text{ vol } S(r)dr < \infty.$$

We also observe that $||\pi_*(\Phi)||$ is bounded on N and hence $\pi_*(\Phi) \wedge j^*(\eta)$ is integrable over N.

The proof of the theorem will occupy the rest of this section.

Definition. For $\lambda \in \mathbf{R}_+^*$ we denote by a_λ the operator on E obtained by exponentiating the operation of multiplication by λ in the fibers of ν. If $\xi \in \nu_x(N)$ we have

$$a_\lambda \exp_x \xi = \exp_x \lambda\xi.$$

We note $r(a_\lambda x) = \lambda r(x)$.

LEMMA 2.1.

$$\lim_{\lambda \to 0} a_\lambda^* \eta = \pi^* j^* \eta.$$

Proof. The proof is by a calculation in Fermi coordinates. We write out η in the coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_k)$ to obtain:

$$\eta(x, y) = \sum_K g_K(x, y) dx_K + \sum_{\substack{K', L \\ |L| \geqq 1}} g_{K', L}(x, y) dx_{K'} \wedge dy_L$$

where K, K', L are multi-indices and $|L|$ denotes the cardinality of L. We have

$$\begin{aligned} a_\lambda^* \eta(x, y)\,|_{\lambda=0} &= \left[\sum_K g_K(x, \lambda y) dx_K\right]\Big|_{\lambda=0} \\ &+ \left[\lambda^{|L|} \sum_{\substack{K', L \\ |L| \geqq 1}} g_{K', L}(x, \lambda y) dx_{K'} \wedge dy_L\right]\Big|_{\lambda=0} \\ &= \sum_K g_K(x, 0) dx_K = \pi^* j^* \eta(x, y). \end{aligned}$$

With this the lemma is proved.

LEMMA 2.2.

$$\lim_{\lambda \to \infty} \int_E \eta \wedge a_\lambda^* \Phi = \int_N j^* \eta \wedge \pi_*(\Phi).$$

Proof. We have:

$$\begin{aligned} \int_E \eta \wedge a_\lambda^* \Phi &= \int_E a_\lambda^*(a_{\lambda^{-1}}^* \eta \wedge \Phi) = \int_{a_\lambda(E)} a_{\lambda^{-1}}^* \eta \wedge \Phi \\ &= \int_E a_{\lambda^{-1}}^* \eta \wedge \Phi. \end{aligned}$$

Hence, it suffices to prove:

$$\lim_{\lambda \to 0} \int_E a_\lambda^* \eta \wedge \Phi = \int_N j^* \eta \wedge \pi_*(\Phi).$$

Now we have $\|a_\lambda^* \eta \wedge \Phi\| \leqq C\|\Phi\|$ with the constant C independent of λ; hence, by the Lebesgue dominated convergence theorem we obtain:

$$\begin{aligned} \lim_{\lambda \to 0} \int_E a_\lambda^* \eta \wedge \Phi &= \int_E \lim_{\lambda \to 0} a_\lambda^* \eta \wedge \Phi = \int_E \pi^* j^* \eta \wedge \Phi \\ &= \int_N j^* \eta \wedge \pi_*(\Phi). \end{aligned}$$

With this the lemma is proved.

Remark 2.3. Consider the function $A(\lambda)$ defined in $[1, \infty)$ by:

$$A(\lambda) = \int_E \eta \wedge a_\lambda^* \Phi.$$

Then $A(\lambda) = B(\lambda^{-1})$ where $B(\lambda)$ is the function defined on $(0, 1]$ by:

$$B(\lambda) = \int_E a_\lambda^* \eta \wedge \Phi.$$

Expanding η in a Taylor series around 0 in the coordinates $(y_1, y_2, \ldots, y_k)$ we see that A has an asymptotic development at ∞ given by:

$$A(\lambda) = \sum_{j=0}^{\infty} a_j \lambda^{-j}$$

with

$$a_0 = \int_N j^*\eta \wedge \pi_*(\Phi).$$

The other a_j cannot be expressed in terms of $j^*\eta$; for example:

$$a_1 = \int_N j^*(\eta) \wedge \pi_*(\mathscr{L}_{r\partial/\partial r}\Phi) + \int_N j^*(\iota_{r\partial/\partial r}\eta) \wedge \pi_*(\Phi \wedge dr).$$

Here $\mathscr{L}_{r\partial/\partial r}$ denotes Lie derivation and $\iota_{r\partial/\partial r}$ denotes interior multiplication by $r\partial/\partial r$.

We now show that if Φ is closed then $A(\lambda)$ is constant. We can give a formal argument for this as follows:

$$\frac{dB}{d\lambda}(\lambda) = \frac{d}{d\mu} \int_E a_\mu^* a_\lambda^* \eta \wedge \Phi|_{\mu=1} = \int_E \mathscr{L}_{r\partial/\partial r} a_\lambda^* \eta \wedge \Phi$$

$$= \int_E d\iota_{r\partial/\partial r} a_\lambda^* \eta \wedge \Phi = \int_E \iota_{r\partial/\partial r} a_\lambda^* \eta \wedge d\Phi = 0.$$

Unfortunately, the next to last inequality requires an application of Stokes' Theorem to E. This involves estimating $\iota_{r\partial/\partial r} a_\lambda^* \eta$ which is easy and estimating $\mathscr{L}_{r\partial/\partial r} a_\lambda^* \eta$ which appears to require that $||\partial\eta/\partial r||$ is slowly increasing in r, a requirement that is hard to restate in such a way that it would be satisfied for the applications we have in mind. Instead we give a direct argument for the constancy of B.

We will construct for each λ in $[1, \infty]$, a form τ_λ satisfying:

(i) $d\tau_\lambda = a_\lambda^* \Phi - \Phi$

(ii) $\displaystyle\lim_{r \to \infty} \int_{S(r)} \eta \wedge \tau_\lambda = 0$

where $S(r)$ is the sub-bundle of E with fiber the sphere of radius r in the corresponding fiber of E.

Let us first suppose that such forms $\{\tau_\lambda\}$ exist. We denote by $D(r)$ the bundle obtained from E by replacing each fiber with the closed ball of radius r.

LEMMA 2.3.

$$\int_{D(r)} \eta \wedge a_\lambda^* \Phi - \int_{D(r)} \eta \wedge \Phi = \int_{S(r)} \eta \wedge \tau_\lambda.$$

Proof. In the following argument we abbreviate $\eta \wedge \tau_\lambda$ by ψ. Since N is complete, there exists a proper smooth function $\mu:N \rightarrow [0, \infty)$ and a constant c so that $|d\mu(x)| < c$ for all $x \in N$ ([**16**], Section 35). Let $B(R) \subset N$ be given by

$$B(R) = \mu^{-1}([0, R]).$$

Then $\{B(R):R \geqq 0\}$ is an increasing family of compact sets which exhaust N. We now define a function ρ on $D(r)$ by $\rho = \mu \circ \pi$ so ρ is constant on the fibers of π. We let

$$C(R) = \rho^{-1}([0, R]).$$

Then $\{C(R):R \geqq 0\}$ is an increasing family of compact sets which exhaust $D(r)$. Of course $C(R)$ is the inverse image of $B(R)$ under $\pi:D(r) \rightarrow N$.

Now let m be a smooth function from $(-\infty, \infty)$ to $[0, 1]$ which is 0 for x negative and 1 for $x \geqq 1$. We define a one parameter family $\{f_R :\geqq 0\}$ of smooth functions on $D(r)$ by the formula:

$$f_R(x) = m(\rho(x) - R + 1).$$

We find that f_R is identically zero on $C(R - 1)$ and identically 1 on $C(R)'$, the complement of $C(R)$ in $D(r)$. Also $|df_R|$ is bounded on $D(r)$ independent of R and is identically zero outside of the annulus $C(R) - C(R - 1)$. We have:

$$\psi = f_R\psi + (1 - f_R)\psi$$

$$d\psi = df_R \wedge \psi + f_R d\psi + d((1 - f_R)\psi).$$

Hence:

$$\int_{D(r)} d\psi = \int_{D(r)} df_R \wedge \psi + \int_{D(r)} f_R d\psi + \int_{D(r)} d((1 - f_R)\psi).$$

We show the first two integrals on the right hand side go to zero as R goes to infinity.

$$\left|\int_{D(r)} df_R \wedge \psi\right| = \left|\int_{C(R)-C(R-1)} df_R \wedge \psi\right|$$

$$\leqq K \int_{C(R-1)'} |\psi| \quad \text{for some } K > 0.$$

But since $|\psi|$ is an integrable function on $D(r)$ and the cylinders $\{C(R)\}$ exhaust $D(r)$ we have:

$$\lim_{R\rightarrow\infty} \int_{C(R-1)'} |\psi| = 0.$$

This fact is proved from the Lebesgue dominated convergence theorem by noting that if χ_R is the characteristic function of $C(R - 1)'$ then $\chi_R|\psi|$ goes to zero pointwise and is dominated by $|\psi|$.

As for the second integral we have:

$$\left|\int_{D(r)} f_R d\psi\right| \leqq \int_{C(R-1)'} \|d\psi\|.$$

The right-hand integral tends to zero by the previous argument.

To evaluate the third integral we note that $1 - f_R$ vanishes outside $C(R)$. We choose R' larger than R so that the vertical sides of $C(R')$ are smooth (such an R' exists by Sard's Theorem applied to ρ). We then have:

$$\int_{D(r)} d((1 - f_R)\psi) = \int_{C(R')} d((1 - f_R)\psi) = \int_{S(r)} (1 - f_R)\psi.$$

The last equality follows because $(1 - f_R)\psi$ vanishes on the vertical sides of $C(R')$. Thus we obtain:

$$\int_{D(r)} d\psi = \lim_{R\to\infty} \int_{S(r)} (1 - f_R)\psi.$$

We apply the Lebesgue dominated convergence theorem to the integral on the right-hand side noting that

$$\lim_{R\to\infty} (1 - f_R)\psi = \psi \quad \text{and} \quad |(1 - f_R)\psi| \leqq |\psi|.$$

With this the lemma is proved.

Under the assumption that the forms $\{\tau_\lambda\}$ exist, we have now proved the theorem. Indeed, passing to the limit in Lemma 2.3 as r goes to infinity we obtain for every λ:

$$\int_E \eta \wedge a_\lambda^*\Phi = \int_E \eta \wedge \Phi.$$

Now passing to the limit in λ we obtain our theorem by Lemma 2.2.

We now construct τ_λ. First recall that $\{a_\lambda:\lambda \in \mathbf{R}_+^*\}$ is the one parameter group with infinitesimal generator $r\partial/\partial r$. We define τ_λ by the formula:

$$\tau_\lambda = \iota_{r\partial/\partial r} \int_1^\lambda a_\mu^*\Phi\frac{d\mu}{\mu}.$$

It is immediate that τ_λ satisfies (i) above. Indeed:

$$d\tau_\lambda = \mathcal{L}_{r\partial/\partial r} \int_1^\lambda a_\mu^*\Phi\frac{d\mu}{\mu} = \frac{d}{ds}\int_1^\lambda a_s^* a_\mu^* \frac{d\mu}{\mu}\bigg|_{s=1}$$

$$= \frac{d}{ds}\int_s^{s\lambda} a_\mu^*\Phi\frac{d\mu}{\mu}\bigg|_{s=1} = a_\lambda^*\Phi - \Phi.$$

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We now prove that τ_λ satisfies (ii). We first note that for any form τ and any finite-volume oriented submanifold N of a Riemannian manifold E such that $||\tau||$ is bounded on N we have:

$$\left|\int_N j^*\tau\right| \leqq \sup_{x\in N}||\tau(x)||\ \mathrm{vol}(N).$$

Indeed since j^* is norm decreasing it is sufficient to prove the above inequality for a top dimensional form (on N) where it is obvious.

We next note that:

$$\begin{aligned}||\tau_\lambda|| &\leqq r\left|\left|\int_1^\lambda a_\mu^*\Phi\frac{d\mu}{\mu}\right|\right| \leqq r\int_1^\lambda ||a_\mu^*\Phi||\frac{d\mu}{\mu}\\ &\leqq rC\int_1^\lambda a_\mu^*||\Phi||\mu^k\frac{d\mu}{\mu} \leqq rC\int_1^\lambda e^{-m\rho\mu r}p(\mu r)\mu^k\frac{d\mu}{\mu}\\ &\leqq e^{-m\rho r}q(r)\end{aligned}$$

where q is a polynomial. This last estimate is obtained by integrating the inequality:

$$e^{-m\rho\mu r}p(\mu r)\mu^{k-1} \leqq e^{-m\rho r}p(\mu r)\mu^{k-1} \quad \text{for } 1 \leqq \mu \leqq \lambda.$$

Hence:

$$||\eta \wedge \tau_\lambda|| \leqq C||\tau_\lambda|| \leqq Ce^{-m\rho r}q(r).$$

But we have seen in Lemma 1.1 that

$$\mathrm{vol}(S(r)) \leqq Cr^{k-1}e^{(m-1)\rho r}.$$

With this (ii) is established and the theorem is proved.

Theorem 2.1 may be generalized to include the case of cycles with coefficients. Let V be a flat bundle over E and s a parallel section of V. Then we may form a cycle $N \otimes s$ with coefficients in V; see [**9**], Section 4. We assume that we have chosen a Riemannian metric on V^* such that $||s||$ is bounded on E.

COROLLARY. *If η is a closed bounded t-form on E with values in V^* we have*:

$$\int_E \eta \wedge (\Phi \otimes s) = \int_N j^*\eta \wedge \pi_*(\Phi \otimes s).$$

Proof.

$$\int_E \eta \wedge (\Phi \otimes s) = \int_E \langle\eta, s\rangle \wedge \Phi = \int_N j^*\langle\eta, s\rangle \wedge \pi_*(\Phi).$$

Here $\langle\ ,\ \rangle$ denotes the pairing between V^* and V. The last equality follows because $\langle \eta, s\rangle$ is a bounded closed form with scalar values so Theorem 1 applies.

3. A variation on a theme of Borel. In this section we give another proof of Theorem 2.1 (in order to put it in its proper context) by considering the analogues for the tubes of Section 1 of Theorem 3.4 and Theorem 5.2 of [**3**]. For simplicity we assume the manifold N of Section 1 is totally geodesic.

We will also assume that N has the following property. Let $\mathscr{A}_b^*(N)$ denote the subcomplex of the de Rham complex $\mathscr{A}^*(N)$ consisting of those forms of η such that η and $d\eta$ are bounded. We assume that the inclusion of $\mathscr{A}_b^*(N)$ into $\mathscr{A}^*(N)$ induces an isomorphism of cohomology. This property is satisfied trivially if N is compact and for arithmetically defined finite volume quotients of symmetric spaces by [**3**], Theorem 3.4.

We have a vector bundle structure $\pi:D \to D_1$. By choosing a system of global Fermi coordinates we can enlarge D to a manifold with boundary $\bar{D}$ by adding a point for each ray in the normal bundle emanating from a point of D_1. We note each ray may be parametrized by the restriction of the function r of Section 2. Since the group Γ acts by isometries it will preserve the set of rays and consequently it will act on $\bar{D}$. The quotient space $\bar{E} = \Gamma\backslash\bar{D}$ is clearly compact along the fibers. We use the Fermi polar coordinates $(u_1, u_2, \dots, u_{k-1})$ to give coordinates in S, the sphere bundle at infinity; observe that the u_j's are constant on rays.

We now construct the analogue of a Siegel set centered around a point ∞_x at infinity in the fiber over $x \in N$. We let ω_1 be a small convex neighborhood of x in N and ω_2 a small disk in the unit sphere in the normal fiber over x intersecting the ray corresponding to ∞_x. Since the covering $p:D \to E$ is trivial over the contractible set $\pi^{-1}(\omega_1)$ we have a product structure on $\pi^{-1}(\omega_1)$ induced by the global Fermi coordinates on D. We let $\omega = \omega_1 \times \omega_2$ and define $S_{t,\omega}$ to be the set of points in E whose x and u coordinates are in ω_1 and ω_2 respectively and such that $r(x) > t$. Clearly, the collection of open sets obtained by varying ω and t in the above construction gives a neighborhood basis for the points in S. We will call such sets, special open sets. Each special open set is stable for the action of a_λ provided $\lambda \geqq 1$.

We now give a precise notion of growth for differential forms on E. Let n be a real number. We define $a:E \to \mathbf{R}_+$ by:

$$a(x) = e^{r(x)}.$$

Definitions. (i) A form η is said to be *n-bounded* if there exists a polynomial $p = p(r)$ in one variable such that:

$$\|\eta(x)\| \leqq a(x)^n p(r(x)).$$

(ii) A form is said to have *moderate growth* if it is n-bounded for some n (hence for all $m \geqq n$).

(iii) A form is said to have *rapid decrease* if it is n-bounded for all n.

(iv) A form η is said to have *compact support along the fiber* if the support of η is contained in the disk bundle $D(r)$ for some r (depending on η).

We now consider the complex $\mathscr{A}_n^*(E)$ consisting of those forms η on E such that η and $d\eta$ are n-bounded. We also have the complexes $\mathscr{A}_{mg}^*(E)$ consisting of those forms η on E such that η and $d\eta$ are moderate growth and $\mathscr{A}_{rd}^*(E)$ consisting of those forms η on E such that η and $d\eta$ are rapidly decreasing and $\mathscr{A}_c^*(E)$ consisting of forms on E that are compactly supported along the fiber. We then have the following theorem, to be compared with [**3**], Theorem 3.4.

THEOREM 3.1. *The cohomology of $\mathscr{A}_n^*(E)$ for $n \geqq 0$ is the cohomology of E with coefficients in* **R**. *In particular, the cohomology of $\mathscr{A}_{mg}^*(E)$ is the cohomology of E with coefficients in* **R**.

Proof. Let $\mathscr{A}_b^*(N)$ denote the complex described in the first paragraph of this section. Then by Lemma 1.3 if $n \geqq 0$ we have maps of complexes:

$$\pi^*:\mathscr{A}_b^*(N) \rightarrow \mathscr{A}_n^*(E)$$

$$j^*:\mathscr{A}_n^*(E) \rightarrow \mathscr{A}_b^*(N).$$

These maps satisfy $j^*\pi^* = \mathrm{id}$; hence, π^* induces an injection of $H^*(N, \mathbf{R})$ into $H(\mathscr{A}_n^*(E))$. To prove surjectivity of π^* it is sufficient to prove π^*j^* induces the identity map on $H(\mathscr{A}_n^*(E))$. But in Lemma 2.1 we have seen that

$$\lim_{\lambda\to 0} a_\lambda^*\eta = \pi^*j^*\eta.$$

Hence, it is sufficient to prove that $a_\lambda^*\eta$ is cohomologous to η in $\mathscr{A}_n^*(E)$ for each λ. But we have seen that if we define

$$\tau_\lambda = \iota_{r\partial/\partial r} \int_1^\lambda a_\mu^*\eta \frac{d\mu}{\mu}$$

then

$$d\tau_\lambda = a_\lambda^*\eta - \eta.$$

But clearly, τ_λ is n-bounded if η is and the first statement of the theorem is proved. The second follows from the first because cohomology commutes with direct limits and

$$\mathscr{A}_{mg}^*(E) = \lim_{\substack{\rightarrow \\ n}} \mathscr{A}_n^*(E).$$

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The next theorem is the analogue of [**3**], Theorem 5.2.

THEOREM 3.2. *There exists a real number* n_0 *so that if* $n \leqq -n_0$ *then the cohomology of* $\mathscr{A}_n^*(E)$ *is the cohomology of* E *with compact support along the fiber. Moreover the cohomology of* $\mathscr{A}_{rd}^*(E)$ *is the cohomology of* E *with compact support along the fiber.*

Proof. Of course we have an inclusion for every n:

$$\iota: \mathscr{A}_c^*(E) \rightarrow \mathscr{A}_n^*(E).$$

To show ι is onto in cohomology, for n is sufficiently negative, is easy, see [**12**], Lemma III.3.1. However, it is somewhat harder to establish injectivity. In fact, we establish injectivity and surjectivity at the same time following the sheaf-theoretic method of [**3**].

As in [**3**], we define presheaves $\mathscr{F}_c^*$ and $\mathscr{F}_n^*$ by assigning to any open set $U \subset \bar{E}$ the space of differential forms on $U \cap E$ which are restrictions of forms compactly supported along the fiber (respectively n-bounded with n-bounded exterior derivatives). The presheaves $\mathscr{F}_c^*$ and $\mathscr{F}_n^*$ are obviously sheaves. $\mathscr{F}_c^*$ and $\mathscr{F}_n^*$ are flabby sheaves by definition. Hence, by the comparison theorem in sheaf theory, [**6**], II, 4.6.2, it is sufficient to prove that the inclusion $\mathscr{F}_c^* \rightarrow \mathscr{F}_n^*$ induces an isomorphism of derived sheaves. To see this it is sufficient to prove a Poincaré lemma for $\mathscr{F}_n^*(U)$ where U is the complement of a tubular neighborhood $D(a)$ of N. Now if $\eta \in \mathscr{F}_n^*(U)$ we consider the following expression:

$$\tau_\lambda = -\iota_{r\partial/\partial r} \int_1^\infty a_\mu^* \eta \frac{d\mu}{\mu}.$$

We extend τ_λ to E by multiplying η by a smooth function σ of r which is zero in a neighborhood of N and 1 on U. Then for $x \in U$ we find τ_λ satisfies:

$$d\tau_\lambda = \eta.$$

Also if η is n-bounded then τ_λ is also clearly n-bounded. Unfortunately, $\eta \in \mathscr{F}_n^*(U)$ with $n < 0$ does not imply that the above integral converges. We must choose n_0 so that $||\eta||$ being n_0-bounded implies that the coefficients of η in Fermi coordinates are integrable along the orbits of a_λ. Since the Fermi coordinate differentials are bounded below by Lemma 1.2, an upper bound on $||\eta||$ implies a (much weaker) upper bound on the coefficients of η. Consequently n_0 exists and the theorem is proved.

The assertion concerning the cohomology of $\mathscr{A}_{rd}^*(E)$ may be proved in a similar fashion.

Remark 3.1. The example discussed in the introduction for $\Gamma\backslash\mathbf{H}$ shows that the cohomology of $\mathscr{A}_n^*(E)$ for $-n_0 < n < 0$ need not coincide with either the absolute cohomology or the cohomology with compact support. Clearly if $s = \deg \eta$ then we may choose $n_0 = s\rho + \epsilon$ for any $\epsilon > 0$. We note that to prove Theorem 3.2 we really needed to compare only two Mayer-Vietoris sequences, not spectral sequences as in [**6**].

We now show how the considerations of this section give a new proof of Theorem 2.1.

PROPOSITION 3.1. *If the inclusion* $\iota:\mathscr{A}_c^s(E) \to \mathscr{A}_{m\rho}^s(E)$ *is surjective then Theorem* 2.1 *holds.*

Proof. Let Φ and η be as in the statement of Theorem 2.1. Then we may find $\tau \in \mathscr{A}_{m\rho}^{s-1}(E)$ and $\psi \in \mathscr{A}_c^s(E)$ such that:

$$\Phi = \psi + d\tau.$$

But then we have by Stokes' Theorem (noting that $\eta \wedge \tau$ is integrable by Remark 2.1):

$$\int_E \eta \wedge \Phi = \int_E \eta \wedge \psi.$$

By a similar argument using Theorem 3.1 we may assume η is a pull-back $\eta = \pi^*\nu$ of a form ν on N. Since $\nu = j^*\eta$ we see that ν is necessarily closed and bounded. But an easy modification of a standard result in topology (see [**4**], Proposition 6.15 and use that supp ψ and N have finite volume) shows that:

$$\int_E \pi^*\nu \wedge \psi = \int_N \nu \wedge \pi_*(\psi).$$

Also

$$\pi_*\psi = \pi_*\Phi + d\pi_*\tau \quad \text{and} \quad \int_N d\pi_*(\tau) \wedge \nu = 0.$$

Hence we obtain:

$$\int_N \nu \wedge \pi_*(\psi) = \int_N \nu \wedge \pi_*(\Phi)$$

and the proposition is proved.

COROLLARY. *Theorem* 2.1 *holds.*

Proof. We have proved ι is surjective in Theorem 3.2.

4. The theta correspondence and cohomology. We begin this section by recalling a cohomological version of the theta correspondence. Let G denote $O(p, q)$, $U(p, q)$ or $Sp(p, q)$ and G' denote respectively the metaplectic covers of $Sp_n(\mathbf{R})$, $U(n, n)$ or $SO^*(4n)$. Let V denote the standard representation of G. Let K denote a maximal compact subgroup of G and K' a maximal subgroup of G'. We let z_0 denote the point in D, the symmetric space of G, with isotropy K. We let D' denote the symmetric space of G'.

Let $\Phi = \Phi(z, x)$ for $z \in D$, $x \in V^n$, be an element of

$$(\mathscr{A}^l(D) \otimes \mathscr{S}(V^n))^{G_0},$$

the superscript G_0 denoting G_0-invariants where G_0 denotes the identity component of G. Hence Φ may be regarded as a differential form on D taking values in the Schwartz space $\mathscr{S}(V^n)$. Recall that G' acts on $\mathscr{S}(V^n)$ via the oscillator representation ω; see for example [**14**]. We now make two assumptions concerning Φ. First we assume Φ is K'-finite (where K' acts by ω). Second we assume Φ is closed as a differential form on D (all components of Φ relative to a basis of $\mathscr{S}(V^n)$ are closed forms on D). This second assumption is equivalent to saying Φ is a cocycle in the standard complex used to calculate the continuous cohomology of G with values in $\mathscr{S}(V^n)$. We may then define a function $\theta_\Phi(g', g)$ on $G' \times G$ by:

$$\theta_\Phi(g', g) = \Theta(\omega(g')\Phi(gz_0, x)).$$

Here Θ is the theta distribution described in the introduction associated to a lattice $L \subset V$ and an element $x_0 \in L^n$. Then θ_Φ is an l-form on $M = \Gamma\backslash D$ where Γ is the subgroup of G leaving the lattice L invariant. Also, θ_Φ is an automorphic form in g' for the (arithmetic) subgroup $\Gamma' \subset G'$ which leaves Θ fixed; see Chapter II of [**14**]. We may use θ_Φ as the kernel of an integral transform Λ_Φ from automorphic forms on G' to closed differential l-forms on M.

Of course the previous considerations are interesting only if there are examples of closed forms Φ with values in $\mathscr{S}(V^n)$ as described above. In fact such forms exist in abundance; see [**12**]. Since we do not have space here to discuss the general theory we apply the integral formula of Section 2 to the integral transform Λ_Φ for $G = SO(p, q)$ and Φ as in [**14**], Chapter III, Section 1. In this case we may regard D is the set of negative q-planes in V, [**12**], I.1. By a negative q-plane in V we mean a subspace $P \subset V$ of dimension q so that $(\ ,\ )$ restricted to P is negative definite. Let $m = p + q$.

We now recall the formula for this Φ. First we let $\mathfrak{g}$ denote the Lie algebra of G, $\mathfrak{k}$ the Lie algebra of K and $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ under the Killing form of $\mathfrak{g}$. Then by Frobenius reciprocity we have an isomorphism:

$$F:(\mathscr{A}^l(D) \otimes \mathscr{S}(V^n))^{G_0} \to (\Lambda^l\mathfrak{p}^* \otimes \mathscr{S}(V^n))^{K_0}.$$

Here the arrow F is evaluation at the negative q-plane z_0 and we have identified $T_{z_0}(D)$ and $\mathfrak{p}$. We will denote the image of an element Φ under the above arrow by ϕ. We note that $\mathfrak{p}$ is canonically isomorphic to $z_0^\perp \otimes z_0$. Also K_0 denotes the identity component of K. We change our notation from Λ_Φ to Λ_ϕ and from θ_Φ to θ_ϕ.

We first construct an element Φ_0 of $(\mathscr{A}^0(D) \otimes \mathscr{S}(V^n))^{G_0}$. We let $(\ ,\ )$ be the standard form of signature (p, q) and $(\ ,\ )_{z_0}$ be the majorant (see [**14**], I.1) corresponding to z_0. We define a positive definite form $((\ ,\ ))$ on V^n by defining $((x, y))$ to be the trace of the n by n matrix with i, j-th entry $(x_i, y_j)_{z_0}$. Here we have $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ and x_i, y_j are elements of V. We define the Gaussian $\phi_0 \in \mathscr{S}(V^n)^{K_0}$ by the formula:

$$\phi_0(x) = e^{-\pi((x,x))} = \prod_{i=1}^{n} e^{-\pi(x_i,x_i)_{z_0}}.$$

Then ϕ_0 transforms by a character under K', the maximal compact subgroup of $Mp_n(\mathbf{R})$ covering $U(n)$.

We now look for a G_0 invariant, K' semi-invariant, operator ∇ such that:

$$\nabla:(\mathscr{A}^*(D) \otimes S(V^n))^{G_0} \rightarrow (\mathscr{A}^{*+k}(D) \otimes S(V^n))^{G_0}.$$

Using the isomorphism F it is sufficient to write down an operator ∇' which is K_0 invariant, K' semi-invariant and satisfies:

$$\nabla':(\Lambda^*(z_0^{\perp} \otimes z_0) \otimes S(V^n))^{K_0} \rightarrow (\wedge^{*+nq}(z_0^{\perp} \otimes z_0) \otimes S(V^n))^{K_0}.$$

Such an operator will give rise to the desired operator ∇.

We give a formula in coordinates for ∇'. We choose a basis $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_m\}$ compatible with the splitting

$$V = z_0^{\perp} \otimes z_0$$

such that $(\ ,\ )$ is in standard diagonal form relative to this basis. Then we let

$$\{x_{ij}:i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$$

denote coordinates relative to the basis $\{e_i \otimes \epsilon_j\}$ for $V \otimes \mathbf{R}$. Here $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ is the standard basis for $\mathbf{R}^n$ and we write $V^n = V \otimes \mathbf{R}^n$. We use the index convention that α, β will stand for indices between 1 and p and μ, ν for those between $p + 1$ and m. We normalize the Riemannian metric on D to coincide on $T_{z_0}(D)$ with the negative of the tensor product $(\ ,\ ) \otimes (\ ,\ )$ restricted to $z_0^{\perp} \otimes z_0$. For this metric

$$\{e_\alpha \otimes e_\mu:1 \leqq \alpha \leqq p, p + 1 \leqq \mu \leqq p + q\}$$

is an orthonormal basis. Using the metric $e_\alpha \otimes e_\mu$ gives rise to an element $(e_\alpha \otimes e_\mu)^{\#}$ in $T^*_{z_0}(D)$ which we identify with the Maurer-Cartan form $\omega_{\alpha\mu}$ in $\mathfrak{p}^*$. This is a K-equivariant identification.

We have operators $\partial/\partial x_{ij}$, $M(x_{ij})$ on $\mathscr{S}(V^n)$ where $M(x_{ij})$ denotes multiplication by x_{ij}. We also have operators $A(\omega_{ij})$ on $\Lambda^*(z_0^{\perp} \otimes z_0)$ where $A(\omega_{ij})$ denotes exterior multiplication by ω_{ij}. Then we define the Howe operator by:

$$\nabla' = \frac{(-1)^{nq}}{2^{nq/2}} \prod_{i=1}^{n} \prod_{\mu=p+1}^{m} \sum_{\alpha=1}^{p} \left(\frac{1}{2\pi} \frac{\partial}{\partial x_{\alpha i}} - M(x_{\alpha i})\right) \otimes A(\omega_{\alpha\mu}).$$

Finally, we define

$$\phi = \nabla' \phi_0 \in (\Lambda^{nq}(z_0^{\perp} \otimes z_0) \otimes \mathscr{S}(V^n))^{K_0}.$$

In what follows we will need somewhat more information concerning the continuous cohomology class ϕ. We may express ϕ in terms of monomials ω_I in the $\omega_{\alpha\mu}$'s according to:

$$\phi(x) = \sum_I \phi_I(x)\omega_I.$$

In the case $n = 1$, it is important to know the coefficient $\phi_{I_0}(x)$ of $\omega_{1p+1} \wedge \ldots \wedge \omega_{1p+q}$. We see that this coefficient is given by:

$$\phi_{I_0}(x) = \frac{1}{2^{q/2}(2\pi)^{q/2}} H_q(\sqrt{2\pi}(e_1, x))\phi_0(x)$$

where $H_q(t)$ is the q-th Hermite polynomial given by:

$$H_q(t) = (-1)^q e^{t^2} \frac{d^q}{dt^q} e^{-t^2}.$$

We have now defined forms $\Phi(z, x)$ on D for any x in V^n. When we are interested in the dependence on n we will denote the above form by Φ_n.

We observe that the form Φ_1 determines the form Φ_n. Indeed, we have an isomorphism $\mathscr{S}(V)^{\otimes n}$ to $\mathscr{S}(V^n)$ sending $f_1 \otimes f_2 \otimes \ldots \otimes f_n$ to

$$\prod_{i=1}^{n} f_i.$$

We also have the n-th exterior power map.

$$\Lambda^q(z_0^{\perp} \otimes z_0) \to \Lambda^{nq}(z_0^{\perp} \otimes z_0).$$

Clearly both of these maps are K-homomorphisms. Combining these two mappings we obtain a K-homomorphism (of degree n):

$$\Lambda^q(z_0^{\perp} \otimes z_0) \otimes S(V) \to \Lambda^{nq}(z_0^{\perp} \otimes z_0) \otimes S(V^n)$$

and consequently a map of K_0-invariants to be denoted $\wedge$:

$$\bigotimes_1^n (\Lambda^q(z_0^{\perp} \otimes z_0) \otimes S(V))^{K_0} \to (\Lambda^{ng}(z_0^{\perp} \otimes z_0) \otimes S(V^n))^{K_0}.$$

If $g \in G$ and $z = gz_0$ we let $(\, , \,)_z$ denote the majorant of $(\, , \,)$ associated to z. Then we have:

$$(x, y)_z = (g^{-1}x, g^{-1}y)_{z_0}.$$

We note the transformed Gaussian satisfies

$$\phi_0(g^{-1}x) = \prod_{i=1}^{n} e^{-\pi(x_i, x_i)_z}.$$

We then have the following lemma whose proof is left to the reader.

LEMMA 4.1.

$$\Phi_n(z, x) = \Phi_1(z, x_1) \wedge \Phi_1(z, x_2) \wedge \ldots \wedge \Phi_1(z, x_n).$$

Notation. From the notation adopted above we see $\Phi_1(z, x_j)$ is the q-form obtained by applying the partial Howe operator

$$\nabla_j = \frac{(-1)^q}{2^{q/2}} \prod_{\mu=p+1}^{p} \sum_{\alpha=1}^{p} \left(\frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha j}} - M(x_{\alpha j})\right) \otimes A(\omega_{\alpha\mu})$$

to the Gaussian in the variable x_j.

In Section 5 we will need a naturality property of the forms $\Phi(z, x)$ under restriction.

Let y be a vector in V of positive length and x another vector of positive length so that $x = x' + x''$ with x' a multiple of y and $(x'', y) = 0$. Let V_y denote the orthogonal complement of y in V, G_y be the subgroup of G which fixes y, D_y the set of negative q-planes contained in V_y and $i_y : D_y \rightarrow D$ the inclusion. We may consider the dual pair

$$Mp_n(R) \times G_y \subset Mp(\mathbf{R}^{2n} \otimes V_y).$$

The theory of the previous section produces an element

$$\Phi'_n \in (\mathscr{A}^q(D_y) \otimes \mathscr{S}(V_y))^{G_y}.$$

We then have the following lemma.

LEMMA 4.2.

$$i_y^*\Phi(z, x) = \phi_0(x')\Phi'(z, x'') \quad (\textit{note } z \in D_y).$$

We now summarize the main properties of Φ. We need some more notation. Let G_x denote the stabilizer of the span of x, G''_x the isotropy of x and G'_x the subgroup of G_x acting trivially on the orthogonal complement of the span of x. We have:

$$G_x = G'_x \times G''_x.$$

We now have the following proposition whose proof may be found in [**12**]. The reader should also be able to verify the following properties by direct calculation.

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PROPOSITION 4.1. (i) *$\Phi(z, x)$ is a closed nq-form on D for every x in V^n.*

(ii) *$\Phi(z, x)$ transforms under $MU(n)$ according to the* $(\sqrt{\det})^m$ *(see Chapter* II *of* [**14**]).

(iii) *$\Phi(z, x)$ is invariant under the group G_x'' (but not under G_x).*

We have now constructed the desired Φ and we consider the element

$$\theta_\phi \in \mathscr{A}^{nq}(\Gamma\backslash D) \otimes C^\infty(Mp_n(\mathbf{R}))$$

defined by:

$$\theta_\phi(g', z) = \Theta(\omega(g')\Phi) = \sum\nolimits'_{x \in L^n} \omega(g')\Phi(z, x).$$

Clearly, θ_ϕ defines a closed differential nq form on $M = \Gamma\backslash D$ for a suitable congruence subgroup (again denoted Γ) of the integral points of $O(p, q)$. The transformation law in g' is very subtle but is well-known, see [**14**], Chapter II. Since Θ is invariant under Γ':

(i) $\theta_\phi(\gamma' g', z) = \theta_\phi(g', z)$

and since ϕ transforms under K' like $(\sqrt{\det})^m$ we have:

(ii) $\theta_\phi(g'k', z') = [\sqrt{\det}(k')]^m \theta_\phi(g', z)$.

The formulas (i) and (ii) imply that θ_ϕ is a section of the line bundle $\mathscr{L}$ over $M = \Gamma'\backslash \mathfrak{h}_n$ (here $\mathscr{L}$ is the m-th power of the $Mp_n(\mathbf{R})$-homogeneous line bundle with isotropy representation $\sqrt{\det}$). We use τ to denote the coordinate in $\mathfrak{h}_n$. Then $\tau = u + iv$ with u and v real n by n symmetric matrices and v positive definite. We define an element $g'_\tau \in Mp_n(\mathbf{R})$, satisfying $g'_\tau(i1_n) = \tau$, by the following formula:

$$g'_\tau = \begin{pmatrix} \sqrt{v} & \sqrt{v}^{-1}u \\ 0 & \sqrt{v}^{-1} \end{pmatrix}.$$

Then we define $\theta_\phi(\tau, z)$ by the formula:

$$\theta_\phi(\tau, z) = j(g'_\tau, i1_n)^{m/2}\theta_\phi(g'_\tau, z) = (\det v)^{-m/4}\theta_\phi(g'_\tau, z).$$

We may use Θ_ϕ as a kernel of an integral transform and we obtain an integral transform:

$$\Lambda: C_0^\infty(M', \mathscr{L}) \to \mathscr{A}^{nq}(M)$$

given by:

$$\Lambda(f) = (\theta_\phi, f)$$

where the inner product on the right is the hermitian L^2-inner product on $C_0^\infty(M', \mathscr{L})$ anti-linear in the second variable. We will often use the notation $\theta_\phi(f)$ instead of $\Lambda(f)$.

Since θ_ϕ is closed we also obtain a map:

$$C_0^\infty(M', \mathscr{L}) \to H^{nq}(M, \mathbf{R}).$$

Now $\mathscr{L}$ is a holomorphic line bundle. Holomorphic sections of $\mathscr{L}$ are classical Siegel modular forms; that is, holomorphic functions on $\mathfrak{h}_n$ satisfying the transformation law:

$$f((a\tau + b)(c\tau + d)^{-1}) = \det(c\tau + d)^{m/2}f(\tau)$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma' \subset Sp_n(\mathbf{Z}).$$

We denote the holomorphic cusp-forms satisfying the above transformation law by $S_{m/2}(\Gamma')$. Since θ_ϕ has moderate growth we can integrate a holomorphic cusp form against θ_ϕ and obtain a lifting

$$\Lambda: S_{m/2}(\Gamma') \to \mathscr{A}^{nq}(M).$$

A computation of Casimir values yields the following theorem (see [**12**]).

THEOREM 4.1. *The lift of a holomorphic cusp form is a closed harmonic nq form on M.*

We have constructed a mapping from spaces of classical Siegel modular forms to closed harmonic forms on locally symmetric spaces of orthogonal groups. We want to relate the image of this map to the dual classes of special cycles which we have described in the introduction.

We consider the pairing, denoted $[\ ,\ ]$, between nq-forms ω with arbitrary support and compactly supported $(p - n)q$ forms η given by:

$$[\eta, \omega] = \int_M \eta \wedge \omega.$$

Now define the Siegel modular form $\theta_\phi(\eta)$ for η compactly-supported of degree $(p - n)$ as $[\eta, \theta_\phi]$. One finds easily that:

$$(\theta_\phi(\eta), f) = [\eta, \theta_\phi(f)]$$

where the inner product on the left is the Petersson inner product on $S_{m/2}(\Gamma')$. Now consider the following two subspaces of the cohomology of degree $(p - n)q$ with compact supports. The first, $H_\theta^\perp$, is the space of all classes of closed compactly supported $(p - n)q$ forms which are orthogonal under $[\ ,\ ]$ to the image of $S_{m/2}(\Gamma')$. The second, $H_{\text{cycle}}^\perp$, is the space of all classes of closed compactly supported $(p - n)q$ forms with period zero on all the special cycles C_β with $\beta > 0$. We now have the following theorem which is the main theorem of this paper.

THEOREM 4.2. *If $n < m/4$ then $H_\theta = H_{\text{cycle}}$.*

The theorem follows easily from a formula for certain Fourier coefficients of $\theta_\phi(\eta)$ which we now describe.

By the transformation law for θ we see that θ_ϕ is periodic with respect to the lattice Γ'_∞. Consequently $\theta_\phi(\eta)$ has a Fourier expansion with respect to the characters of Γ'_∞. Let $a_\beta(\theta_\phi(\eta))$ denote the β-th Fourier coefficient for β an element of the dual lattice L' of Γ'_∞ (β will be a symmetric n by n matrix with rational entries). a_β is a function of v where $\tau = u + iv$. Then, for β positive definite, we have the following formula, to be proved in the next section:

$$\text{(S)} \qquad a_\beta(\theta_\phi(\eta))(v) = e^{-2\pi \operatorname{tr} \beta v} \int_{C_\beta} \eta.$$

We now show (S) implies the theorem. Clearly, it is enough to show

$$H_\theta^\perp = H_{\text{cycle}}^\perp.$$

This later equality we establish by proving two inclusions.

We first establish $H_{\text{cycle}} \subset H_\theta^\perp$. Accordingly, we assume η is orthogonal to the dual forms of the cycles C_β. Hence $\int_{C_\beta} \eta = 0$ for all cycles C_β with β positive definite and accordingly $a_\beta(\theta_\phi(\eta)) = 0$ for all such β. But then any f in $S_{m/2}(\Gamma')$ has Fourier coefficients disjoint from $\theta_\phi(\eta)$ and consequently we have

$$[\eta, \theta_\phi(f)] = (\theta_\phi(\eta), f) = 0.$$

We now establish $H_\theta^\perp \subset H_{\text{cycle}}^\perp$. We assume that $\theta_\phi(\eta)$ is orthogonal to all holomorphic cusp forms. We introduce the Poincaré series (convergent provided $n < (p + q)/4$):

$$p_\beta(\tau) = c \sum_{\Gamma'_\infty \backslash \Gamma'} \frac{\gamma^* e^{2\pi i \operatorname{tr} \beta \tau}}{j(\gamma, \tau)^m}.$$

Here c is a constant chosen so that

$$(f, p_\beta) = a_\beta(f) \quad \text{for } f \in S_{m/2}(\Gamma').$$

We recall that $p_\beta(\tau)$ is a holomorphic cusp form.

We will also need the series:

$$p_\beta(\tau, s) = c(s) \sum_{\Gamma'_\infty \backslash \Gamma'} \frac{\gamma^* e^{2\pi i \operatorname{tr} \beta \tau}}{j(\gamma, \tau)^m} \det v(\gamma\tau)^s.$$

Here $c(s)$ is chosen so that

$$(f, p_\beta(\tau, s)) = a_\beta(f)$$

for f a holomorphic cusp form. Then assuming $n < (p + q)/4$ we have $p_\beta(\tau, s)$ is holomorphic in s in a vertical half-plane containing 0 and $p_\beta(\tau, 0) = p_\beta(\tau)$.

Since $p_\beta(\tau)$ is a holomorphic cusp form we have:

$$(\theta_\phi(\eta), p_\beta(\tau, s)) |_{s=0} = (\theta_\phi(\eta), p_\beta(\tau)) = 0.$$

We now compute the first inner product directly. By the usual unfolding argument (valid for Re s sufficiently large) we obtain:

$$\begin{aligned}
&(\theta_\phi(\eta), p_\beta(\tau, s)) \\
&= c(s) \int_{\mathscr{F}_\infty} e^{-2\pi i \operatorname{tr} \beta\tau} (\det v)^{m/2+s} \theta_\phi(\eta) \frac{dudv}{(\det v)^{n+1}} \\
&= c(s) \int_{\mathscr{P}_n} e^{-2\pi \operatorname{tr} \beta v} (\det v)^{m/2+s-(n+1)/2} a_\beta(\theta_\phi(\eta)) \frac{dv}{(\det v)^{(n+1)/2}} \\
&= c(s) \left(\int_{C_\beta} \eta \right) \int_{\mathscr{P}_n} e^{-4\pi \operatorname{tr} \beta v} (\det v)^{m/2+s-(n+1)/2} \frac{dv}{(\det v)^{(n+1)/2}}.
\end{aligned}$$

Here $\mathscr{F}_\infty$ is a fundamental domain for Γ'_∞ in $\mathfrak{h}_n$ and $\mathscr{P}_n$ is the space of positive definite symmetric n by n matrices.

The above integral formula coincides with $(\theta_\phi(\eta), p_\beta(\tau, s))$ a priori only for Re s large, but, by the principle of unique analytic continuation, it must coincide with $(\theta_\phi(\eta), p_\beta(\tau, s))$ in any region where they are both defined. The second integral has been computed in **[17]**, Hilfsatz 37, and is convergent and non-zero provided Re $s > n - m/2$. This region includes zero under our assumption on n and m and consequently both the inner product and the integral are regular at $s = 0$. Evaluating the integral at $s = 0$ we obtain a non-zero constant c' and find:

$$cc' \int_{C_\beta} \eta = (\theta_\phi(\eta), p_\beta(\tau)) = 0.$$

Hence the period of η over C_β is zero and the theorem is proved.

5. The positive-definite Fourier coefficients of $\theta_\phi(\eta)$. The purpose of this section is to prove the formula (S), that is the formula:

$$a_\beta(\theta_\phi(\eta)) = e^{-2\pi \operatorname{tr} \beta v} \int_{C_\beta} \eta \quad \text{for } \beta > 0.$$

For any x in V^n we have defined groups G_x, G'_x and G''_x such that $G_x = G'_x \times G''_x$. We define Γ_x, Γ'_x and Γ''_x by intersecting Γ. We assume henceforth that (,) restricted to the span of x is positive definite. Since we are also assuming that Γ is neat we find $\Gamma_x = \Gamma''_x$ (see for example the remark following Lemma 7.1 of **[9]**).

For x as above, we have defined D_x to be the set of negative q-planes contained in the orthogonal complement of the span of x. Recall we are taking for D the set of negative q-planes in V. We note that G_x acts transitively on D_x and also G''_x acts transitively on D_x. We have defined C_x by $C_x = \Gamma_x \backslash D_x$. We also define $E = \Gamma_x \backslash D$ and denote the covering map $D \to E$ by p. We may identify C_x with $p(D_x)$. C_x is a totally geodesic submanifold of E. We observe that the form $\Phi(z, x)$ is invariant under G''_x, hence under Γ_x and consequently induces a form, also to be denoted $\Phi(z, x)$ on E. We assume that the base-point z_0 of Section 4 is chosen to lie on D_x; that is, $u \subset z_0^{\perp}$ where u denotes the span of x.

The critical observation for what follows is that the space E is, in a natural way, a vector bundle over C_x. Indeed, since D_x is totally geodesic and D is non-positively curved, there are no focal points of D_x in D (see the proof of Lemma A1 in the appendix). Thus, the Riemannian exponential map of the total space of the normal bundle of D_x in D is a diffeomorphism. We obtain a vector bundle structure $\pi:D \to D_x$ as in Section 1. On passing to the quotient by Γ_x we obtain a vector bundle structure $E \to C_x$ also to be denoted π. We are now in the situation considered in Sections 1 and 2.

In fact there is a great deal more structure here, the fibers of π are also totally geodesic sub-symmetric spaces of D, see [**14**], Chapter I. We will require considerably more notation. We let the rank of a fiber of π (as a symmetric space) be l. We see $l = \min(n, q)$. We let A be a split torus in the fiber of π with Lie algebra $\mathfrak{A}$ and we let $r = (r_1, r_2, \ldots, r_l)$ be coordinates in $\mathfrak{A}$. More precisely, we define $\mathfrak{A}$ as follows. Choose an orthonormal basis $(x'_1, x'_2, \ldots, x'_n)$ for u. Let $v = (u \oplus z_0)^{\perp}$. Define $h_r: V \to V$ by:

$$h_r(x) = 0 \quad \text{if } x = x'_i \text{ for } i > l \text{ or } = e_{p+i} \text{ for } i > l$$

$$h_r(x'_i) = r_i e_{p+i} \quad \text{for } i = l, 2, \ldots, l$$

$$h_r(e_{p+i}) = r_i x'_i \quad \text{for } i = 1, 2, \ldots, l.$$

Then $\mathfrak{A} = \{h_r : r \in \mathbf{R}^l\}$. We put $a_r = \exp h_r$. Note that h_r and a_r are symmetric relative to $(\ ,\ )_{z_0}$. We say the split torus above is adapted to the partial frame $\{x'_1 \ldots, x'_l, e_{p+1}, \ldots, e_{p+l}\}$. We say a split torus A is adapted to the pair u, z_0 if there exist orthonormal bases as above for u and z_0 so that A may be put in the above form.

We let p_u denote the orthogonal projection on u (here "orthogonal" is interpreted as orthogonal for either $(\ ,\ )$ or $(\ .\ )_{z_0}$, each gives the same p_u). We abbreviate the norm on V or on any tensor space of V associated to $(\ ,\ )_{z_0}$ by $\|\ \|_0$. The symbol $\|\ \|$ will, as usual, denote the pointwise norm associated to the Riemannian metric on D. Let β_0 be the smallest eigenvalue of the matrix β where β is the length of x.

We wish to estimate $\|\Phi\|$ along all split tori A so that $A \cdot z_0$ is normal to D_x. These are just the various split tori adapted to u and z_0. They depend on the choice of partial frame and are permuted transitively by the compact group $G'_x \times SO(z_0)$. Of course Φ is invariant under $SO(z_0)$ but it is not invariant under G'_x.

PROPOSITION 5.1. *For any split torus A in G adapted to u and z_0, any $x \in u^n$ of length β and $a \in A$ there exists a positive constant $C_1(\beta)$ independent of a and the choice of x (of length β) but depending on β such that*:

$$||\Phi||\,(z_0, a^{-1}x) \leqq C_1(\beta)e^{-(\pi\beta_0/4)||a||_0^2}.$$

Proof. To prove the proposition we may work on D. We claim it is sufficient to prove the proposition for some split torus A adapted to u and z_0. To see this assume the proposition is proved for some such torus A. Now let B be any other split torus adapted to the pair u, z_0. Then there exists $k \in K$ such that k stabilizes u and such that $kBk^{-1} = A$. Let b be in B. We claim we have:

$$||\Phi||\,(z_0, b^{-1}x) \leqq C_1(\beta)e^{-(\pi\beta_0/4)||b||_0^2}.$$

Indeed write $b = k^{-1}ak$ with $a \in A$. Then:

$$||\Phi||\,(z_0, b^{-1}x) = ||\Phi||\,(z_0, k^{-1}a^{-1}kx) = ||\Phi||\,(z_0, a^{-1}kx).$$

But kx has length β if x does so:

$$||\Phi||\,(z_0, b^{-1}x) \leqq C_1(\beta)e^{-(\pi\beta_0/4)||a||_0^2}.$$

But $||a||_0^2 = ||b||_0^2$ and the claim is proved.

We next claim we may assume $u = \text{span } e$ where $e = \{e_1, e_2, \ldots, e_n\}$. Let us denote this latter span by w. We may choose $k \in K_0$ so that $ku = w$. Suppose we have proved the above formula for w and a split torus B adapted to w, z_0. Let x be an element of u^n of length β. Then $kx \in w^n$ and $(kx, kx) = \beta$. We claim the above formula holds for x with $A = k^{-1}Bk$. By assumption we have:

$$||\Phi||\,(z_0, kak^{-1}kx) \leqq C_1(\beta)e^{-(\pi(\beta)_0/4)||kak^{-1}||_0^2}.$$

Hence:

$$||\Phi||\,(z_0, kax) \leqq C_1(\beta)e^{-(\pi\beta_0/4)||kak^{-1}||_0^2}.$$

Using the equivariance of Φ and the invariance of z_0 and $||\ ||_0$ we obtain the claim.

The above reduction is convenient because we may now take A to be the split torus adapted to the partial frame $\{e_1, \ldots, e_l, e_{p+1}, \ldots, e_{p+l}\}$. We rename w by u.

Clearly we have

$$||\Phi||^2(z_0, a^{-1}x) = p(a, x)\phi_0(a^{-1}x)^2$$

where p is a polynomial in the entries of a with coefficients which are themselves polynomials in the x_{ij} where

$$x_i = \sum_{j=1}^{n} x_{ij} e_j.$$

The coefficient polynomials are universal in the sense that they depend only on Φ and not on a or x. We observe that the set of all $x \in u^n$ of length β is compact since $(\, , \,)|u$ is positive definite. We replace each of these coefficient polynomials by the maximum value it takes on the set of all x in u^n of length β. We let q be the resulting polynomial. Clearly we have for $x \in u^n$ of length β:

$$\|\Phi\|^2(z_0, a^{-1}x) \leqq q(a)\phi_0(a^{-1}x)^2.$$

But $q(a_r)$ is a polynomial in the chr_i and shr_i. Consequently for any $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that

$$q(a_r) \leqq C(\epsilon)e^{\epsilon(ch^2r_1 + \ldots + ch^2r_l + sh^2r_1 + \ldots + sh^2r_l)}.$$

But

$$\frac{1}{2}\|a_r\|_0^2 = ch^2r_1 + \ldots + ch^2r_l + sh^2r_1 + \ldots + sh^2r_l + \frac{1}{2}(p + q) - l$$

and we obtain

$$p(a, x) \leqq C(\epsilon, \beta)e^{\epsilon\|a\|_0^2}.$$

We now estimate

$$\phi_0(a^{-1}x) = e^{-\pi\|a^{-1}x\|_0^2}.$$

Clearly:

$$\|a^{-1}x\|_0^2 = \sum_{i=1}^{n} (a^{-1}x_i, a^{-1}x_i)_{z_0} = \sum_{i=1}^{n} (a^{-2}x_i, x_i)_{z_0}.$$

Now choose $m \in \text{End } u$ so that $me_i = x_i$. We extend m to be the identity on $u^\perp$. We have:

$$\sum_{i=1}^{n} (a^{-2}x_i, x_i)_{z_0} = \sum_{i=1}^{n} (a^{-2}me_i, me_i)_{z_0}.$$

We note that the quadratic form q on u given by $q(x) = (mx, mx)$ is just the quadratic form corresponding to the matrix β and so ${}^t mm = \beta$ where we identify β with an element of End u by using the basis $\{e_1, e_2, \ldots e_n\}$. We have:

$$\sum_{i=1}^{n} (a^{-2}me_i, me_i)_i = \sum_{i=1}^{n} ({}^t ma^{-2}me_i, e_i)_{z_0}$$

$$= \operatorname{tr}(p_u{}^t m a^{-2} m p_u) = \operatorname{tr}({}^t m p_u a^{-2} p_u m)$$

$$= \operatorname{tr}(m{}^t m p_u a^{-2} p_u) \geqq \beta_0 \operatorname{tr}(p_u a^{-2} p_u).$$

Putting $a = a_r$ we find:

$$\operatorname{tr}(p_u a^{-2} p_u) = \sum_{i=1}^{l} ch 2r_i + n - l = \sum_{i=1}^{l} ch^2 r_i + \sum_{i=1}^{l} sh^2 r_i + n - l$$

$$= \frac{1}{2}\|a_r\|_0^2 + n - \frac{1}{2}(p + q).$$

We find:

$$\|\phi_0(a^{-1}x)\| \leqq C(\beta)e^{-(\pi\beta_0/2)\|a\|_0^2}.$$

Combining this estimate with the previous estimate with $\epsilon = \pi\beta_0/4$ we find for $a \in A$ and all x in u^n of length β we have:

$$\|\Phi(z_0, a^{-1}x)\| \leqq C_1(\beta)e^{-(\pi\beta_0/4)\|a\|_0^2}.$$

With this the proposition is proved.

COROLLARY. *$\Phi(z, x)$ is rapidly decreasing along the fibers of*

$$\pi:\Gamma_e\backslash D \rightarrow \Gamma_e\backslash D_e.$$

Proof. Given z in D we may send it to the fiber over z_0 by an element of G_e''. Since ϕ is invariant under G_e'' we see that it is sufficient to establish that $\|\Phi\|$ is rapidly decreasing along the fiber over z_0. But if $z \in \pi^{-1}(z_0)$ we choose a split torus A adapted to u and z_0 and an element $a_r \in A$ such that $z = a_r z_0$. Then the distance $d(z, z_0)$ from z to z_0 (and hence from z to D_e) is given by:

$$d(z, z_0)^2 = \sum_{i=1}^{n} r_i^2.$$

But we have:

$$\|\Phi\|(a_r z_0, x) \leqq C_1(\beta)e^{-1/2\beta_0\pi(ch^2 r_1 + \ldots + ch^2 r_l + sh^2 r_1 + sh^2 r_1 + \ldots + sh^2 r_l)}.$$

The corollary is now obvious.

We now apply the results of Section 2.

PROPOSITION 5.2. *Let ϕ be a rapidly decreasing closed nq-form on E and η a bounded $(p - n)q$-form on E. Let*

$$\kappa = \int_{\text{fiber}} \phi.$$

Then:

$$\int_E \eta \wedge \phi = \kappa \int_{C_x} \eta.$$

Proof. We apply Theorem 2.1 and observe that since ϕ is invariant under G''_x we have $\pi_*(\phi) = \kappa$, a constant.

COROLLARY. *Let* $\Omega = \sum_{\Gamma_x \backslash \Gamma} \gamma^* \phi$. *Then we have*:

$$\int_M \eta \wedge \Omega = \kappa \int_{C_x} \eta.$$

Proof. The corollary follows from a routine unfolding argument, see [**11**], Lemma 2.1.

We now study the integral for the β-th Fourier coefficient of $\theta_\phi(\eta)$ where η is a compactly supported form on M. This integral is given by:

$$a_\beta(\theta_\phi(\eta))(v) = \frac{1}{\operatorname{vol} \mathscr{D}(v)} \int_{\mathscr{D}(v)} \theta_\phi(\eta)(u + iv) e^{-2\pi i \operatorname{tr} \beta u} du.$$

Let $\theta_\phi(\tau, z, \beta)$ be the function defined by:

$$\theta_\phi(\tau, z, \beta) = (\det v)^{-m/4} \sum_{\substack{x \in L^n \\ (x,x) = 2\beta}}{}' \omega(g'_\tau)\Phi(z, x).$$

Here the superscript prime indicates that we sum only over those x congruent to x_0 mod N. Then an argument identical to that of [**11**], page 254, yields the following lemma.

LEMMA 5.1.

$$a_\beta(\theta_\phi(\eta))(v) = \int_M \eta \wedge \theta_\phi(iv, z, \beta).$$

We note that $\theta_\phi(\tau, z, \beta)$ is Γ invariant but is no longer Γ' invariant. We now rewrite $\theta_\phi(\tau, z, \beta)$ as follows. Recall that we have chosen a set of representatives $\mathscr{C}'_\beta$ for the Γ-orbits of frames in L^n of length 2β which are congruent to x_0 mod N. We define for $x \in \mathscr{C}'_\beta$:

$$\theta_\phi(\tau, z, x) = (\det v)^{-m/4} \sum_{\gamma \in \Gamma_x \backslash \Gamma} \omega(g'_\tau)\gamma^* \Phi(z, x).$$

We define:

$$\kappa(g', x) = \int_{\text{fiber}} \omega(g')\Phi(z, x).$$

We define $\kappa'(\tau, x)$ for $\tau = u + iv$ by the formula:

$$\kappa'(\tau, x) = (\det v)^{-m/4} \ldots$$

$$F(g) = e^{-\epsilon(\|g^{-1}e_1\|_0^2 + \ldots + \|g^{-1}e_{n-1}\|_0^2)}.$$

We observe that if $e' = (e_1, e_2, \ldots, e_{n-1})$ and $h \in G''_{e'}$ and $k \in K$ we

$$a_\beta(\theta_\phi(\eta))(v) = \sum_{x \in \mathscr{C}'_\beta} \int_M \eta \wedge (\det v)^{-m/4} \sum_{\Gamma_x \backslash \Gamma} \omega(g'_{iv})\gamma^*\Phi(z, x)$$

hence, by Propositions 5.1 and 5.2, (since η is compactly-supported on M, $||\eta||$ is bounded on M and hence the pull-back of $||\eta||$ is bounded on E):

$$\alpha_\beta(\theta_\phi(\eta))(v) = \sum_{x \in \mathscr{C}'_\beta} \kappa'(iv, x) \int_{C_x} \eta.$$

Thus, to prove (S) it suffices to prove the following theorem.

THEOREM 5.1.

$$\kappa'(iv, x) = e^{-2\pi \mathrm{tr} \beta v}.$$

The previous formula is a local formula, it depends only on computing an integral at infinity. This integral was computed in [**14**], first for $n = 1$ by a local computation, then global considerations were used to prove a product formula, the "Main Lemma" of [**14**], Chapter III, Section 3 expressing κ's for general n and v diagonal as a product of κ for the diagonal entries of v. The arguments in [**14**] reduce computing $\kappa'(iv, x)$ to computing $\kappa'(i1_n, ea_\mu)$ where $e = (e_1, e_2, \ldots, e_n)$ and a_μ is the diagonal matrix with diagonal entries $(\mu_1, \mu_2, \ldots, \mu_n)$. This calculation is independent of the original discrete group Γ. In order to effect it we introduce a new discrete group Γ satisfying the hypotheses:

(i) Γ is cocompact in G

(ii) Γ_{e_i} is cocompact in G_{e_i} for all $i = 1, 2, \ldots, n$.

Such groups Γ are easy to construct, see [**14**]. Thus we have reduced our problem to the case considered in [**14**]. However, we take this occasion to give full details for the necessary estimates for the main lemma of [**14**] (which were only sketched at the end of [**14**]). We incorporate the necessary estimates into Lemma 5.1 (below).

In order to state our lemma we recall the forms Φ_n of Section 4 and the relation:

$$\Phi_n(z, e) = \Phi_1(z, e_1) \wedge \Phi_1(z, e_2) \wedge \ldots \wedge \Phi_1(z, e_n).$$

We define forms Φ_{n-1} and Ω_{n-1} by:

$$\Phi_{n-1}(z) = \Phi_1(z, e_1) \wedge \ldots \wedge \Phi_{n-1}(z, e_{n-1})$$

and

$$\Omega_{n-1}(z) = \sum_{\Gamma_e \backslash \Gamma_{e_n}} \gamma^*\Phi_{n-1}(z).$$

We will see below that the series for Ω_{n-1} is absolutely convergent, see (i) and (ii) below.

We now introduce a family of partial Gaussian functions F_ϵ on G depending on a parameter $\epsilon > 0$ by:

$$F_\epsilon(g) = e^{-\epsilon(\|g^{-1}e_1\|_0^2 + \ldots + \|g^{-1}e_{n-1}\|_0^2)}.$$

We observe that if $e' = (e_1, e_2, \ldots, e_{n-1})$ and $h \in G''_{e'}$ and $k \in K$ we have:

$$F_\epsilon(h'gk) = F_\epsilon(g).$$

Remark 5.3. Here we have dropped the μ_i's of [**14**]. The function $F(g)$ of [**14**] which depends on the μ_i's may be majorized by $F_\epsilon(g)$ for suitable ϵ, just replace each μ_i by

$$\mu_0 = \min\{\mu_1, \mu_2, \ldots, \mu_n\}.$$

We leave the details to the reader. We will henceforth ignore the μ_i's as they play no role in the following estimates.

LEMMA 5.1. *The family F_ϵ satisfies the following*:
(i) *There exists ϵ and C so that*

$$\|\Phi_{n-1}\| \leqq CF_\epsilon.$$

(ii) *The series*

$$\sum_{\Gamma_e \backslash \Gamma_{e_n}} \gamma^* F_\epsilon(g)$$

converges for all $\epsilon > 0$.
(iii) *h^*F_ϵ is a non-increasing function along normal geodesics to D_{e_n} for all $\epsilon > 0$ and $h \in G_{e_n}$.*

Proof. To prove (i) we apply Proposition 5.1 noting

$$\sum_{i=1}^{n-1} \|ae_i\|_0^2 = \|a\|_0^2$$

to obtain that for any split torus A adapted to e', z_0 and $a \in A$ we have:

$$\|\Phi\|(z_0, a^{-1}e') \leqq C_1 e^{-\pi/4(\|a^{-1}e_1\|_0^2 + \ldots + \|a^{-1}e_{n-1}\|_0^2)}.$$

We extend the inequality to all g by writing $g = hk'ak$ with $h \in G''_{e'}$, $a \in A$, a fixed split torus, $k \in K$ and k' a rotation leaving fixed the orthogonal complement of span e'. Both sides are independent of h and k and k' just changes A to a new split torus A'.

Since F_ϵ is rapidly decreasing along the fibers of $E = \Gamma_{e'} \backslash D$ it follows that

$$\sum_{\Gamma_{e'} \backslash \Gamma} \gamma^* F_\epsilon$$

converges. But now observe that the inclusion of Γ_{e_n} into Γ induces an embedding of $\Gamma_e \backslash \Gamma_{e_n}$ into $\Gamma_{e'} \backslash \Gamma$. Hence the series

$$\sum_{\Gamma_e \backslash \Gamma_{e_n}} \gamma^* F_\epsilon$$

is a sub-series of the above series and (ii) is proved.

To prove the third statement let a_t be the element of G defined for $t \in \mathbf{R}$ by:

$$a_t e_i = e_i \quad \text{for } i \neq n \text{ or } i \neq p + 1$$

$$a_t e_n = chte_n + shte_{p+1}$$

$$a_t e_{p+1} = shte_n + chte_{p+1}.$$

Then $a_t \cdot z_0$ sweeps out a normal geodesic to D_{e_n} as t varies. Moreover, every normal geodesic is of the form $h^{-1}a_t \cdot z_0$ for some $h \in G_{e_n}$ (observe that G_{e_n} acts transitively on normal unit spheres). But we have for $h \in G_{e_n}$:

$$F_\epsilon(ha_t z_0) = e^{-\epsilon(\|a_t^{-1}he_1\|_0^2 + \ldots + \|a_t^{-1}he_{n-1}\|_0^2)}.$$

We now make two observations. First we observe that since $h \in G_{e_n}$ we have

$$(he_i, e_n) = 0 \quad \text{for } i = 1, 2, \ldots, n - 1.$$

Also we observe that if $(x, e_n) = 0$ then

$$\|a_t^{-1}x\|_0^2 \geqq \|x\|_0^2.$$

To see this write $x = be_{p+1} + x'$ where a_t leaves x' fixed. Then

$$a_t^{-1}x = chtbe_{p+1} + x' \quad \text{and} \quad \|a_t^{-1}x\|_0^2 = ch^2tb^2 + \|x'\|_0^2.$$

The statement (iii) is now immediate.

COROLLARY. *$\|\Omega_{n-1}\|$ is bounded on $\Gamma_{e_n} \backslash D$.*

Proof. We define a function T on $\Gamma_{e_n} \backslash D$ by:

$$T = \sum_{\Gamma_e \backslash \Gamma_{e_n}} \gamma^* F_\epsilon.$$

Then by (i) we have:

$$\|\Omega_{n-1}\| \leqq CT.$$

Hence, it is sufficient to prove that T is bounded on $\Gamma_{e_n} \backslash D$. But T is nonincreasing along geodesics normal to D_{e_n} since each term in its defining series is by (iii). Hence, it is sufficient to prove that T is bounded on $\Gamma_{e_n} \backslash D_{e_n}$. But this is a compact space. The corollary is now proved.

Appendix. Some standard estimates. In this appendix we prove Lemma 1.2. We recall that we are assuming N is totally geodesic. In this case, D_1 is also totally geodesic and we may assume that the frame field $E_1, \ldots, E_k$ of

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Section 1 is parallel along D_1. In this case we have for any point $x \in D_1$ and any i, j:

$$\nabla_{\partial/\partial y_j} \frac{\partial}{\partial x_i}\Big|_x = \nabla_{\partial/\partial x_i} \frac{\partial}{\partial y_j}\Big|_x = 0.$$

Hence if γ_0 is a geodesic normal to N and emanating from x and T is the unit tangent to γ_0 then $T(0)$ is a linear combination of the $\partial/\partial y_j$'s and we obtain:

$$\nabla T \frac{\partial}{\partial x_i}\Big|_x = 0 \quad \text{for } i = 1, 2, \ldots, n.$$

Since, by [7], Corollary 2.3 we know $r\partial/\partial y_j$ is a Jacobi field for $j = 1, 2, \ldots, k$ the estimates we need are a consequence of comparison theorems for lengths of Jacobi fields. To prove the lower bounds in Lemma 1.2 we compare D to hyperbolic m-space of curvature $-\rho^2$.

For convenience we restate Lemma 1.2.

LEMMA A.1. *Let γ_0 be a geodesic normal to N emanating from $x \in N$ which is parametrized by arc length. Then along γ_0 we have the estimates*:

(a) $$\left\|\frac{\partial}{\partial x_j}\Big|_{\gamma_0(t)}\right\| \leqq \left\|\frac{\partial}{\partial x_j}\Big|_x\right\| e^{\rho t}$$

(b) $$\left\|\frac{\partial}{\partial y_j}\Big|_{\gamma_0(t)}\right\| \leqq \rho\left\|\frac{\partial}{\partial y_j}\Big|_x\right\| e^{\rho t}.$$

Proof. Suppose $\gamma_0(t) = \exp t\xi$ with $\|\xi\| = 1$. We rename ξ by T_0. We estimate $\partial/\partial x_j$ and $\partial/\partial y_j$ along $\gamma_0(t)$ by using the Rauch comparison theorems.

We first use [5], 1.28, to estimate any Jacobi field V along γ_0 satisfying $V(0) = 0$. We apply 1.28 with $M_0 = D$ and M equal to hyperbolic n-space with constant curvature $-\rho^2$. We obtain the estimate:

(*) $\quad \|V(t)\| \leqq \|V'(0)\| sh \rho t.$

We note that there are no points conjugate to x along γ_0 because D has non-positive curvature. We apply (*) with

$$V(t) = r\partial/\partial y_j|_{\gamma_0(t)}$$

and note

$$V'(0) = \partial/\partial y_j|_x.$$

Hence:

$$\left\|t\frac{\partial}{\partial y_j}\Big|_{\gamma_0(t)}\right\| \leqq \left\|\frac{\partial}{\partial y_j}\Big|_x\right\|$$

and

$$\left\| t\frac{\partial}{\partial y_j}\Big|_{\gamma_0(t)} \right\| \leqq \left\| \frac{\partial}{\partial y_j}\Big|_x \right\| \frac{sh\rho t}{t} \leqq \rho \left\| \frac{\partial}{\partial y_j}\Big|_x \right\| e^{\rho t}.$$

We next apply **[5]**, 1.29, to estimate any Jacobi field along γ_0 satisfying $V'(0) = 0$. We take M and M_0 as before. We observe that the manifold N_0 of the hypothesis of their theorem coincides with D_1 since D_1 is totally geodesic. Since N_0 is totally geodesic and D is non-positively curved there can be no focal points of N_0 along γ_0. This can be proved by applying **[20]**, Theorem 3.2 with $(V, S) = (D, D_1)$ and $('V, 'S) = (\mathbf{R}^m, \mathbf{R}^n)$ where $\mathbf{R}^m$ has the flat metric and $\mathbf{R}^n$ is linearly embedded. We obtain the estimate

(**) $\quad \|V(t)\| \leqq \|V(0)\| ch\rho t \leqq \|V(0)\| e^{\rho t}.$

We apply (**) with $V(t) = \partial/\partial x_j|_{\gamma_0(t)}$ and note

$$V(0) = \partial/\partial x_i|_x.$$

Hence, we obtain statement (a) of the lemma.

In our application of Lemma 1.2 in Section 3 we also will assume that $N \cap U$ has compact closure. In that case it is sufficient to let C denote the bigger of the maximum value of the functions $\|\partial/\partial x_j\|$ and $\|\partial/\partial x_j\|$ on $N \cap U$ and the maximal value for $\|\nabla T\partial/\partial x_j\|$ and $\|\nabla T\partial/\partial y_j\|$ on $\nu(N \cap U)$, to obtain the following corollary.

COROLLARY 1. *There exists a constant C so that for any $x \in U$ we have*:

$$\left\| \frac{\partial}{\partial x_j}\Big|_x \right\| \leqq Ce^{\rho r(x)}$$

$$\left\| \frac{\partial}{\partial y_j}\Big|_x \right\| \leqq Ce^{\rho r(x)}.$$

By duality we obtain the following lower bounds on the coordinate differentials.

COROLLARY 2.

(a) $\quad \|dx_i|_x\| \geqq Ce^{-\rho r(x)}$

(b) $\quad \|dy_j|_x\| \geqq Ce^{-\rho r(x)}.$

We now find upper bounds for the lengths of the coordinate differentials, or lower bounds for the lengths of the coordinate vector fields.

LEMMA A.2. *Let γ be a geodesic normal to N emanating from $x \in N$ which is parametrized by arc length. Then along γ we have the estimates*:

(a) $\|dx_i|_{\gamma(t)}\| \leqq \|dx_i|_x\|$ *for* $i = 1, 2, \ldots, n$

(b) $\|dy_j|_{\gamma(t)}\| \leqq \|dy_j|_x\|$ *for* $j = 1, 2, \ldots, k$.

Proof. We again apply [**5**] but this time with $M = D$ and M_0 equal to $\mathbf{R}^m$ with the flat metric. In the case $V(0) = 0$ we obtain:

$$\left\| t \frac{\partial}{\partial y_j}\Big|_{\gamma(t)} \right\| \geqq \left\| \frac{\partial}{\partial y_j}\Big|_x \right\| t.$$

We apply * with

$$V(t) = r \frac{\partial}{\partial y_j}\Big|_{\gamma(t)}$$

and obtain:

$$\left\| t \frac{\partial}{\partial y_j}\Big|_{\gamma(t)} \right\| \geqq \left\| \frac{\partial}{\partial y_j}\Big|_x \right\| t.$$

The statement (b) of Lemma A2 follows:

To prove the statement (a) we note that a Jacobi field $V(t)$ in Euclidean space satisfying $V'(0) = 0$ is a constant field. The value of the constant must be $V(0)$ and the lemma follows.

COROLLARY 1. *In any standard coordinate patch we have*:

(a) $\|dx_i\| \leqq C$ *for* $i = 1, 2, \ldots, n$

(b) $\|dy_j\| \leqq C$ *for* $1, 2, \ldots, k$.

REFERENCES

1. A. Ash, *Non-square-integrable cohomology of arithmetic groups*, Duke Math. J. *47* (1980), 435-449.
2. A. Borel, *Introduction aux groupes arithmetiques* (Hermann, 1969).
3. ——— *Stable real cohomology of arithmetic groups II*, Collected Papers III, Springer, 650-684.
4. R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Math. *82*, Springer.
5. J. Cheeger and D. G. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland Mathematical Library *9* (1975).
6. R. Godement, *Theorie des faisceaux*, Actualités Sci. Indust. 1252 (Hermann, Paris, 1958).
7. A. Gray, *Comparison theorems for the volumes of tubes as generalizations of the Weyl tube formula*, Topology *21* (1982), 201-228.
8. R. Howe, *Automorphic forms of low rank*, preprint.

9. D. Johnson and J. Millson, *Deformation spaces associated to compact hyperbolic manifolds*, to appear in Discrete Groups in Geometry and Analysis, Birkhauser, Progress in Math.

10. S. Kudla and J. Millson, *Harmonic differentials and closed geodesics on a Riemann surface*, Invent. Math. *54* (1979), 193-211.

11. ——— *Geodesic cycles and the Weil representation I; quotients of hyperbolic space and Siegel modular forms*, Comp. Math. *45* (1982), 207-271.

12. ——— *The theta correspondence and harmonic forms I*, Math. Ann. *274* (1986), 353-378.

13. ——— *The theta correspondence and harmonic forms II*, Math. Ann. *277* (1987), 267-314.

14. J. Millson, *Cycles and harmonic forms on locally symmetric spaces*, Can. Math. Bull. *28* (1985), 3-38.

15. J. Millson and M. S. Raghunathan, *Geometric construction of cohomology for arithmetic groups I*, in *Papers dedicated to the memory of V. K. Patodi*, Indian Academy of Sciences, Bangalore 560080.

16. G. de Rham, *Variétés differentiables*, Actualités Sci. Indust. 1222b (Hermann, Paris, 1973).

17. C. L. Siegel, *Uber die analytische theorie der quadratischen formen*, Annals of Math. *36* (1935), 527-606.

18. Y. L. Tong and S. P. Wang, *Period integrals in non compact quotients of $SU(p, 1)$*, Duke Math. J. *52* (1985), 649-688.

19. S. P. Wang, *Correspondence of modular forms to cycles associated to $O(p, q)$*, preprint.

20. F. Warner, *Extension of the Rauch comparison theorems to submanifolds*, Trans. Amer. Math. Soc. *122* (1966), 341-356.

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