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A NEW BITOPOLOGICAL PARACOMPACTNESS

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Abstract

In this paper we define a new generalization of paracompactness for bitopological spaces. (X, τ_1, τ_2) is δ-pairwise paracompact if and only if every τ_i open cover admits a $\tau_1 \vee \tau_2$ open refinement which is $\tau_1 \vee \tau_2$ locally finite. Every quasimetric space (X, τ_p, τ_q) is δ-pairwise paracompact. An analogue of Michael's characterization of regular paracompact spaces is proved for δ-pairwise paracompact spaces.

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1. Introduction

The study of bitopological spaces was initiated by Kelly [5] when he investigated a nonsymmetric generalization of metric spaces, namely, the quasimetric spaces of Wilson [10]. Every quasimetric led to the consideration of a conjugate quasimetric and to the study of the structure of bitopological spaces. Since then, considerable effort has been expended in obtaining appropriate generalizations of standard topological properties in the bitopological category. This process of generalization met with considerable difficulty with the covering properties. The basic problem is the stability of bitopological spaces satisfying some covering axiom in the presence of the pairwise Hausdorff separation property, when the two topologies on the space may collapse and revert to the unitopological setting. With particular reference to bitopological compactness, some of the problems, and various definitions and their scope, have been discussed by Cooke and Reilly [1].

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Paracompactness in the bitopological setting appears to be the most intractable of all. In this paper, we introduce and study a new notion of pairwise paracompactness. Our main result is that our new notion allows for the first time a bitopological version of Michael's classical characterization of regular paracompact spaces. Let us write the abbreviation *b.t.s.* for bitopological space and *nbd* for neighbourhood. If τ_i is a topology for the set X, and if $A \subset X$, we write $\tau_i \mathrm{cl}\, A$ for the closure of A taken in the topology τ_i, $\tau_i\langle x\rangle$ for the set of all τ_i open nbds of x, and $\tau_i|A$ for the relative topology of τ_i on A.

2. Some initial results

Let (X, τ) be a topological space. Let $\mathscr{U}$ be an open cover of X. Any open cover $\mathscr{V}$ (of X) is called a *refinement* of $\mathscr{U}$ if every $V \in \mathscr{V}$ is a subset of some $U \in \mathscr{U}$. Further, $\mathscr{V}$ is called $\mathscr{G}$ *locally finite* if every point $x(\in X)$ belongs to a set $G \in \mathscr{G}$ such that G meets only finitely many members of $\mathscr{V}$. It should be noted that $\mathscr{G}$ is a cover of X. We say X is τ *paracompact with respect to* $\mathscr{G}$ if every τ open cover of X admits a τ open refinement which is $\mathscr{G}$ locally finite. Obviously this leads to our definition [6, Definition 2] of τ_1 *paracompact with respect to* τ_2 for a b.t.s. (X, τ_1, τ_2). Let us call (X, τ_1, τ_2) *RR-pairwise paracompact* if X is τ_i paracompact with respect to τ_j, $(i, j = 1, 2;\ i \neq j)$. Another definition of pairwise paracompactness which appeared early was given by Fletcher, Hoyle III and Patty [3]. A b.t.s. (X, τ_1, τ_2) is *FHP-paracompact* if and only if every τ_i open cover of X has a τ_j open τ_j locally finite refinement $(i, j = 1, 2;\ i \neq j)$.

It is clear that there is no need to restrict the refinement to be a subset of τ. Indeed one can take the members of the refinement from any collection of subsets of X that covers X, or better still from a different topology, a third topology on X.

For a b.t.s. (X, τ_1, τ_2) we denote by L the lattice of topologies on X indicated in Figure 1.

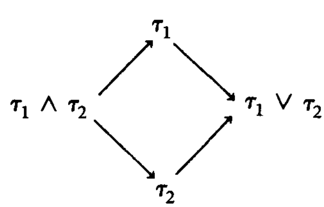

Figure 1

2.1. DEFINITION. Let (X, τ_1, τ_2) be a b.t.s. Then X is called $(\tau - \sigma)$ *paracompact with respect to* ρ if and only if every τ open cover has a σ open refinement which is ρ locally finite, where τ, σ, $\rho \in L$.

In particular, X is "τ_1 paracompact with respect to τ_2", if, in Definition 2.1, the choice is $(\tau, \sigma, \rho) \equiv (\tau_1, \tau_1, \tau_2)$, and X is RR-pairwise paracompact if the choices are $(\tau, \sigma, \rho) \equiv (\tau_1, \tau_1, \tau_2)$ and $(\tau, \sigma, \rho) \equiv (\tau_2, \tau_2, \tau_1)$. The choice in the definition of *FHP*-pairwise paracompactness is $(\tau, \sigma, \rho) \equiv (\tau_i, \tau_j, \tau_j)$, $(i, j = 1, 2;$ $i \neq j)$.

2.2. THEOREM. *Let (X, τ_1, τ_2) be a b.t.s. Let $\tau, \tau', \sigma, \sigma', \rho, \rho' \in L$. Suppose X is $(\tau - \sigma)$ paracompact with respect to ρ. Then X is $(\tau' - \sigma')$ paracompact with respect to ρ' if $\tau' \leqslant \tau$, $\sigma \leqslant \sigma'$ and $\rho \leqslant \rho'$.*

The following result is a consequence of [3, Theorem 9].

2.3. THEOREM. *Let (X, τ_1, τ_2) be pairwise Hausdorff. Suppose X is $(\tau - \sigma)$ paracompact with respect to ρ. Then $\tau_1 \subset \tau_2$ if $\tau_1 \leqslant \tau \leqslant \tau_1 \vee \tau_2$ and $\tau_1 \wedge \tau_2 \leqslant \sigma$, $\rho \leqslant \tau_2$.*

The next result is a consequence of [6, Proposition 4].

2.4. THEOREM. *Let (X, τ_1, τ_2) be pairwise Hausdorff. Suppose X is $(\tau - \sigma)$ paracompact with respect to ρ. Themn $\tau_1 \subset \tau_2$ if $\tau_1 \wedge \tau_2 \leqslant \sigma \leqslant \tau_1 \leqslant \tau \leqslant \tau_1 \vee \tau_2$ and $\tau_1 \wedge \tau_2 \leqslant \rho \leqslant \tau_2$.*

Recall [7, Definition] that (X, τ_1, τ_2) is $\tau_1 R \tau_2$ if for each $x \in X$, $\tau_1 \text{cl}\{x\} = \bigcap\{U | U \in \tau_2\langle x \rangle\} = \bigcap\{\tau_1 \text{cl}\, U | U \in \tau_2\langle x \rangle\}$. Notice in particular that a pairwise Hausdorff space is $\tau_1 R \tau_2$. Recall that a topological space (X, τ) is called R_0 if $x \in U \in \tau$ implies $\text{cl}\{x\} \subset U$.

An immediate consequence of [7, Proposition 2] is the following.

2.5. THEOREM. *If (X, τ_1, τ_2) is such that (X, τ_1) is R_0, and if (X, τ_1, τ_2) is $\tau_1 R \tau_2$ and $(\tau - \sigma)$ paracompact with respect to ρ, then $\tau_1 \subset \tau_2$ if $\tau_1 \wedge \tau_2 \leqslant \sigma \leqslant \tau_1 \leqslant \tau \leqslant \tau_1 \vee \tau_2$ and $\tau_1 \wedge \tau_2 \leqslant \rho \leqslant \tau_2$.*

3. Paracompactness in bitopological spaces

From the remarks in Section 2, we are naturally led to consider the following situations for the property "$(\tau - \sigma)$ paracompact with respect to ρ" in Definition 2.1.

Case (i). $\tau = \tau_i$, $\sigma = \tau_1 \vee \tau_2$, $\tau = \tau_j$ $(i, j = 1, 2;\ i \neq j)$.
Case (ii). $\tau = \tau_i$, $\sigma = \tau_1 \vee \tau_2$, $\rho = \rho_i$ $(i = 1, 2)$.
Case (iii). $\tau = \rho_i$, $\sigma = \tau_i$, $\rho = \tau_1 \vee \tau_2$ $(i = 1, 2)$.
Case (iv). $\tau = \tau_i$, $\sigma = \tau_1 \vee \tau_2$, $\rho = \tau_1 \vee \tau_2$ $(i = 1, 2)$.

3.1. Definition. Let (X, τ_1, τ_2) be a b.t.s. Let X be $(\tau - \sigma)$ paracompact with respect to ρ. Then

in case (i) X is called *α-pairwise paracompact*,
in case (i) X is called *β-pairwise paracompact*,
in case (iii) X is called *γ-pairwise paracompact*, and
in case (iv) X is called *δ-pairwise paracompact*.

Note that either of *RR*-pairwise paracompactness or *FHP*-pairwise paracompactness implies α-pairwise paracompactness. Also each of α, β and γ-pairwise paracompactness implies δ-pairwise paracompactness.

3.2. Theorem. *Let (X, τ_1, τ_2) be a b.t.s. Let (X, τ_1) and (X, τ_2) be a pair of regular spaces. Then X is β-pairwise paracompact if and only if (X, τ_1) and (X, τ_2) are paracompact.*

Proof. This follows from Michael [2, Theorem 2.3(3), page 163].

3.3. Theorem. *Let (X, τ_1, τ_2) be pairwise Hausdorff and α-pairwise paracompact. Then $\tau = \tau_2$.*

Proof. Let U be a τ_1 open set and let $x \in U$. For each $y \in X - U$, there exists a τ_1 open nbd V_y of y such that $x \notin \tau_2 \mathrm{cl}\, V_y$, since the space is pairwise Hausdorff. Then $\mathscr{C} = \{V_y | y \in X - U\} \cup \{U\}$ is a τ_1 open cover of X, so that it has $\tau_1 \vee \tau_2$ open refinement $\mathscr{D} = \{W_y | y \in X - U\} \cup \{G\}$. (Such a 'precise' refinement exists.) Further, $\mathscr{D}$ is τ_2 locally finite. Let $W = \bigcup\{W_y | y \in X - U\}$. Then $X - U \subset \bigcup\{\tau_2 \mathrm{cl}\, W_y | y \in X - U\} = \tau_2 \mathrm{cl}\, W \subset \bigcup\{\tau_2 \mathrm{cl}\, V_y | y \in X - U\}$, so that $x \in X - \tau_2 \mathrm{cl}\, W \subset U$, where $X - \tau_2 \mathrm{cl}\, W$ is τ_2 open. Thus $\tau_1 \subset \tau_2$. Reversing the role of τ_1 and τ_2 in the proof, we get $\tau_1 \subset \tau_2$, so that $\tau_1 = \tau_2$.

3.4. Theorem. *Let (X, τ_1, τ_2) be pairwise Hausdorff. Let X be $(\tau - \sigma)$ paracompact with respect to ρ. Then $\tau_1 \subset \tau_2$ if $\tau_1 \leqslant \tau \leqslant \tau_1 \vee \tau_2$, $\sigma \in L$ and $\tau_1 \wedge \tau_2 \leqslant \rho \leqslant \tau_2$.*

Suppose we define (X, τ_1, τ_2) to be τ_1 *α-paracompact with respect to* τ_2 if every τ_1-open cover admits a $\tau_1 \vee \tau_2$ open refinement which is τ_2 locally finite. Then a modification of the proof of Theorem 3.3 leads to the following result, the proof of which is indicated briefly.

3.5. Theorem. *If (X, τ_1, τ_2) is such that (X, τ_1) is R_0, and if (X, τ_1, τ_2) is $\tau_1 R \tau_2$ and τ_1 α-paracompact with respect to τ_2, then $\tau_1 \subset \tau_2$.*

PROOF. Let V be a τ_1 open set and let $x \in V$. Then $\tau_1 \text{cl}\{x\} \subset V$. Also $\tau_1 \text{cl}\{x\} = \bigcap\{U | U \in \tau_2\langle x\rangle\} = \bigcap\{\tau_1 \text{cl}\, U | U \in \tau_2\langle x\rangle\}$. Then $\mathscr{C} = \{X - \tau_1 \text{cl}\, U | U \in \tau_1\langle x\rangle\} \cup \{V\}$ is a τ_1 open cover of X. Proceeding as in Theorem 3.3, with suitable modifications, we arrive at a τ_2 open nbd Q of x such that $x \in Q \in V$, whence $\tau_1 \subset \tau_2$.

A consequence of this result is the following result.

3.6. THEOREM. *If* (X, τ_1, τ_2) *is such that* (X, τ_1) *is* R_0, *and if* (X, τ_1, τ_2) *is* $\tau_1 R \tau_2$ *and* $(\tau - \sigma)$ *paracompact with respect to* ρ, *then* $\tau_1 \subset \tau_2$ *if* $\tau_1 \leqslant \tau \leqslant \tau_1 \vee \tau_2$, $\sigma \in L$ *and* $\tau_1 \wedge \tau_2 \leqslant \rho \leqslant \tau_2$.

An immediate consequence of Definition 3.1 (iv) and of the fact that every metrizable space is paracompact is the following result.

3.7. THEOREM. *If* (X, τ_1, τ_2) *is such that* τ_1 *is induced by a quasimetric* p *on* X *and* τ_2 *by its conjugate* q *on* X *(i.e.,* $\tau_1 = \tau_p$ *and* $\tau_2 = \tau_q$*), then* (X, τ_1, τ_2) *is* δ*-pairwise paracompact.*

3.8. REMARKS. By Theorem 3.7, it is clear that there exists a pairwise Hausdorff δ-pairwise paracompact b.t.s. (X, τ_1, τ_2) with $\tau_1 \neq \tau_2$. Indeed, such spaces are not α-pairwise paracompact.

If $\mathscr{R}$ is the set of real numbers, and if p is the quasimetric defined by $p(x, y) = \min\{1, |x - y|\}$ if $x \leqslant y$ and $p(x, y) = 1$ if $y < x$, then we get a quasimetric space (R, τ_q, τ_q) such that τ_p and τ_q are regular, and such that (R, τ_p) and (R, τ_q) are paracompact. Thus, by Theorem 3.2, (R, τ_p, τ_q) is β-pairwise paracompact. It should be noted that neither τ_p nor τ_q is metrizable.

It has been proved by Stoltenberg [9, Corollary 4.3] that every sequentially compact Hausdorff quasimetric space is metrizable, and by Sion and Zelmar [8, Theorem 2.5] that every compact regular quasimetric space is metrizable. If we consider Helly's space H [4, Example M, page 164], we see that it is compact, Hausdorff, first countable (and hence sequentially compact) and separable; but then it is not metrizable, so that it is not quasimetrizable. It we take this topology to be τ_1 and the discrete topology on H to be τ_2, then (H, τ_1, τ_2) is a β-pairwise paracompact space, and hence δ-pairwise paracompact. However, (H, τ_1, τ_2) is not quasimetrizable.

There is an example given by Dieudonné of a locally compact Hausdorff space which is not normal. This space is quasimetrizable, as observed by Stoltenberg [9, Example 4.6]. Let us describe this example briefly. Let $X = \{(m/n^2, 1/n) | m$ is an integer; n a positive integer$\} \cup \{(x, 0) | x \in \mathbb{R}\}$. Let us define a topology τ on

X as follows: if $(x, y) \in X$ and $y \neq 0$, define $U_n(x, y) = \{(x, y)\}$ and $U(x, 0) = \{(u, v) | u \leqslant 1/2^n$ and $|x - u| \leqslant v\}$ for each positive integer n. Then $\{U_n(z)\}$ forms a nbd base for each $z \in X$ in the topology τ. Thus we get a locally compact Hausdorff but non-normal topological space (X, τ). Let $U_0(x, y) = X$. Define p: $X \times X \to R$ by $p(z, z') = 1/2^n$ when $z \in U_n(z') - U_{n+1}(z')$, where $z, z' \in X$. Then p is a quasimetric on X such that $\tau = \tau_p$. The topology τ_q induced by the conjugate q is discrete. Thus we see that (X, τ_p, τ_q) is an example of δ-pairwise paracompact space which is not β-pairwise paracompact.

It is also clear that if (X, τ_1) and (X, τ_2) are regular, then (X, τ_1, τ_2) is γ-pairwise paracompact (and hence δ-pairwise paracompact), provided it is β-pairwise paracompact. However, any space X which has a paracompact topology for τ_2 and a non-paracompact one for τ_1, with $\tau_1 \subset \tau_2$, is an example of a δ-pairwise paracompact space which is not β-pairwise paracompact. In particular, if $X = [0, \Omega)$, and if τ_1 is the order topology on X and τ_2 the discrete topology, then (X, τ_1, τ_2) is an example of a pairwise Hausdorff *biregular*, (i.e., τ_1 and τ_2 are regular) δ-pairwise paracompact space which is not β-pairwise paracompact. It is to be noted that (X, τ_1, τ_2) is not quasimetrizable.

We now provide the promised analogue of Michael's theorem for δ-pairwise paracompact spaces which are pairwise regular.

3.9. THEOREM. *Let (X, τ_1, τ_2) be pairwise regular. Then the following are equivalent* $(i = 1, 2)$.

(i) *X is δ-pairwise paracompact.*

(ii) *Every τ_i open cover has a $\tau_1 \vee \tau_2$ open refinement that can be decomposed into an at most countable collection of families of $\tau_1 \vee \tau_2$ open sets which are $\tau_1 \vee \tau_2$ locally finite.*

(iii) *Every τ_i open cover has a $\tau_1 \vee \tau_2$ locally finite refinement whose members need neither be $\tau_1 \vee \tau_2$ open nor $\tau_1 \vee \tau_2$ closed.*

(iv) *Every τ_i open cover has a $\tau_1 \vee \tau_2$ locally finite refinement whose members are $\tau_1 \vee \tau_2$ closed subsets of X.*

PROOF. That (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Let $\mathscr{U} = \{U_\alpha | \alpha \in \Gamma\}$ be any τ_i open cover of X. There exists, by hypothesis in (ii), a $\tau_1 \vee \tau_2$ open refinement $\mathscr{V} = \{\mathscr{V}_n | n \in Z^+\}$, where each $\mathscr{V}_n = \{V_{n\beta} | \beta \in \Delta_n\}$ is a $\tau_1 \vee \tau_2$ locally finite family (but need not be a cover). For fixed n, let us write $W_n = \bigcup\{V_{n\beta} | \beta \in \Delta_n\}$. Then $\{W_n | n \in Z^+\}$ is a $\tau_1 \vee \tau_2$ open cover of X. Let $S_i = W_i - \bigcup_{j=1}^{i-1} W_j$. Then $\{S_n | n \in Z^+\}$ is a cover which is $\tau_1 \vee \tau_2$ locally finite. Thus $\{T_{n\alpha} = S_n \cap V_{n\alpha} | (n, \alpha) \in Z^+ \times \Gamma\}$ is a refinement of $\mathscr{U}$, and it is $\tau_1 \vee \tau_2$ locally finite.

(iii) $\Rightarrow$ (iv). Let $\mathscr{U} = \{U_\alpha | \alpha \in \Gamma\}$ be a τ_i open cover of X. For each x there exists a τ_i open nbd U_x of x with $U_x \in \mathscr{U}$. Since X is pairwise regular, there exists a τ_i open nbd V_x of x such that $x \in V_x \subset \tau_j \operatorname{cl} V_x \subset U_x$. Indeed, $x \in V_x \subset (\tau_1 \vee \tau_2) \operatorname{cl} V_x \subset \tau_j \operatorname{cl} V_x \subset U_x$. Now $\{V_x | x \in X\}$ is a τ_i open cover of X. Hence it has a "precise" refinement $\{A_x | x \in X\}$ which is $\tau_1 \vee \tau_2$ locally finite, by the hypothesis in (iii). Hence $\{(\tau_1 \vee \tau_2) \operatorname{cl} A_x | x \in X\}$ is a refinement of $\mathscr{U}$ (since $(\tau_1 \vee \tau_2) \operatorname{cl} A_x \subset (\tau_1 \vee \tau_2) \operatorname{cl} V_x \subset U_x$) which is also $\tau_1 \vee \tau_2$ locally finite.

(iv) $\Rightarrow$ (i). Let $\mathscr{U} = \{U_\alpha | \alpha \in \Gamma\}$ be a τ_i open cover of X. Let $\mathscr{V}$ be a $\tau_1 \vee \tau_2$ closed $\tau_1 \vee \tau_2$ locally finite refinement of $\mathscr{U}$. Then each point $x \in X$ has a $\tau_1 \vee \tau_2$ open nbd W_x which meets only finitely many members of $\mathscr{V}$. Also, since the space is pairwise regular, there exist τ_i open sets A_i and B_i $(i = 1, 2)$ such that $x \in B_1 \cap B_2 \subset \tau_2 \operatorname{cl} B_1 \cap \tau_1 \operatorname{cl} B_2 \subset A_1 \cap A_2 = W_x$. Since $(\tau_1 \vee \tau_2) \operatorname{cl}(B_1 \cap B_2) \subset W_x$, if we set $(\tau_1 \vee \tau_2) \operatorname{cl}(B_1 \cap B_2) = P_x$, then $\{P_x | x \in X\}$ is a $\tau_1 \vee \tau_2$ closed covering of X such that each P_x meets only finitely many members of $\mathscr{V}$. Then, by [2, Theorem 1.5, page 162], there exists, for each $V \in \mathscr{V}$, an enlargement $E(V)$ in $\tau_1 \vee \tau_2$ and, since $\mathscr{V}$ is a refinement of $\mathscr{U}$, a $U(V)$ in $\mathscr{U}$ such that $V \subset E(V) \cap U(V)$, such that $E(V) \cap U(V)$ is $\tau_1 \vee \tau_2$ open, and such that $\{E(V) | V \in \mathscr{V}\}$ is $\tau_1 \vee \tau_2$ locally finite. Therefore $\mathscr{G} = \{E(V) \cap U(V) | V \in \mathscr{V}\}$ is a $\tau_1 \vee \tau_2$ open $\tau_1 \vee \tau_2$ locally finite refinement of $\mathscr{U}$.

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