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TWO-PARAMETER NONLINEAR STURM-LIOUVILLE PROBLEMS

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We study two-parameter nonlinear Sturm-Liouville problems. We shall establish the continuity of the variational eigencurve $\lambda(\mu)$ and asymptotic formulas of $\lambda(\mu)$ as $\mu \to \infty, \mu \to \pi^2$.

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1. Introduction

We consider the following two-parameter nonlinear Sturm-Liouville problem:

$$\begin{cases} -u(x)'' = \mu u(x) - \lambda\{u(x)^p + f(u(x))\}, & x \in I = (0, 1), \\ u(x) > 0, \quad x \in I, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $p > 1, \mu > \pi^2, \lambda > 0$ and f is a real-valued, increasing, odd and locally Lipschitz continuous function on R.

The purpose of this paper is to investigate the behaviour of a variational eigencurve $\lambda = \lambda(\mu)$ obtained by variational theory on a general level set

$$N_\mu := \left\{u \in W_0^{1,2}(I) : \int_0^1 (u'(x)^2 - \mu u(x)^2)dx = -2\alpha\right\}, \tag{1.2}$$

where $\alpha > 0$ is a fixed number.

In order to motivate the results of this paper, let us briefly recall the known results concerning linear and nonlinear two-parameter eigenvalue problems. Linear two-parameter eigenvalue problems in ordinary differential equations began with the analysis of Lamé's equation and there are many works. We refer to Binding and Volkmer **[5]**, Faierman **[11, 12, 13]**, Langer **[16]**, Rynne **[18]**, Volkmer **[22]**, and the references cited therein. Especially, the study of the asymptotic behaviour of eigenvalues has found considerable interest, and one of the main object is to find the asymptotic directions of the spectrum (the limit of the ratio of two eigenvalue parameters). In particular, Binding and Browne **[3, 4]** studied the linear two-parameter Sturm-Liouville equation

$$u''(x)+\mu a(x)u(x)=\lambda b(x)u(x), \quad x\in I,$$

where $\mu, \lambda \in R$ are parameters. In [3, 4], under suitable boundary conditions, the asymptotic formulas of $\lambda_n(\mu)/\mu$ as $\mu\to\infty$ have been given, where $\lambda_n(\mu)$ is the n-th eigenvalue when $\mu>0$ is given. Concerning nonlinear two-parameter problems, interest has been directed mainly at bifurcation problems. We refer to Browne and Sleeman [7, **8**, **9**], Gómez [**14**], Rynne [**17**], Chow and Hale [**10**], and the references cited therein. In this paper, motivated by the work [3, 4], we focus our attention on the asymptotic behaviour of the variational eigencurve $\lambda(\mu)$ as $\mu\to\infty, \pi^2$, which is regarded as a nonlinear version of the study of the asymptotic directions. We note here that since (1.1) is nonlinear, λ is parameterized by μ and an additional parameter α. More precisely, $\lambda=\lambda(\mu,\alpha)$ and α is a parameter of general level sets defined in (1.2), which is developed by Zeidler [**23**], and it seems effective for us to consider the equation (1.1) under the variational framework of general level sets.

Recently, the following nonlinear two-parameter problems were considered in Shibata [**20**, **21**]:

$$\begin{cases} -u''(x)=\mu u(x)-\lambda(1+|u(x)|^{p-1})u(x), & x\in I,\\ u(0)=u(1)=0, \end{cases} \tag{1.3}$$

$$\begin{cases} -u''(x)=\mu u(x)-\lambda|u(x)|^{p-1}u(x), & x\in I,\\ u(0)=u(1)=0, \end{cases} \tag{1.4}$$

where $p>1$. It was shown in [**20**] that as $\mu\to\infty$

$$\frac{\lambda(\mu,\alpha)}{\mu}\to 1. \tag{1.5}$$

In [**21**], the asymptotic behaviour of $\lambda(\mu)$ as $\mu\to\infty$ was obtained by using a simple scaling technique.

Motivated by these facts, we shall establish asymptotic formulas for our non-linear problem (1.1) by using variational theory on general level sets.

2. Main results

We use the following notation. Let

$$\|u\|_q := \left(\int_0^1 |u(x)|^q dx\right)^{\frac{1}{q}} \quad (q\ge 1), \tag{2.1}$$

$$A(u,\mu) := \|u'\|_2^2 - \mu\|u\|_2^2, \tag{2.2}$$

$$E(u, v) := \int_0^1 f(u) v dx, \quad E(u) := E(u, u), \quad F(u) := \int_0^1 \int_0^{u(x)} f(s) ds dx, \tag{2.3}$$

$$G(u) := \frac{1}{p+1} \|u\|_{p+1}^{p+1} + F(u), \tag{2.4}$$

$$H(u) := \|u\|_{p+1}^{p+1} + E(u). \tag{2.5}$$

Now we define the variational eigenvalues $\lambda(\mu)$ of (1.1): $\lambda(\mu)$ are called the variational eigenvalues of (1.1) if there exists $u_\mu(x) \in N_\mu$ satisfying the following conditions (2.6)–(2.8)

$$(u_\mu(x), \lambda(\mu)) \in N_\mu \times R \qquad \text{satisfies (1.1).} \tag{2.6}$$

$$u_\mu(x) > 0, \quad x \in I. \tag{2.7}$$

$$G(u_\mu(x)) = \beta(\mu) := \inf_{u \in N_\mu} G(u). \tag{2.8}$$

By Sobolev's embedding theorem, $u_\mu \in W_0^{1,2}(I) \subset C(\bar{I})$. Then, by (1.1), $u_\mu'' \in C(\bar{I})$, and consequently, $u_\mu \in C^2(\bar{I})$. (cf. Brezis [6, p. 136].) Now we state our main results.

Theorem 1. *Assume that f satisfies the following conditions:*

$$s \mapsto \frac{f(s)}{s} \quad \text{is strictly increasing on} \quad R_+ := (0, \infty), \tag{2.9}$$

$$\lim_{s \to \infty} \frac{f(s)}{s} = +\infty, \tag{2.10}$$

$$\lim_{s \to 0} \frac{f(s)}{s} = 0. \tag{2.11}$$

There exists a constant $1 < m < \infty$, $C_1, C_2 > 0$ such that

$$|f(s)| \le C_1 + C_2 |s|^m. \tag{2.12}$$

Then there exists $\lambda(\mu)$ for $\mu > \pi^2$. Furthermore, $\mu \mapsto \lambda(\mu)$ is continuous.

Theorem 2. *Assume* (2.9)–(2.11) *and* (2.12) *for $m < p$. Then the following asymptotic formula holds as $\mu \to \pi^2$:*

$$\sqrt{\mu - \pi^2}\, u_\mu(x) \to 2\sqrt{\alpha} \sin \pi x \quad \text{in} \quad W_0^{1,2}(I), \tag{2.13}$$

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$$\frac{\lambda(\mu)}{(\mu-\pi^2)^{\frac{p+1}{2}}} \to \sqrt{\pi}2^{-p}\alpha^{\frac{1-p}{2}}\frac{\Gamma\left(\frac{p+3}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)}. \tag{2.14}$$

Theorem 3. *Assume* (2.9), (2.10) *and* (2.11). *Furthermore, assume that there exists a constant* $C_3, \delta > 0$ *such that for* $0 \le s \le \delta$ *and* $q > p$

$$f(s) \le C_3 s^q. \tag{2.15}$$

Then there exists a constant $C_4 > 0$ *such that*

(1) *If* $q > p+1$, *then the following asymptotic formula holds as* $\mu \to \infty$:

$$C_4^{-1}\mu^{\frac{p}{2}} \le \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}} - \lambda(\mu) \le C_4\mu^{\frac{p}{2}}. \tag{2.16}$$

(2) *If* $p < q \le p+1$, *then the following asymptotic formula holds as* $\mu \to \infty$:

$$\left|\lambda(\mu) - \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}}\right| \le C_4\mu^{\frac{2p+1-q}{2}}. \tag{2.17}$$

The remainder of this paper is organized as follows. In Section 3, we shall prove Theorem 1. Section 4 is devoted to the proof of Theorem 2. Finally, we shall prove Theorem 3 in Section 5.

3. Proof of Theorem 1

The existence of $(u_\mu(x), \lambda(\mu))$ which satisfies (2.6) and (2.8) is due to Zeidler [**23**, Proposition 6a]. Since $G(u_\mu) = G(|u_\mu|)$, we assume that $u_\mu \ge 0$ in I. If there exists $x_0 \in I$ such that $u_\mu(x_0) = 0$, then clearly $u'_\mu(x_0) = 0$, since $u_\mu(x) \ge 0$ in I. Therefore, by the uniqueness theorem of ODE we obtain that $u_\mu \equiv 0$ in I. However, this is impossible, since $0 \notin N_\mu$. Thus $u_\mu > 0$ in I, and the existence of a variational eigenvalue is completed.

Now we shall show the uniqueness of $\lambda(\mu)$ for $\mu > \pi^2$. We begin with some fundamental lemmas.

Lemma 3.1. *Let* (λ_0, u_0) *and* (λ_1, u_1) *satisfy* (2.6)–(2.8). *Then* $\lambda_0 = \lambda_1$.

Proof. Multiply (1.1) by $u_j (j = 0, 1)$. Then it follows from (2.6) and integration by parts that for $j = 0, 1$

$$-2\alpha = A(u_j, \mu) = -\lambda_j H(u_j). \tag{3.1}$$

Since $H(u_j) > 0$ by (2.5), we obtain by (3.1) that $\lambda_j > 0$.

Assume that $\lambda_0 < \lambda_1$. Then it follows from (1.1) that

$$\begin{aligned}-u_0'' + \lambda_1(u_0^p + f(u_0)) &= -u_0'' + \lambda_0(u_0^p + f(u_0)) + (\lambda_1 - \lambda_0)(u_0^p + f(u_0)) \\ &\geq \mu u_0.\end{aligned}$$

Therefore, u_0 is a supersolution of the equation

$$\begin{cases} -u'' + \lambda_1(u^p + f(u)) = \mu u & \text{in} \quad I, \\ u(0) = u(1) = 0, \end{cases}$$

that is, u_0 satisfies

$$\begin{cases} -u'' + \lambda_1(u^p + f(u)) \geq \mu u & \text{in} \quad I, \\ u(0), u(1) \geq 0. \end{cases} \tag{3.2}$$

Hence, by [**2**, Théorème 4] we obtain that $u_1 \leq u_0$ in I. If $u_0 \equiv u_1$, then it follows from (3.1) that $\lambda_0 = \lambda_1$, which is a contradiction. Hence there exists a compact non-empty interval $I_1 \subset I$ such that $u_1 < u_0$ in I_1. Since f is increasing, it is clear from (2.4) that $G(u_1) < G(u_0)$, which is a contradiction, since we have $G(u_0) = G(u_1) = \beta(\mu)$ by (2.8). Thus the proof is complete. □

Lemma 3.2. *Let $\{\mu_k\}_{k=1}^{\infty}$ be a sequence satisfying $\mu_k > \pi^2$ and $\mu_k \to \mu_0 > \pi^2$ as $k \to \infty$. Then there exists a constant $C_7 > 0$ such that for any $k \in N$*

$$C_7^{-1} \leq \beta(\mu_k) \leq C_7. \tag{3.3}$$

Proof. We assume that

$$\beta(\mu_k) \to 0 \quad \text{as} \quad k \to \infty \tag{3.4}$$

and drive a contradiction. Let $u_k = u_{\mu_k} \in N_{\mu_k}$. Then it follows from (2.4), (2.8), (3.4) and Hölder's inequality that as $k \to \infty$

$$\|u_k\|_{p+1} \to 0, \quad \|u_k\|_2 \leq \|u_k\|_{p+1} \to 0; \tag{3.5}$$

this along with (2.6) implies that for $k \gg 1$

$$\|u_k'\|_2^2 = \mu_k \|u_k\|_2^2 - 2\alpha < 0, \tag{3.6}$$

which is a contradiction. Thus we obtain the estimate from below.

Next, we shall show the estimate from above. Since $2\sqrt{\frac{\alpha}{\mu_k - \pi^2}} \sin \pi x \in N_{\mu_k}$ and $f(s)$ is increasing, we obtain by (2.8) that

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$$\beta(\mu_k) \le \frac{1}{p+1}\left(2\sqrt{\frac{\alpha}{\mu_k-\pi^2}}\right)^{\frac{p+1}{2}} \int_0^1 \sin^{p+1}\pi x dx + F\left(2\sqrt{\frac{\alpha}{\mu_k-\pi^2}}\sin \pi x\right)$$
$$\le C\left(\xi^{-\frac{p+1}{2}} + f\left(2\sqrt{\frac{\alpha}{\xi}}\right)\right),$$

where $\xi = \max_{k\in N}|\mu_k - \pi^2| > 0$. This is the desired estimate. Thus the proof is complete. □

Lemma 3.3. *$\beta(\mu)$ is continuous in μ for $\mu > \pi^2$.*

Proof. Let $\{\mu_k\}_{k=1}^{\infty}$ be a sequence satisfying $\mu_k > \pi^2$ and $\mu_k \to \mu_0 > \pi^2$ as $k \to \infty$. Furthermore, let $(u_k, \lambda_k) = (u_{\mu_k}(x), \lambda(\mu_k))$. Put

$$-2\alpha_k = A(u_k, \mu_0) = A(u_k, \mu_k) + (\mu_k - \mu_0)\|u_k\|_2^2 = -2\alpha + (\mu_k - \mu_0)\|u_k\|_2^2. \quad (3.7)$$

It follows from Lemma 3.2 that

$$\|u_k\|_1^2 \le \|u_k\|_2^2 \le \|u_k\|_{p+1}^2 \le \{(p+1)\beta(\mu_k)\}^{\frac{2}{p+1}} \le C_8^2, \quad (3.8)$$

where $C_8 = \{(p+1)C_7\}^{\frac{1}{p+1}}$. Then (3.7) and (3.8) imply that $\alpha_k \to \alpha$ as $k \to \infty$. Hence we see that $\alpha_k > 0$ for $k \gg 1$. Now we put $v_k = \sqrt{\frac{\alpha}{\alpha_k}}u_k \in N_{\mu_0}$. We obtain by (2.8) that

$$\begin{aligned}\beta(\mu_0) &\le \left(\frac{\alpha}{\alpha_k}\right)^{\frac{p+1}{2}} \frac{1}{p+1}\|u_k\|_{p+1}^{p+1} + F(v_k)\\ &\le G(u_k) + \frac{1}{p+1}\left\{\left(\frac{\alpha}{\alpha_k}\right)^{\frac{p+1}{2}} - 1\right\}\|u_k\|_{p+1}^{p+1} + |F(v_k) - F(u_k)|.\end{aligned} \quad (3.9)$$

By (3.6) and (3.8) we obtain

$$\|u_k\|_\infty^2 \le \|u_k'\|_2^2 \le \mu_k\|u_k\|_{p+1}^2 \le C_9^2, \quad (3.10)$$

where $C_9^2 = \max_k \mu_k C_8^2$. Since $\alpha_k \to \alpha$ as $k \to \infty$, we obtain by (3.8) and (3.10) that by choosing another constant $C_9 > 0$ if necessary, $\|v_k\|_\infty^2 \le C_9^2$ for $k \gg 1$. Therefore, it follows from (3.8) and (3.10) that

$$\begin{aligned}|F(v_k) - F(u_k)| &\le f(C_9)\|u_k - v_k\|_1 \le f(C_9)\left|\sqrt{\frac{\alpha}{\alpha_k}} - 1\right|\|u_k\|_1\\ &\le C_8 f(C_9)\left|\sqrt{\frac{\alpha}{\alpha_k}} - 1\right|.\end{aligned} \quad (3.11)$$

We obtain by (3.9), (3.10) and (3.11) that

$$\beta(\mu_0) \le \liminf_{k\to\infty} \beta(\mu_k). \tag{3.12}$$

Next, we put

$$-2\gamma_k = A(u_0, \mu_k) = -2\alpha + (\mu_0 - \mu_k)\|u_0\|_2^2. \tag{3.13}$$

Then for $k \gg 1$ we have $\gamma_k > 0$ and $\gamma_k \to \alpha$ as $k \to \infty$. Let

$$w_k = \sqrt{\frac{\alpha}{\gamma_k}}u_0 \in N_{\mu_k}. \tag{3.14}$$

It follows from (2.8) and (3.14) that

$$\beta(\mu_k) \le G(w_k) \le G(u_0) + \left|\left(\frac{\alpha}{\gamma_k}\right)^{\frac{p+1}{2}} - 1\right| \|u_0\|_{p+1}^{p+1} + |F(w_k) - F(u_0)|. \tag{3.15}$$

Then we obtain by the same calculation as (3.11) that $|F(w_k) - F(u_0)| \to 0$ as $k \to \infty$. Therefore, it follows from (3.15) that

$$\limsup_{k\to\infty} \beta(\mu_k) \le \beta(\mu_0). \tag{3.16}$$

Thus, our assertion follows from (3.12) and (3.16). □

Proof of Theorem 1: Continuity.

Assume that there exists $\mu_0 > \pi^2$ such that $\lambda(\mu)$ is not continuous at $\mu = \mu_0$. Then there exists a sequence $\{\mu_k\}_{k\in N}^{\infty}$ and $\delta > 0$ such that $\mu_k \to \mu_0$ as $k \to \infty$ and

$$|\lambda(\mu_k) - \lambda(\mu_0)| \ge \delta. \tag{3.17}$$

We fix such μ_0 and a sequence $\{\mu_k\}_{k\in N}$, and derive a contradiction. Let u_k be the eigenfunction corresponding to $\lambda(\mu_k)$. We have by (2.3) and the mean value theorem that for $s > 0$

$$\int_0^s f(t)dt = f(s_1)s \le f(s)s, \tag{3.18}$$

where $0 < s_1 < s$. Then we obtain by (2.3)–(2.5) and (3.18) that $F(u) \le E(u)$, so consequently

$$G(u) \le H(u). \tag{3.19}$$

It follows from (3.1), Lemma 3.2 and (3.19) that

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$$0 \le \lambda(\mu_k) = \frac{2\alpha}{H(u_k)} \le \frac{2\alpha}{G(u_k)} = \frac{2\alpha}{\beta(\mu_k)} \le 2\alpha C_7.$$

Hence, by choosing a subsequence of $\{\lambda(\mu_k)\}$ if necessary, we may assume without loss of generality that there exists a constant λ_0 such that $\lambda(\mu_k) \to \lambda_0$ as $k \to \infty$. Furthermore, since $\mu_k \to \mu_0$ as $k \to \infty$, we may assume without loss of generality that $\mu_k \le 2\mu_0$ and $\beta(\mu_k) \le 2\beta(\mu_0)$ for $k \in N$. Then by (2.4) and (2.8), we see that

$$\begin{aligned} \|u_k\|_\infty^2 &\le \|u_k'\|_2^2 \le \mu_k \|u_k\|_2^2 \le 2\mu_0 \|u_k\|_{p+1}^2 \le 2\mu_0((p+1)G(u_k))^{\frac{2}{p+1}} \\ &= 2\mu_0((p+1)\beta(\mu_k))^{\frac{2}{p+1}} \le C_{10}^2 := 2\mu_0(2(p+1)\beta(\mu_0))^{\frac{2}{p+1}}; \end{aligned} \tag{3.20}$$

this implies that we can choose a weakly convergent subsequence of $\{u_k\}_{k=1}^\infty$ in $W_0^{1,2}(I)$, which we write $\{u_k\}_{k=1}^\infty$ again. Let $u_0 = w - \lim_{k\to\infty} u_k$ in $W_0^{1,2}(I)$. Then by Sobolev's embedding theorem we obtain that $u_0 = \lim_{k\to\infty} u_k$ in $C(I)$ and hence $\|u_0\|_\infty \le C_{10}$. Since u_k satisfies (1.1), we see that for $\phi \in W_0^{1,2}(I)$

$$\int_0^1 u_k'\phi' dx - \mu_k \int_0^1 u_k \phi dx = -\lambda(\mu_k)\left\{\int_0^1 (u_k^p \phi + f(u_k)\phi) dx\right\}. \tag{3.21}$$

Since f is locally Lipschitz continuous, there exists a constant $C_{11} > 0$ such that for $s_1, s_2 \in [0, C_{10}]$

$$|f(s_1) - f(s_2)| \le C_{11}|s_1 - s_2|. \tag{3.22}$$

Since $\|u_0\|_\infty, \|u_k\|_\infty \le C_{10}$, we obtain by (3.22), Hölder's inequality and Sobolev's embedding theorem that as $k \to \infty$

$$\begin{aligned} \left|\int_0^1 (f(u_k) - f(u_0))\phi dx\right| &\le C_{11} \int_0^1 |u_k - u_0|\phi dx \\ &\le C_{11}\|u_k - u_0\|_2 \|\phi\|_2 \to 0. \end{aligned} \tag{3.23}$$

Since $\|u_0\|_\infty, \|u_k\|_\infty \le C_{10}$, we obtain by the mean value theorem that there exists $\theta(x) \in [0, 1]$ for $x \in I$ such that

$$\begin{aligned} |u_k(x)^p - u_0(x)^p| &= p|\theta(x)u_k(x)^{p-1} + (1-\theta(x))u_0(x)^{p-1}||u_k(x) - u_0(x)| \\ &\le pC_{10}^{p-1}|u_k(x) - u_0(x)|; \end{aligned}$$

this along with Hölder's inequality implies that as $k \to \infty$

$$\left|\int_0^1 (u_k^p - u_0^p)\phi dx\right| \le pC_{10}^{p-1} \int_0^1 |(u_k - u_0)\phi| dx \le pC_{10}^{p-1}\|u_k - u_0\|_2 \|\phi\|_2 \to 0. \tag{3.24}$$

Let $k \to \infty$ in (3.21). Then we obtain by using (3.23), (3.24) and Sobolev's embedding theorem that

$$\int_0^1 u_0'(x)\phi'(x)dx - \mu_0 \int_0^1 u_0(x)\phi(x)dx = -\lambda_0 \left\{ \int_0^1 (u_0(x)^p \phi(x) + f(u_0(x))\phi(x))dx \right\}. \quad (3.25)$$

Now we shall show that $u_0 = \lim_{k\to\infty} u_k$ in $W_0^{1,2}(I)$. By (3.20) and (3.23) we obtain that as $k \to \infty$

$$\begin{aligned} |E(u_k) - E(u_0)| &\le \left| \int_0^1 f(u_k)(u_k - u_0)dx \right| + \left| \int_0^1 (f(u_k) - f(u_0))u_0 dx \right| \\ &\le f(C_{10})\|u_k - u_0\|_1 + C_{11}\|u_k - u_0\|_2 \|u_0\|_2 \to 0. \end{aligned} \quad (3.26)$$

Now, we put $\phi = u_0$ in (3.25). Then we obtain by (3.6), (3.25) and (3.26) that

$$\|u_0'\|_2^2 = \mu_0 \|u_0\|_2^2 - \lambda_0 H(u_0) = \lim_{k\to\infty} \{\mu_k \|u_k\|_2^2 - \lambda(\mu_k)H(u_k)\} = \lim_{k\to\infty} \|u_k'\|_2^2. \quad (3.27)$$

Hence we obtain that $u_0 = \lim_{k\to\infty} u_k$ in $W_0^{1,2}(I)$. Consequently, it follows from Sobolev's embedding theorem that $u_0 \in N_{\mu_0}$. Furthermore, since $\|u_0\|_\infty, \|u_k\|_\infty \le C_{10}$, we obtain that as $k \to \infty$

$$\begin{aligned} |F(u_k) - F(u_0)| &\le \left| \int_0^1 dx \int_{u_0}^{u_k} f(s)ds \right| \le \max_{0\le s\le C_{10}} f(s) \int_0^1 |u_k(x) - u_0(x)|dx \\ &= f(C_{10})\|u_k - u_0\|_1 \to 0; \end{aligned}$$

this along with Lemma 3.3 implies that

$$G(u_0) = \lim_{k\to\infty} G(u_k) = \lim_{k\to\infty} \beta(\mu_k) = \beta(\mu_0).$$

Therefore, (u_0, λ_0) satisfies (2.6)–(2.8) so that $\lambda_0 = \lambda(\mu_0)$. This contradicts (3.17) and Lemma 3.1. Thus the proof is complete. □

4. Proof of Theorem 2

We begin with preparing some lemmas.

Lemma 4.1. *There exists a constant* $C_{12} > 0$ *such that for* $0 < \mu - \pi^2 \ll 1$

$$\lambda(\mu) \le C_{12}(\mu - \pi^2)^{\frac{p+1}{2}}. \quad (4.1)$$

Proof. Since $u_\mu \in N_\mu$, we obtain by Poincaré's inequality that

$$2\alpha = \mu\|u_\mu\|_2^2 - \|u'_\mu\|_2^2 \le (\mu - \pi^2)\|u_\mu\|_2^2,$$

which together with Hölder's inequality implies that

$$\frac{2\alpha}{\mu - \pi^2} \le \|u_\mu\|_2^2 \le \|u_\mu\|_{p+1}^2. \tag{4.2}$$

We obtain by (2.6), (3.1) and (4.2) that

$$\lambda(\mu) = \frac{2\alpha}{H(u_\mu)} \le \frac{2\alpha}{\|u_\mu\|_{p+1}^{p+1}} \le (2\alpha)^{\frac{1-p}{2}}(\mu - \pi^2)^{\frac{p+1}{2}}.$$

Thus the proof is complete.□

Let

$$s_\mu := \frac{2\sqrt{\alpha}}{(\mu - \pi^2)^{\frac{1}{2}}} \sin \pi x \in N_\mu. \tag{4.3}$$

Lemma 4.2. *There exists a constant $L > 0$ such that as $\mu \to \pi^2$*

$$\sqrt{\mu - \pi^2}\, u_\mu(x) \to L \sin \pi x \quad \text{in} \quad W_0^{1,2}(I).$$

Proof. We put $v_\mu = (\mu - \pi^2)^{\frac{1}{2}} u_\mu(x)$. It follows from (3.21) that for $\phi \in C_0^\infty(I)$

$$\frac{1}{(\mu - \pi^2)^{\frac{1}{2}}} \int_0^1 v'_\mu \phi' dx - \frac{\mu}{(\mu - \pi^2)^{\frac{1}{2}}} \int_0^1 v_\mu \phi dx \tag{4.4}$$
$$= -\lambda(\mu)\left\{ \frac{1}{(\mu - \pi^2)^{\frac{p}{2}}} \int_0^1 v_\mu^p \phi dx + \int_0^1 f\left(\frac{v_\mu}{(\mu - \pi^2)^{\frac{1}{2}}}\right)\phi dx \right\}.$$

We shall show that there exists a constant $C_{13} > 0$ such that

$$\|v_\mu\|_\infty \le C_{13}. \tag{4.5}$$

Since (2.12) holds for $1 < m < p$, we obtain that there exists a constant $C_{14} > 0$ such that

$$F(s_\mu) \le \int_0^1 \left\{ C_1 s_\mu + \frac{C_2}{m+1} s_\mu^{m+1} \right\} dx \le C_{14}\{(\mu - \pi^2)^{-\frac{1}{2}} + (\mu - \pi^2)^{-\frac{m+1}{2}}\}. \tag{4.6}$$

We find by (2.8), (4.3) and (4.6) that there exists a constant $C_{15} > 0$ such that

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$$\begin{aligned}\frac{1}{p+1}\|u_\mu\|_{p+1}^{p+1} \le G(u_\mu) \le G(s_\mu) = \frac{1}{p+1}\|s_\mu\|_{p+1}^{p+1} + F(s_\mu) \\ \le C_{15}\left\{(\mu-\pi^2)^{-\frac{p+1}{2}} + (\mu-\pi^2)^{-\frac{1}{2}} + (\mu-\pi^2)^{-\frac{m+1}{2}}\right\}.\end{aligned} \tag{4.7}$$

Now, (4.7) implies that there exists a constant $C_{16} > 0$ such that as $\mu \to \pi^2$

$$\|v_\mu\|_{p+1}^{p+1} = \|(\mu-\pi^2)^{\frac{1}{2}} u_\mu\|_{p+1}^{p+1} \le C_{16}^{p+1},$$

from which, (3.6) and Hölder's inequality it follows that

$$\|v_\mu\|_\infty^2 \le \|v'_\mu\|_2^2 \le \mu\|v_\mu\|_2^2 \le \mu\|v_\mu\|_{p+1}^2 \le \mu C_{16}^2. \tag{4.8}$$

Since $\mu \to \pi^2$, we obtain (4.5) for $C_{13} = \max \sqrt{\mu} C_{16}$. Now (2.12), Lemma 4.1 and (4.5) imply that as $\mu \to \pi^2$

$$\lambda(\mu)\int_0^1 f\left(\frac{v_\mu}{(\mu-\pi^2)^{\frac{1}{2}}}\right)\phi dx \le C_{12}(\mu-\pi^2)^{\frac{p+1}{2}}\left\{C_1 + C_2\frac{\|v_\mu\|_\infty^m}{(\mu-\pi^2)^{\frac{m}{2}}}\right\}\|\phi\|_\infty \to 0. \tag{4.10}$$

By (4.8) we can choose a weakly convergent subsequence of $\{v_\mu\}$ in $W_0^{1,2}(I)$, which we write as $\{v_\mu\}$ again. Let $v_0 = \mathrm{w}-\lim_{\mu\to\pi^2} v_\mu$. Then by Lemma 4.1, (4.4) and (4.10) we obtain that for $\phi \in C_0^\infty(I)$

$$\int_0^1 v'_0\phi' dx = \pi^2\int_0^1 v_0\phi dx. \tag{4.11}$$

Hence, $v_0 = L\sin \pi x$ for some $L > 0$.

Finally, we shall show that $v_0 = \lim_{\mu\to\pi^2} v_\mu$ is in $W_0^{1,2}(I)$. Put $\phi = v_0$ in (4.11). Then

$$\|v'_0\|_2^2 = \pi^2\|v_0\|_2^2 = \lim_{\mu\to\pi^2}\mu\|v_\mu\|_2^2 = \lim_{\mu\to\pi^2}\{\|v'_\mu\|_2^2 + 2(\mu-\pi^2)\alpha\} = \lim_{\mu\to\pi^2}\|v'_\mu\|_2^2.$$

Since $v_0 = w - \lim_{\mu\to\pi^2} v_\mu$, this implies that our assertion is true. □

Proof of Theorem 2. First, we shall prove (2.13). To this end, we show that $L = 2\sqrt{\alpha}$ in Lemma 4.2. Let $\mu \to \pi^2$ in (4.2). Then we obtain

$$2\alpha \le \|v_\mu\|_2^2 \to \|L\sin\pi x\|_2^2 = \frac{L^2}{2}. \tag{4.12}$$

Thus we obtain $2\sqrt{\alpha} \le L$.

Now, we shall show $L \le 2\sqrt{\alpha}$. It follows from (4.7) that

$$\frac{1}{p+1}\|v_\mu\|_{p+1}^{p+1} \le \frac{1}{p+1}\|2\sqrt{\alpha}\sin \pi x\|_{p+1}^{p+1} + (\mu-\pi^2)^{\frac{p+1}{2}}F(s_\mu). \tag{4.13}$$

Let $\mu \to \pi^2$ in (4.13). Then we obtain by (4.6) that

$$\frac{1}{p+1}L^{p+1} \le \frac{1}{p+1}(2\sqrt{\alpha})^{p+1}.$$

Thus we get (2.13) in Theorem 2.

Finally, we shall show (2.14). It follows from (3.1) that

$$\frac{\lambda(\mu)}{(\mu-\pi^2)^{\frac{p+1}{2}}} = \frac{2\alpha}{\|v_\mu\|_{p+1}^{p+1} + (\mu-\pi^2)^{\frac{p}{2}}E\left(\dfrac{v_\mu}{(\mu-\pi^2)^{\frac{1}{2}}}, v_\mu\right)}. \tag{4.14}$$

We obtain by (2.12) and (4.5) that as $\mu \to \pi^2$

$$(\mu-\pi^2)^{\frac{p}{2}}E\left(\frac{v_\mu}{(\mu-\pi^2)^{\frac{1}{2}}}, v_\mu\right) \le (\mu-\pi^2)^{\frac{p}{2}}\left\{C_1 + C_2\frac{\|v_\mu\|_\infty^m}{(\mu-\pi^2)^{\frac{m}{2}}}\right\}\|v_\mu\|_\infty \to 0. \tag{4.15}$$

Furthermore, we obtain by (2.13) and (4.15) that

$$\|v_\mu\|_{p+1}^{p+1} \to (2\sqrt{\alpha})^{p+1}\int_0^1 \sin^{p+1}\pi x dx = (2\sqrt{\alpha})^{p+1}\pi^{-\frac{1}{2}}\Gamma\left(\frac{p}{2}+1\right)/\Gamma\left(\frac{p+3}{2}\right). \tag{4.16}$$

Substitute (4.15) and (4.16) into (4.14) and let $\mu \to \pi^2$. Then (2.14) follows immediately. □

5. Proof of Theorem 3

Let $w_\mu = \lambda(\mu)^{\frac{1}{p-1}}u_\mu$. Then it follows from (1.1) that w_μ satisfies

$$\begin{cases} -w_\mu'' = \mu w_\mu - \{w_\mu^p + \lambda(\mu)^{\frac{p}{p-1}}f(\lambda(\mu)^{-\frac{1}{p-1}}w_\mu)\} \quad \text{in} \quad I, \\ w_\mu(x) > 0, \qquad x \in I, \\ w_\mu(0) = w_\mu(1) = 0. \end{cases} \tag{5.1}$$

Lemma 5.1. *There exists a constant $C_{17} > 0$ such that for $\mu \gg 1$*

$$\mu^{\frac{p+1}{2}} \le C_{17}\lambda(\mu).$$

Proof. We first show that $u_\mu(x)$ is bounded for $\mu \gg 1$. It follows from (2.8), (2.15) and (4.3) that there exists a constant $C_{18} > 0$ such that

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$$\frac{1}{p+1}\|u_\mu\|_{p+1}^{p+1} \le \beta(\mu) \le G(s_\mu) = \left\{\frac{1}{p+1}\|s_\mu\|_{p+1}^{p+1} + F(s_\mu)\right\} \le C_{18}\{(\mu-\pi^2)^{-\frac{p+1}{2}} + (\mu-\pi^2)^{-\frac{q+1}{2}}\}. \tag{5.2}$$

Then we obtain by (1.2), (3.5), (5.2) and Hölder's inequality that there exists a constant $C_{19} > 0$ such that for $\mu \gg 1$

$$\|u_\mu\|_\infty^2 \le \|u'_\mu\|_2^2 \le \mu\|u_\mu\|_2^2 \le \mu\|u_\mu\|_{p+1}^2 \le C_{19}^2. \tag{5.3}$$

Let

$$I_1 := \{x \in I : u_\mu(x) \le \delta\}, \quad I_2 := \{x \in I : \delta < u_\mu(x) \le C_{19}\},$$

where δ is defined by (2.15). Then it is clear from (5.3) that $I = I_1 \cup I_2$. Now we obtain by (2.15), (5.2) and (5.3) that

$$\begin{aligned}\frac{2\alpha}{\lambda(\mu)} &= H(u_\mu) = \|u_\mu\|_{p+1}^{p+1} + \int_0^1 f(u_\mu)u_\mu dx \\ &\le 2(p+1)C_{18}\mu^{-\frac{p+1}{2}} + \int_0^1 \frac{f(u_\mu)}{u_\mu^p}u_\mu^{p+1}dx \\ &\le 2(p+1)C_{18}\mu^{-\frac{p+1}{2}} + C_3\delta^{q-p}\int_{I_1} u_\mu^{p+1}dx + \frac{f(C_{19})}{\delta^p}\int_{I_2} u_\mu^{p+1}dx \\ &\le 2(p+1)C_{18}\mu^{-\frac{p+1}{2}} + \left(C_3\delta^{q-p} + \frac{f(C_{19})}{\delta^p}\right)\|u_\mu\|_{p+1}^{p+1} \\ &\le 2(p+1)C_{18}\left(1 + C_3\delta^{q-p} + \frac{f(C_{19})}{\delta^p}\right)\mu^{-\frac{p+1}{2}}.\end{aligned}$$

Thus the proof is complete. □

We find from (5.1) that w_μ satisfies

$$\begin{cases} -w''_\mu + w_\mu^p \le \mu w_\mu & \text{in} \quad I, \\ w_\mu(0) = w_\mu(1) = 0, \end{cases} \tag{5.4}$$

that is, w_μ is a subsolution of the equation

$$\begin{cases} -v'' + v^p = \mu v & \text{in} \quad I, \\ v(0) = v(1) = 0. \end{cases} \tag{5.5}$$

We choose a constant $K_\mu > 0$ such that $K_\mu > \|w_\mu\|_\infty$ and $K_\mu^{p-1} > \mu$. Then $K_\mu(x) = K_\mu$ $(x \in \bar{I})$ is a supersolution of (5.5), that is, $K_\mu(x)$ satisfies

$$\begin{cases} -K_\mu''(x) + K_\mu(x)^p \geq \mu K_\mu(x) & \text{in} \quad I, \\ \quad K_\mu(0), K_\mu(1) \geq 0. \end{cases}$$

Furthermore, it is clear that $w_\mu(x) < K_\mu(x)$ in I. Hence, by Amann [**1**, (1.1) Theorem], we find that there exists a solution v_μ of (5.5) satisfying

$$w_\mu \leq v_\mu \leq K_\mu$$

in I. Since $\mu > \pi^2$, we know from Berestycki [**2**, Théorème 6] that this v_μ is a unique positive solution of (5.5). Furthermore, it is easy to see that $W_1(x) \equiv \mu^{1/(p-1)}$ is a supersolution of (5.5), and $W_2(x) = (\mu - \pi^2)^{1/(p-1)} \sin \pi x$ is a subsolution of (5.5) with $W_2(x) \leq W_1(x)$. Therefore, by Amann [**1**, (1.1) Theorem], there exists a solution V_μ of (5.5) such that

$$(\mu - \pi^2)^{1/(p-1)} \sin \pi x \leq V_\mu(x) \leq \mu^{1/(p-1)}$$

for $x \in I$. Since the positive solution of (5.5) is unique, we have $v_\mu \equiv V_\mu$. Consequently, we obtain

$$w_\mu \leq v_\mu \leq \mu^{\frac{1}{p-1}}. \tag{5.6}$$

Now, we put

$$r = r(\mu) = \|v_\mu\|_2.$$

Then we know from Heinz [**15**, Proposition 2.1] that $r(\mu)$ is a strictly increasing function of $\mu > \pi^2$, since $v_{\mu_1} < v_{\mu_2}$ in I if $\mu_1 < \mu_2$. Hence, μ is a function of $r > 0$, that is, $\mu = \mu(r)$. More precisely, $\mu(r)$ is increasing for $r \in (0, \infty)$, and $\mu(r) \to \pi^2$ as $r \to 0$ and $\mu(r) \to \infty$ as $r \to \infty$. We refer to Berestycki [**2**] for these properties. Therefore, v_μ is parameterized by $r > 0$. Now, we introduce the auxiliary functions $C_1(r)$ and $R(r)$ for $r > 0$:

$$C_1(r) := \|v_\mu'\|_2^2 + \frac{2}{p+1}\|v_\mu\|_{p+1}^{p+1}, \quad R(r) := \mu(r) - r^{p-1}. \tag{5.7}$$

For these functions, we know the following properties:

Lemma 5.2. ([**19**, Lemma 1.1, Theorem]) *$C_1(r)$ is differentiable in $r > 0$ and the following equality holds:*

$$\frac{dC_1(r)}{dr} = 2r\mu(r). \tag{5.8}$$

Furthermore, for $r \gg 1$

$$C_{20}^{-1} r^{\frac{p-1}{2}} \le R(r) \le C_{20} r^{\frac{p-1}{2}}. \tag{5.9}$$

Then we obtain by (5.7), (5.9), and direct calculation that there exists a constant $C_{21} > 0$ such that for $\mu \gg 1$ (i.e., $r \gg 1$)

$$C_{21}^{-1} \mu(r)^{\frac{1}{2}} \le R(r) \le C_{21} \mu(r)^{\frac{1}{2}}. \tag{5.10}$$

Multiply (5.5) by v_μ. Then integration by parts yields

$$\mu(r) r^2 = \|v'_\mu\|_2^2 + \|v_\mu\|_{p+1}^{p+1}. \tag{5.11}$$

Since $\mu(r) \to \pi^2$ as $r \to 0$, we see from (5.11) that as $r \to 0$

$$\|v'_\mu\|_2^2,\ \|v_\mu\|_{p+1}^{p+1} \to 0;$$

this along with (5.7) implies that $C_1(r) \to 0$ as $r \to 0$. Therefore, we have by (5.8) that

$$C_1(r) = \int_0^r 2\mu(s) s\, ds = \frac{2}{p+1} r^{p+1} + \int_0^r 2sR(s)\, ds. \tag{5.12}$$

Lemma 5.3. *There exists a constant $C_{22} > 0$ such that for $\mu \gg 1$*

$$\mu^{\frac{p+1}{p-1}}(1 - C_{22}^{-1}\mu^{-\frac{1}{2}}) \le \|v_\mu\|_{p+1}^{p+1} \le \mu^{\frac{p+1}{p-1}}(1 - C_{22}\mu^{-\frac{1}{2}}).$$

Proof. By (5.7) and (5.10), we obtain

$$C_{21}^{-1} \mu(r)^{1/2} \le \mu(r) - r^{p-1} \le C_{21} \mu(r)^{1/2}. \tag{5.13}$$

This implies that

$$\mu(r)^{1/(p-1)}(1 - C_{21}\mu^{-1/2})^{1/(p-1)} \le r \le \mu(r)^{1/(p-1)}(1 - C_{21}^{-1}\mu^{-1/2})^{1/(p-1)}.$$

Therefore, we have

$$\mu(r)^{1/(p-1)}(1 - C_{23}\mu(r)^{-1/2}) \le r \le \mu(r)^{1/(p-1)}(1 - C_{23}^{-1}\mu(r)^{-1/2}). \tag{5.14}$$

Now, by (5.11) and (5.14)

$$\begin{aligned} \|v_\mu\|_{p+1}^{p+1} &\le \mu(r) r^2 \le \mu(r)\mu(r)^{2/(p-1)}(1 - C_{23}^{-1}\mu(r)^{-1/2})^2 \\ &\le \mu(r)^{(p+1)/(p-1)}(1 - C_{22}\mu(r)^{-1/2}). \end{aligned} \tag{5.15}$$

Next, we obtain by (5.7), (5.11), and (5.12) that

$$
\begin{aligned}
\|v_\mu\|_{p+1}^{p+1} &= \frac{p+1}{p-1}\{\mu(r)r^2 - C_1(r)\} \\
&= r^{p+1} + \frac{p+1}{p-1}\left\{r^2R(r) - \int_0^r 2sR(s)ds\right\}.
\end{aligned} \tag{5.16}
$$

There exists a constant $r_0 > 0$ such that (5.9) holds for $r \geq r_0$. Then, for $r \geq r_0$

$$
\begin{aligned}
\int_0^r 2sR(s)ds &= \int_0^{r_0} 2sR(s)ds + \int_{r_0}^r 2sR(s)ds \\
&\leq 2\int_0^{r_0} (\mu(s)s + s^p)ds + 2C_{20}\int_{r_0}^r s^{(p+1)/2}ds \\
&\leq C_{24} + \frac{4}{p+3}C_{20}r^{(p+3)/2} \leq C_{25}r^{(p+3)/2},
\end{aligned}
$$

where $C_{24} = \mu(r_0)r_0^2 + 2r_0^{p+1}/(p+1) - 4C_{20}r_0^{(p+3)/2}/(p+3)$. This along with (5.7), (5.9), (5.10), (5.14) and (5.16) implies that for $r \gg 1$

$$
\begin{aligned}
\|v_\mu\|_{p+1}^{p+1} &\geq r^{p+1} - \int_0^r 2sR(s)ds \geq r^{p+1} - C_{25}r^{(p+3)/2} \\
&= r^2(r^{p-1} - C_{25}r^{(p-1)/2}) \geq r^2(\mu(r) - (1 + C_{20}C_{25})R(r)) \\
&\geq \mu(r)^{2/(p-1)}(1 - C_{23}\mu(r)^{-1/2})^2(\mu(r) - C_{26}\mu(r)^{1/2}) \\
&\geq \mu(r)^{(p+1)/(p-1)}(1 - C_{27}\mu(r)^{-1/2}).
\end{aligned} \tag{5.17}
$$

Thus, the proof is complete. □

Lemma 5.4. *There exist constants $C_{28}, C_{29} > 0$ such that for $\mu \gg 1$*

$$
2\alpha\lambda(\mu)^{\frac{2}{p-1}} \leq \mu^{\frac{p+1}{p-1}}\{1 - C_{28}\mu^{-\frac{1}{2}} + C_{29}\mu^{\frac{p-q}{2}}\}. \tag{5.18}
$$

Proof. It follows from Lemma 5.1 and (5.6) that for $\mu \gg 1$

$$
\lambda(\mu)^{-\frac{1}{p-1}}w_\mu \leq C_{30}\mu^{-\frac{1}{2}}, \tag{5.19}
$$

where $C_{30} = C_{17}^{\frac{1}{p-1}}$. We obtain by (2.15), (3.1), (5.6), Lemma 5.3 and (5.19) that

$$
\begin{aligned}
\frac{2\alpha}{\lambda(\mu)} &= H(u_\mu) = \lambda(\mu)^{-\frac{p+1}{p-1}}\|w_\mu\|_{p+1}^{p+1} + \lambda(\mu)^{-\frac{1}{p-1}}E(\lambda(\mu)^{-\frac{1}{p-1}}w_\mu, w_\mu) \\
&\leq \lambda(\mu)^{-\frac{p+1}{p-1}}\|v_\mu\|_{p+1}^{p+1} + C_3\lambda(\mu)^{-\frac{q+1}{p-1}}\|v_\mu\|_{q+1}^{q+1} \\
&\leq \lambda(\mu)^{-\frac{p+1}{p-1}}\left\{\left(\mu^{\frac{p+1}{p-1}}\right) - C_{22}^{-1}\left(\mu^{\frac{p+3}{2(p-1)}}\right)\right\} + C_3\lambda(\mu)^{-\frac{q+1}{p-1}}\mu^{\frac{q+1}{p-1}};
\end{aligned} \tag{5.20}
$$

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this along with Lemma 5.1 implies that

$$2\alpha\lambda(\mu)^{\frac{2}{p-1}} \le \mu^{\frac{p+1}{p-1}} - C_{20}^{-1}\left(\mu^{\frac{p+3}{2(p-1)}}\right) + C_{29}\mu^{\frac{p+1}{p-1}}\mu^{\frac{p-q}{2}},$$

where $C_{29} = C_3 C_{17}^{\frac{q-p}{p-1}}$. This is the desired inequality. □

We obtain by (2.15), (5.1), (5.6) and Lemma 5.1 that for $\mu \gg 1$

$$-w_\mu'' + w_\mu^p \ge w_\mu\left\{\mu - C_3\lambda(\mu)^{\frac{p-q}{p-1}} w_\mu^{q-1}\right\} \ge w_\mu\left\{\mu - C_3\lambda(\mu)^{\frac{p-q}{p-1}}\mu^{\frac{q-1}{p-1}}\right\} \ge \mu\left(1 - C_{31}\mu^{\frac{p-q}{2}}\right)w_\mu, \quad (5.21)$$

where $C_{31} = C_3 C_{17}^{\frac{q-p}{p-1}}$. Put $\nu := \mu(1 - C_{31}\mu^{\frac{p-q}{2}})$. Then w_μ is a supersolution of (5.5), in which μ is replaced by ν, that is, w_μ satisfies

$$\begin{cases} -w_\mu'' + w_\mu^p \ge \nu w_\mu & \text{in} \quad I, \\ w_\mu(0), w_\mu(1) \ge 0. \end{cases} \quad (5.22)$$

We can choose a constant $\epsilon_\mu > 0$ so small that $z_\mu(x) := \epsilon_\mu \sin \pi x$ satisfies

$$\begin{cases} -z_\mu'' + z_\mu^p \le \nu z_\mu & \text{in} \quad I, \\ z_\mu(0) = z_\mu(1) = 0 \end{cases}$$

and $z_\mu < w_\mu$ in I. Then by [**1**, (1.1) Theorem] we obtain that there exists a positive solution v_ν of (5.5), in which μ is replaced by ν and satisfies $z_\mu \le v_\nu \le w_\mu$ in I. Then we obtain by [**2**, Théorème 4] that

$$(\nu - \pi^2)^{\frac{1}{p-1}} \sin \pi x \le v_\nu \le w_\mu. \quad (5.23)$$

Lemma 5.5. There exists a constant $C_{32} > 0$ such that for $\mu \gg 1$

$$2\alpha\lambda(\mu)^{\frac{2}{p-1}} \ge \mu^{\frac{p+1}{p-1}}(1 - C_{32}\mu^{-\frac{1}{2}} - C_{32}\mu^{\frac{p-q}{2}}). \quad (5.24)$$

Proof. It follows from (3.1), Lemma 5.3 and (5.23) that

$$\begin{aligned} \frac{2\alpha}{\lambda(\mu)} &= H(u_\mu) \ge \lambda(\mu)^{-\frac{p+1}{p-1}}\|w_\mu\|_{p+1}^{p+1} \ge \lambda(\mu)^{-\frac{p+1}{p-1}}\|v_\nu\|_{p+1}^{p+1} \\ &\ge \lambda(\mu)^{-\frac{p+1}{p-1}}\left(\nu^{\frac{p+1}{p-1}} - C_{22}\left(\nu^{\frac{p+3}{2(p-1)}}\right)\right); \end{aligned} \quad (5.25)$$

this along with Lemma 5.4 implies that

$$\begin{aligned} 2\alpha\lambda(\mu)^{\frac{2}{p-1}} &\ge \mu^{\frac{p+1}{p-1}}(1 - C_{31}\mu^{\frac{p-q}{2}})^{\frac{p+1}{p-1}} - C_{22}\left(\mu^{\frac{p+3}{2(p-1)}}(1 - C_{31}\mu^{\frac{p-q}{2}})^{\frac{p+3}{2(p-1)}}\right) \\ &\ge \mu^{\frac{p+1}{p-1}}(1 - C_{32}\mu^{\frac{p-q}{2}}) - C_{32}\mu^{\frac{p+3}{2(p-1)}}. \end{aligned} \quad (5.26)$$

Thus the proof is complete. □

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Proof of Theorem 3. We know that there exist constants $C_{33}, C_{34}, C_{35} > 0$ such that for $0 \le t \ll 1$

$$1 - C_{33}t \le (1-t)^{\frac{p-1}{2}} \le 1 - C_{34}t \tag{5.27}$$

and for $|t| \ll 1$

$$(1+t)^{\frac{p-1}{2}} \le 1 + C_{35}|t|. \tag{5.28}$$

(1) Let $q > p+1$. Then we see that for $\mu \gg 1$

$$0 < C_{28}\mu^{-\frac{1}{2}} - C_{29}\mu^{\frac{p-q}{2}} \ll 1. \tag{5.29}$$

Then by Lemma 5.4 and (5.27) we ob.ain that for $\mu \gg 1$

$$\begin{aligned}\lambda(\mu) &\le \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}}\left\{1 - \left(C_{28}\mu^{-\frac{1}{2}} - C_{29}\mu^{\frac{p-q}{2}}\right)\right\}^{\frac{p-1}{2}} \\ &\le \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}}\left\{1 - C_{34}\left(C_{28}\mu^{-\frac{1}{2}} - C_{29}\mu^{\frac{p-q}{2}}\right)\right\};\end{aligned} \tag{5.30}$$

this implies that for $\mu \gg 1$

$$\frac{1}{2(2\alpha)^{\frac{p-1}{2}}}C_{28}C_{34}\mu^{\frac{p}{2}} \le \frac{1}{(2\alpha)^{\frac{p-1}{2}}}C_{34}\left(C_{28}\mu^{\frac{p}{2}} - C_{29}\mu^{\frac{2p+1-q}{2}}\right) \le \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}} - \lambda(\mu). \tag{5.31}$$

Similarly, we obtain by Lemma 5.5 and (5.27) that for $\mu \gg 1$

$$\begin{aligned}\lambda(\mu) &\ge \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}}\left\{1 - C_{32}\left(\mu^{-\frac{1}{2}} + \mu^{\frac{p-q}{2}}\right)\right\}^{\frac{p-1}{2}} \\ &\ge \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}}\left\{1 - C_{32}C_{33}\left(\mu^{-\frac{1}{2}} + \mu^{\frac{p-q}{2}}\right)\right\};\end{aligned} \tag{5.32}$$

this implies that for $\mu \gg 1$

$$\frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}} - \lambda(\mu) \le C_{36}\left(\mu^{\frac{p}{2}} + \mu^{\frac{2p+1-q}{2}}\right) \le 2C_{36}\mu^{\frac{p}{2}}, \tag{5.33}$$

where $C_{36} = \frac{C_{32}C_{33}}{(2\alpha)^{\frac{p-1}{2}}}$. Now we obtain (2.16) by (5.31) and (5.33).

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(2) Let $p < q \le p+1$ and $\mu \gg 1$. Then we obtain by Lemma 5.4 and (5.28) that

$$\begin{aligned}\lambda(\mu) &\le \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}}\left\{1-\left(C_{28}\mu^{-\frac{1}{2}}-C_{29}\mu^{\frac{p-q}{2}}\right)\right\}^{\frac{p-1}{2}}\\ &\le \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}}\left\{1+C_{35}\left|C_{28}\mu^{-\frac{1}{2}}-C_{29}\mu^{\frac{p-q}{2}}\right|\right\}\\ &\le \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}}\left\{1+C_{35}\left(C_{28}\mu^{-\frac{1}{2}}+C_{29}\mu^{\frac{p-q}{2}}\right)\right\};\end{aligned} \tag{5.34}$$

this implies that

$$\lambda(\mu)-\frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}} \le \frac{1}{(2\alpha)^{\frac{p-1}{2}}}\left(C_{28}C_{35}\mu^{\frac{p}{2}}+C_{29}C_{35}\mu^{\frac{2p+1-q}{2}}\right) \le C_{37}\mu^{\frac{2p+1-q}{2}}, \tag{5.35}$$

where $C_{37} = \frac{1}{(2\alpha)^{\frac{p-1}{2}}}C_{35}(C_{28}+C_{29})$. Similarly, we obtain by Lemma 5.5 and (5.27) that

$$\begin{aligned}\lambda(\mu) &\ge \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}}\left\{1-C_{32}\left(\mu^{-\frac{1}{2}}+\mu^{\frac{p-q}{2}}\right)\right\}^{\frac{p-1}{2}}\\ &\ge \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}}\left\{1-C_{32}C_{33}\left(\mu^{-\frac{1}{2}}+\mu^{\frac{p-q}{2}}\right)\right\}\\ &\ge \frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}}\left(1-2C_{32}C_{33}\mu^{\frac{p-q}{2}}\right);\end{aligned} \tag{5.36}$$

this implies that

$$\lambda(\mu)-\frac{\mu^{\frac{p+1}{2}}}{(2\alpha)^{\frac{p-1}{2}}} \ge -\frac{2C_{32}C_{33}}{(2\alpha)^{\frac{p-1}{2}}}\mu^{\frac{2p+1-q}{2}}. \tag{5.37}$$

Hence, we obtain (2.17) by (5.35) and (5.37). Thus the proof is complete. □

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