NOTES ON SPHERE PACKINGS

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These notes are to supplement my paper **(4)**, and should be read in conjunction with it. Both are divided into three parts, and in these notes the section numbers have a further digit added; thus §1.41 here supplements §1.4 of **(4)**. References by section numbers are always to **(4)** or to the present notes, but references to other papers are numbered independently.

The principal results of these notes are the following. New sphere packings are given in $[2^m]$, $m \geqslant 6$, and in [24], which are twice as dense as those of §§1.6, 2.3. Others are given in $[2^m]$, $m \geqslant 5$, with the same density as those of §1.6, but in which each sphere touches fewer other spheres than in the earlier packings. Acknowledgment is made of lattice packings in $[2^m]$ given by Barnes and Wall **(2)** in anticipation of §1.6 (in which the packings were lattice packings only for $m \leqslant 6$). Denser packings than those of §2.4 are given for [11], as a section of K_{12}, and for [22] and [23], as sections of the new packing in [24]; that in [11] is due to Barnes **(1)**. A proof is given that the packing in [16] (§1.2) is a section of those in [24] (§2.3 or §2.31). Table I supersedes those of **(4)**, giving values of Rogers' bound for the density of packings, and Coxeter's bound for the number of spheres that may touch any one, for spaces of up to 24 dimensions, in comparison with the best figures achieved by known packings in these spaces.

1. Packings in $[2^m]$

1.31. This section and the next two are preliminary to the constructions of sphere packings in §§1.61–1.63 below. It was shown in §1.3 that any two rows of 2^m binary digits having k-parity differ in at least 2^k places. It follows that any row having exactly k-parity differs in at least 2^k places from any row having $(k + 1)$-parity. We now investigate whether there exist rows differing in more than 2^k places from every row having $(k + 1)$-parity.

For $k = 0$ there is clearly no such row, since every row either has 1-parity (simple even parity) or can be altered in any one place so as to give it 1-parity. There is also no such row for $k = 1$. If a row does not have 1-parity, then it can be altered in one place to give it 2-parity; we reverse the digit whose position, when expressed as an integer in the binary scale, has 1's in just those positions whose significance corresponds to those binary constituent rows (§1.3) of the given row which do not have 1-parity. If a row has 1-parity, we can alter it in two places so as to give it 2-parity; one may be chosen arbitrarily, and the other is then uniquely determined as above for a row not having 1-parity.

Received June 30, 1965. Revised version received August 5, 1966.

https://doi.org/10.4153/CJM-1967-017-0 Published online by Cambridge University Press

For $k \geqslant 2$ there exist rows having k-parity which differ in more than 2^k places from each row having $(k+1)$-parity, as we shall now show. Define recursively a row R_n of $n = 2^{k+2}$ binary digits, for even $k \geqslant -2$, by the relations

$$R_1 = 0, \qquad R_{4n} = R_n\, R_n\, R_n\, \bar{R}_n;$$

thus

$$R_4 = 0001 \quad \text{and} \quad R_{16} = 0001000100011110.$$

We show that, for $k \geqslant 0$, the row R_n has exactly k-parity, and differs from every row of $\mathbf{A}_n$, $\bar{\mathbf{A}}_n$ (§1.1) in $\frac{1}{2}n \pm \frac{1}{2}\sqrt{n}$ places. Since

$$\tfrac{1}{2}n - \tfrac{1}{2}\sqrt{n} = 2^{k+1} - 2^{\frac{1}{2}k} > 2^k$$

for $k > 0$, this will prove the result for rows of length $n = 2^{k+2}$.

The parity result is immediate by induction, or can be seen by noticing that the characteristic sum (§1.4) for the row R_n is the first $\frac{1}{2}k + 1$ terms of the series $ab + cd + ef + \ldots$ of products of pairs of consecutive letters.

Let us define that *a row* A_n designates an arbitrary row of one or other of the two matrices $\mathbf{A}_n$, $\bar{\mathbf{A}}_n$ as defined in §1.1; there are $2n$ such rows. We observe that any row A_{4n} comprises either four copies of a row A_n or two copies of a row A_n and two copies of the complementary row. In either case the last quarter of A_{4n} is the same as an earlier quarter. These two quarters thus differ from the corresponding quarters R_n, $\bar{R}_n$ of R_{4n} in half their places, i.e. n places, since these quarters of R_{4n} are complementary. Assuming inductively that the other quarter rows A_n differ from R_n in $\frac{1}{2}n \pm \frac{1}{2}\sqrt{n}$ places each, we have a total of

$$n + 2(\tfrac{1}{2}n \pm \tfrac{1}{2}\sqrt{n}) = 2n \pm \sqrt{n}$$

places, which is the required result for R_{4n}. Thus the result, being valid for $n = 4$, is valid for $n = 2^{k+2}$ for all even $k \geqslant 0$. We notice incidentally that half of the rows A_n differ from R_n in $\frac{1}{2}n - \frac{1}{2}\sqrt{n}$ places and half, their complements, in $\frac{1}{2}n + \frac{1}{2}\sqrt{n}$ places.

To complete the proof, we have to show that the row R_n followed by $(2^m - n)$ 0's differs from every row of 2^m binary digits having $(k+1)$-parity in at least $\frac{1}{2}n - \frac{1}{2}\sqrt{n}$ places. To do this we form from the given row of 2^m digits a row of n digits by adding modulo 2 the 2^{m-k-2} blocks of $n = 2^{k+2}$ digits forming the given row. This row has at least the same order of parity as the given row, as is seen inductively from the effect of the operation on the binary constituent rows (§1.3). It is also easily seen that this shorter row cannot differ in more places from R_n than does the original row from R_n extended by 0's. Hence the original row differs from R_n extended by 0's in at least $\frac{1}{2}n - \frac{1}{2}\sqrt{n}$ places, i.e. the row R_n followed by 0's differs from every row having $(k+1)$-parity, of whatever total length, in at least $\frac{1}{2}n - \frac{1}{2}\sqrt{n}$ places.

For odd values of $k \geqslant 3$ the row comprising $R_{\frac{1}{2}n}\, R_{\frac{1}{2}n}$ followed by 0's differs from any row having $(k+1)$-parity in at least $\frac{1}{2}n - \sqrt{(\frac{1}{2}n)}$ places. We have not established whether this inequality, or the previous inequality for even k, is the best possible for any value of $k \geqslant 2$. However, it is not difficult to see that any 2-parity row having exactly six 1's shares the property of differing from all 3-parity rows in at least six places.

We note for future reference some properties of 2-parity rows having exactly six 1's. First, no subset of four of the 1's has 2-parity; otherwise we would have a 2-parity row with only two 1's by adding these two rows modulo 2. So if we choose three of the six 1's, these determine a tetrad (a 2-parity set of four 1's) whose fourth member is not one of the six 1's. By addition modulo 2 we see that the same fourth member is determined by the other three of the six 1's. It is also clear that the ten partitions of the six 1's into two sets of three determine ten distinct fourth members. Thus if the row is of sixteen digits, each of the ten 0's determines a partition of the six 1's into two sets of three; if not, it may be shown that the six 1's and the ten fourth members together form a 4-parity set, i.e. a 1 in each place gives a 4-parity row.

1.41. I should have noted that the row which has 1's where both of two given rows have 1's, and 0's where either or both of them have 0's, has a characteristic sum that is formed in the following way. The characteristic sums for the two given rows are multiplied together, every term of one being multiplied by every term of the other. The letters, being binary-valued variables, are idempotent, so that $a^2 = a$, etc., and terms in the product which are equal, after removing indices from squares where possible, are added modulo 2. We see from this that the greatest number of letters in any term of the product cannot exceed the sum of the greatest number of letters in any two terms, one from each of the characteristic sums for the given rows.

This operation of forming the Boolean product of two given rows is used in forming the binary constituent rows of a given row in §1.3, and in investigating the order of parity of the row of carry digits when rows of digits of the same significance in the coordinates of two points are added, as in §1.6 when showing that the packing may not be a lattice. We recall from §1.4 that the maximum number of letters in a term of the characteristic sum for a row of 2^m digits having k-parity is $m - k$. Thus if two rows of 2^m digits having k_1-parity and k_2-parity are added, the sum row has parity of order $\min(k_1, k_2)$, this order being exact if the orders k_1, k_2 are exact and unequal, while the carry row has parity of order $\max(0, k_1 + k_2 - m)$.

1.42. We shall have occasion in §§1.61, 1.62 below to consider points whose coordinates are the sums of the coordinates of pairs of given points. To facilitate this investigation, we consider an array $\mathbf{p}$ of binary digits which is obtained from the coordinates of a point by setting out in columns the values of the coordinates in the binary scale. The first row p_1 comprises the ones digits of the coordinates, and thus has 0's for coordinates with even values and 1's for coordinates with odd values. Subsequent rows p_2, p_4, p_8, . . . comprise the twos, fours, eights, . . . digits of the coordinates. The number of rows is

https://doi.org/10.4153/CJM-1967-017-0 Published online by Cambridge University Press

potentially infinite, but in practice we are interested in only the first few rows. An example of such an array is set out in §1.62.

Now suppose we have two points, whose coordinates are set out in arrays **p**, **c**, as above, and we wish to investigate the *sum point* whose coordinates are the sums of those of the given points; symbolically we write $\mathbf{s} = \mathbf{p} + \mathbf{c}$, where **s** is the corresponding array for the sum point. The first row s_1 of **s** is the sum modulo 2 of p_1 and c_1, which we write $s_1 = p_1 + c_1$, with the convention that sums of rows (but not of whole arrays) are always to be interpreted modulo 2. The carry from this addition is $p_1 \times c_1$, where the multiplication sign denotes Boolean multiplication as described in §1.41. (Juxtaposition without a sign, such as $p_1 c_1$, would mean forming an extended row by following the digits of p_1 with those of c_1.) We thus have that the twos digits row s_2 is given by $s_2 = p_2 + c_2 + p_1 \times c_1$. The carries formed by this addition are $p_2 \times c_2 + (p_2 + c_2) \times p_1 \times c_1$, so the fours digits row s_4 is given by

$$s_4 = p_4 + c_4 + p_2 \times c_2 + (p_2 + c_2) \times p_1 \times c_1.$$

Corresponding expressions can be obtained for $s_8, \ldots$ without difficulty; they are of increasing complexity.

1.61. Barnes and Wall **(2)** have given a number of quadratic forms in 2^m variables. Interpreted as results on sphere packings, these give lattice packings in $[2^m]$ of which the densest have the same density as those of §1.6. This anticipates the suggestion at the end of §1.6 that there should be such lattice packings for all m, not only for $m \leqslant 6$ as obtained there. They also give a series for the number of minimal vectors (which is half the number of spheres touched by each) equivalent to that of §1.7, but they do not sum the series explicitly.

My statement **(4**, Introduction and §1.7**)** that *each* sphere touches

$$(2 + 2)(2 + 2^2) \ldots (2 + 2^m)$$

others is not correct for non-lattice packings. This number was correctly obtained for the number of spheres touching the sphere centred at the origin, and is correct also for certain others, including those equivalent to the sphere centred at the origin under any translation that takes all spheres of the packing into spheres of the packing. But if the packing is not a lattice packing, not all of the spheres are equivalent in this sense, and certain of the other spheres touch fewer than this number. We shall show this for the packing in which the coordinates of the centres are either all even or all odd; a very similar argument gives the corresponding result for the other packing of §1.6.

The centres of the spheres which touch that centred at the origin include $2^{1+\frac{1}{2}m+\frac{1}{2}m^2}$ whose coordinates are all ± 1, this number being the number of $(m-2)$-parity rows which the twos digits may form. The array **p** for any such centre thus comprises a row p_1 of all 1's, a row p_2 with $(m-2)$-parity, and rows $p_4, p_8, \ldots$ identical with p_2. Now consider a sphere whose centre has coordinates giving an array **c** of digits such that c_1 is all 0's but c_2 has exactly $(m-2)$-parity. The spheres touching this sphere include some whose centres have coordinates differing from those of the given sphere by ± 1. For a point to be the centre of such a sphere, it is necessary (but not sufficient, as we shall see) that its coordinates form a binary array **s** satisfying $\mathbf{s} = \mathbf{p} + \mathbf{c}$ with **p** as above, otherwise s_2 will not have $(m-2)$-parity. Since c_1 is all 0's,

$$s_2 = p_2 + c_2, \qquad s_4 = p_4 + c_4 + p_2 \times c_2,$$

and

$$s_8 = p_8 + c_8 + p_4 \times c_4 + (p_4 + c_4) \times p_2 \times c_2.$$

Now p_2, p_4, and p_8 have $(m-2)$-parity, being identical, and c_2, c_4, and c_8 have $(m-2)$-parity, $(m-4)$-parity, and $(m-6)$-parity respectively, so every term in s_8 has $(m-6)$-parity except possibly for $c_4 \times p_2 \times c_2$, which may have only $(m-8)$-parity (§1.41). So if $m > 6$, we may have s_8 lacking $(m-6)$-parity, in which case the sum point is not the centre of a sphere of the packing; there are choices of **p** and **c** for which this happens. Thus there are spheres which touch fewer than $2^{1+\frac{1}{2}m+\frac{1}{2}m^2}$ others whose centres have coordinates differing from their own by ± 1, and the total numbers of spheres which they touch are correspondingly smaller.

For $m \geqslant 5$ we can modify the original packing so that it retains the same density but no sphere touches as many as

$$(2+2)(2+2^2)\ldots(2+2^m)$$

others. Using the coordinates of §1.6 in which the coordinates of centres are all even or all odd, we make the modification that if c_1 consists of 0's then c_4 has $(m-4)$-parity, but if c_1 consists of 1's then c_4 is a row which differs from an $(m-4)$-parity row in some selected digit, say the first digit. Other rows of **c** are as specified in §1.6. With this modification it is impossible for the centres of two spheres of the packing to have all their coordinates differing by ± 1, and so there is no contact between any sphere whose centre has even coordinates and any whose centre has odd coordinates. The lattice packings in [32] and [64] are modified into lattice packings in which each sphere touches $2^{1+\frac{1}{2}m+\frac{1}{2}m^2}$ fewer, namely 81,344 instead of 146,880 in [32] and 5,499,776 instead of 9,694,080 in [64]. (The number 146,880 has been superseded as contender for the maximum number of spheres that may touch any one in [32] by the number 196,560 obtained in §2.31 below for [24], which can itself be further increased in advancing to [32].)

I state without proof a corresponding result for lattice packings in [128]. The coordinates of §1.6 do not give a lattice packing in [128], as we have seen, but it can be shown that the following adaptation gives the lattice packing in which each sphere touches

$$(2+2)(2+2^2)\ldots(2+2^7) = 1{,}260{,}230{,}400$$

others. The coordinates of each centre form an array **c** whose rows c_1, c_2, c_4 have 7-parity (all 0's or all 1's), 5-parity and 3-parity respectively, as in §1.6; c_8 has 1-parity if the number of 1's in c_4 is a multiple of 4, but lacks it if the number is an odd multiple of 2. It can be shown that the sum (§1.42) of any two arrays satisfying these requirements is another array satisfying them. The modification to obtain a lattice in which no sphere whose centre has even coordinates touches any whose centre has odd coordinates is the following. If c_1 is all 0's, then c_8 is as above, but if c_1 is all 1's, then the condition on c_8 is reversed, so that c_8 has 1-parity if the number of 1's in c_4 is an odd multiple of 2, but lacks it if the number is a multiple of 4. Each sphere then touches 2^{29} fewer others, namely 723,359,488 spheres.

I have no doubt that the corresponding constructions of lattice packings are possible in $[2^m]$ for $m > 7$, but have not obtained explicit formulations for the coordinates, and have not pursued the possibility. It seems that the concept of parity of the binary rows is not ideal for discussing such coordinates. It is probable that for such values of m the spheres can be divided into more than two sets with each sphere touching only those of its own set, the number of spheres touched by each being further reduced. In view of the denser packings of §§1.62, 1.63 below, these possibilities are of less interest.

1.62. For $m \geqslant 6$ I have found packings in $[2^m]$ of twice the density of those of §1.6. For each m the spheres are those of the packing of §1.6 in which the centres have their coordinates either all even or all odd (for $m \geqslant 6$ the two sets of coordinates of §1.6 give different packings), together with an equivalent set obtained by translation of the original spheres by adding to the coordinates of their centres the coordinates of a suitable fixed point. With these coordinates the packing is a lattice packing for $m = 6$ but not for $m > 6$ as the original packing is not then a lattice packing. For $m = 7$, however, a lattice packing is obtained if we begin with the lattice packing as given in §1.61 above, and I have no doubt that equally dense lattice packings can be obtained similarly for $m > 7$ by using the lattice packings of Barnes and Wall **(2)** in these cases.

In [64] the fixed point has the coordinates given in the first displayed row below. Beneath this is the binary array **p** set out as described in §1.42. Thus the row p_1 of ones digits is $0_{32}\,1_{32}$ (where 0_{32} means 32 consecutive 0's, etc.), the row p_2 is $R_{16}\,0_{16}\,R_{16}\,0_{16}$, the row p_4 is $0_{31}\,1\,0_{31}\,1$, subsequent rows being omitted as their digits are all 0.

	0002000200022220000000000000000041113111311133331111111111111115
p_1	0000000000000000000000000000000011111111111111111111111111111111
p_2	0001000100011110000000000000000000010001000111100000000000000000
p_4	0000000000000000000000000000000100000000000000000000000000000001

We have to show that no centre of one of the translated spheres is closer to that of an original sphere than the minimum distance 8 between centres of the original spheres. Since if we double the coordinates of the fixed point we get the centre of an original sphere, the new packing is also a lattice packing. An equivalent requirement, then, is to show that if we define a sum point, as in §1.42, whose coordinates are the sums of the coordinates of the fixed point and of the centre of any of the original spheres, then this sum point is at a distance not less than 8 from the origin. Thus we have to show that the sum of the squares of the coordinates of the sum point is at least 64. The sum points at a distance 8 from the origin are in fact centres of spheres touching that whose centre is the origin.

Let us set out the coordinates of the centre of the original sphere and of the sum point in binary arrays $\mathbf{c}$ and $\mathbf{s}$, as in §1.42, so that $\mathbf{s} = \mathbf{p} + \mathbf{c}$. We have $s_1 = p_1 + c_1$; since $p_1 = 0_{32}\,1_{32}$ and c_1 is either 0_{64} or 1_{64}, s_1 is either $0_{32}\,1_{32}$ or $1_{32}\,0_{32}$, so that half the coordinates of the sum point are even and half odd. The odd coordinates contribute at least 32 to the sum of the squares of the coordinates of the sum point. Let us call the half where s_1 has digits 0_{32} the *even half* and where it has digits 1_{32} the *odd half*. We denote the halves of rows by prefixes e, o, so that $es_1 = 0_{32}$ and $os_1 = 1_{32}$, and we use the same description, even or odd, and the same prefix to designate the corresponding half of any other row; thus the even half of any row comprises the 32 digits at the same end of it as the 0's of s_1.

We now examine the row s_2 of twos digits of the coordinates of the sum point. We have $s_2 = p_2 + c_2 + p_1 \times c_1$. Now c_2 has 4-parity and $p_1 \times c_1$ has 5-parity, so the sum of these has 4-parity and each half of it has 3-parity. We shall have frequent occasion to refer to the halves of $c_2 + p_1 \times c_1$. Its odd half is simply oc_2 as there is never any carry into this half. Its even half is $ep_1 + ec_2$, which is either ec_2 or $ec_2 + 1_{32}$; these have the same parity, and will be denoted by $e'c_2$ without discrimination. In this notation we have

$$es_2 = ep_2 + e'c_2.$$

Now $e'c_2$ has 3-parity and $ep_2 = R_{16}\,0_{16}$ which differs in at least six places from every 3-parity row (§1.31), so es_2 includes at least six 1's. The sum point therefore has at least six non-zero even coordinates, and these contribute at least 24 to the sum of the squares of the coordinates of the sum point. Further, since es_2 has 2-parity, there are six or at least eight non-zero digits in it, and the contribution is either 24 or at least 32.

Lastly we show that if the even coordinates contribute exactly 24 to the sum of the squares of the coordinates, then the odd coordinates contribute at least 40 to the sum. We obtained $es_2 = ep_2 + e'c_2$ above; thus es_2 is the sum modulo 2 of $R_{16}\,0_{16}$, each half of which has 2-parity, and $e'c_2$, a 3-parity row each half of which also has 2-parity. Thus each half of es_2 has 2-parity, and so there are either no 1's or at least four 1's in each half. As there are only six 1's in the whole of es_2 in this case, these must all be in one half with none in the other. If the six 1's are in the R_{16} half, we must have $e'c_2 = A_{16}\,0_{16}$; if in the other half, we must have $e'c_2 = R_{16}(R + A)_{16}$, where $(R + A)_{16}$ denotes the sum modulo 2 of R_{16} and an A_{16} (and juxtaposition continues to mean the concatenation of the half rows). Now c_2 has 4-parity, so its halves differ by an A_{32}, and we have $e'c_2 + oc_2 = A_{32}$; hence the halves of oc_2 are each of the form A_{16} or of the form $(R + A)_{16}$. Since $op_2 = R_{16}\,0_{16}$, the carry into os_4, which is $op_2 \times oc_2$, is entirely in the R_{16} half and is either $R_{16} \times A_{16}$ or $R_{16} \times (R + A)_{16}$. As R_{16} and A_{16} are rows of 16 digits and have 2-parity and 3-parity respectively, $R_{16} \times A_{16}$ has 1-parity (§1.41), and $R_{16} \times (R + A)_{16} = R_{16} + R_{16} \times A_{16}$, so this also has 1-parity. Thus in the equation $os_4 = op_4 + oc_4 + op_2 \times oc_2$, oc_4 and $op_2 \times oc_2$ have 1-parity, but $op_4 = 0_{31}\,1$ which lacks 1-parity, so os_4 lacks 1-parity. Hence os_4 differs in at least one place from os_2 (which has 2-parity), and there is at least one odd coordinate which has its twos and fours digits unequal. Now the twos and fours digits are equal both for $+1$ and for -1, so there is at least one odd coordinate that is not ± 1, and this coordinate contributes at least 9 to the sum of the squares of the coordinates.

We conclude then that the sum of the squares of the coordinates of the sum point receives contributions of at least 32 from the odd coordinates and of either 24 or at least 32 from the even coordinates, but that if the even coordinates contribute exactly 24 then the odd coordinates contribute at least 40. Thus in either case the sum of the squares is at least 64, and so the sum point is distant at least 8 from the origin, as required. We have then a sphere packing in [64] which is twice as dense as that of §1.6.

1.63. For $m > 6$ we can construct similar packings by the same method. As already stated, we use the packing of §1.6 in which the coordinates are either all even or all odd as a basis; constructions are also possible for some values of m using the other packing, but these have been examined less fully. The fixed point has the following coordinates, where we write $\nu = 2^{m-3}$ to avoid fractions in subscripts. The ones digits form a row $p_1 = 0_{4\nu}\,1_{4\nu}$ in all cases. If m is even, $p_2 = R_{2\nu}\,0_{2\nu}\,R_{2\nu}\,0_{2\nu}$, but if m is odd, $p_2 = R_\nu\,R_\nu\,0_{2\nu}\,R_\nu\,R_\nu\,0_{2\nu}$. The fours digits form a row p_4 consisting of $0_{31}\,1$ repeated 2^{m-5} times. More significant digits are all 0.

We can show similarly that the points so obtained are centres of a packing of spheres of the same size as those of the packing of §1.6, which thus has twice the density. The analysis is somewhat more complicated than that of §1.62 because the original packing is non-lattice (it can be shown that this does not affect discussions involving only the ones, twos, and fours digits of the coordinates), but the conclusion is stronger, namely that the newly introduced spheres have no contact with the original spheres. We find, in fact, that the sum of the squares of the coordinates of the sum point exceeds 2^m in all cases with $m > 6$.

As an example, we indicate the nature of the result for $m = 7$. We find that the 64 odd coordinates of the sum point include at least two that are not ± 1, so their squares contribute at least 80 to the sum of the squares of the coordinates. The even coordinates include either 12 or at least 16 whose twos digits are non-zero, and if exactly 12 coordinates are ± 2, then there is at least

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one that is not smaller than ± 4. The even coordinates thus contribute at least 64 to the sum of the squares of the coordinates of the sum point, so this is at least 144. We then have that the centres of the introduced spheres are distant at least 12 from the centres of the original spheres; the spheres, being of radius $4\sqrt{2}$, are thus not in contact.

As this argument has no reference to the eight digits of the coordinates of the centres of the spheres, we can perform the same constructions with the lattice packings of §1.61 in [128], and we obtain lattice packings of twice the density of the earlier packings with either the greater or the smaller numbers of spheres touched by each. We thus have three lattice packings with each sphere touching 723,359,488 others, namely the packing of §1.61, the present packing of twice its density, and a packing of half its density comprising only those spheres whose centres have all their coordinates even.

For sufficiently large values of m, it is probable that more than one further set of spheres can be introduced into the original packing, more than doubling its density. It would probably be advantageous in investigating such packings to work with the lattice packings of Barnes and Wall **(2)** rather than the non-lattice packings of §1.6. This would complicate the work, as the coordinates are less simply expressible, and I have not pursued this possibility.

1.71. In this section the number of spheres touched by each in the new packing in [64] given in §1.62 above is evaluated. Each of the original spheres, and each of the introduced spheres, touches 9,694,080 others of its own kind, as found in §1.7. We have to find how many spheres of the opposite kind each touches, and this we shall do by finding how many sum points are distant 8 from the origin. As found in §1.62, these fall into two classes. In the first class, the sum point has one coordinate ± 3, six coordinates ± 2, 31 coordinates ± 1, and the rest 0. In the second class, the sum point has eight coordinates ± 2, 32 coordinates ± 1, and the rest 0. The multiplying factors in parentheses state the number of choices available at each stage; their product is the appropriate subtotal for the number of spheres of each class.

We begin with the first class. The row c_1 may be 0_{64} or $1_{64}(\times 2)$. The half row $e'c_2$ may be of the form $A_{16}\,0_{16}$ or of the form

$$R_{16}(R + A)_{16}$$

$(\times 2)$. In either case there is a choice of sixteen A_{16}'s, this being the number that give sum rows es_2 with six 1's $(\times 16)$. The half row oc_2 differs from $e'c_2$ by an arbitrary A_{32} $(\times 64)$. For ec_4 we choose any even subset of the six 1's of es_2, if these include the final digit $0_{31}\,1$; otherwise we choose this 1 and any odd subset of the six 1's $(\times 32$ in either case). This ensures that es_4 has 1's only where es_2 has 1's, so there are no coordinates ± 4. We now choose

$$oc_4 = os_2 + op_2 \times oc_2,$$

so that os_4 differs from os_2 only by $op_4 = 0_{31}\,1$; if by this choice c_4 does not have

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2-parity, we add to oc_4 the row comprising the final digit $0_{31}\,1$ and the further 1 which gives 2-parity to c_4 (this is uniquely determined; cf. §1.31); os_4 now differs from os_2 only in this further place. Our freedom of choice has now been exhausted; the rows $c_8, c_{16}, \ldots$ are uniquely determined to be such that the rows $s_8, s_{16}, \ldots$ are identical with s_4. Thus there are $2^{17} = 131{,}072$ sum points of this class.

Before we can deal with the second class, we need to know the number of 3-parity rows $e'c_2$ which, when added to $ep_2 = R_{16}\,0_{16}$, give a sum row having exactly eight 1's. These 1's cannot be all in the same half, otherwise $e'c_2$ would have to be of the form $A_{16}\,0_{16}$ or of the form $R_{16}(R + A)_{16}$, but then $ep_2 + e'c_2$ would have either six or ten 1's. So, since each half of es_2 has 2-parity, each half must be a tetrad (a 2-parity set of four 1's), and the second half of $e'c_2$ is the second of these. Since $e'c_2$ has 3-parity, its halves differ by an A_{16}. This implies that the first half, being a 2-parity set differing in four places from an A_{16}, cannot have six 1's, nor can it have ten 1's, else its complement would have six 1's and would differ in four places from the complementary A_{16}. Since it also differs in four places from R_{16}, the only possibilities are that it has four or eight 1's.

Thus the sum tetrad has one or three 1's in common with the 1's of R_{16}; in either case it has a 1 coinciding with a 1 of R_{16} and a 1 coinciding with a 0 of R_{16}. Let us choose these 1's arbitrarily, which may be done in 60 ways. The remaining 14 digits are divided into seven pairs, each forming a tetrad with our chosen pair (our pair and any third digit determine the fourth member of a tetrad uniquely, and this gives the required pairing). Only one of these pairs coincides with two 1's of R_{16}, since the chosen 0 determines a partition of the 1's of R_{16} into two sets of three (§1.31) forming tetrads with it, the chosen 1 and the pair forming one of these sets. Three of the pairs coincide with a 1 and a 0 of R_{16}, and the remaining three pairs coincide with 0's of R_{16}. The four pairs coinciding with two 1's or two 0's of R_{16} complete suitable tetrads, giving 240 choices, but each tetrad will have been chosen three times, according to which of the three 1's or three 0's of R_{16} was first chosen, so there are only 80 distinct sum tetrads.

The second half of $e'c_2$ is now any one of the four tetrads which will complete a 3-parity row with the first half. There are thus 320 possible choices for $e'c_2$. We notice that in all cases the carry $ep_2 \times e'c_2$ has three or five 1's and thus lacks 1-parity.

We can now deal with the second class of sum points. As with the first class, c_1 may be 0_{64} or 1_{64} ($\times 2$). We have found above that $e'c_2$ may be chosen in 320 ways ($\times 320$). The half row oc_2 differs from $e'c_2$ by an arbitrary A_{32} ($\times 64$). Since the odd coordinates of the sum point are all ± 1, os_4 agrees exactly with os_2. This determines oc_4 since $os_4 = op_4 + oc_4 + op_2 \times oc_2$, and is consistent with oc_4 having 1-parity since both op_4 and the carry $op_2 \times oc_2$ lack 1-parity, the latter carry differing from $ep_2 \times e'c_2$ by $ep_2 \times A_{32}$, which is a 1-parity row. We take $ec_4 = oc_4 + os_4 + e'c_2 \times A_{32}$; since os_4 and $e'c_2 \times A_{32}$ are 2-parity

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sets, this makes the whole row c_4 have 2-parity. Since

$$os_4 = op_4 + oc_4 + op_2 \times oc_2,$$

this may be expressed as $ec_4 = op_4 + op_2 \times oc_2 + e'c_2 \times A_{32}$, and we have

$$\begin{aligned} es_4 &= ep_4 + ec_4 + ep_2 \times e'c_2 \\ &= ep_4 + op_4 + op_2 \times oc_2 + e'c_2 \times A_{32} + ep_2 \times e'c_2 \\ &= ep_2 \times oc_2 + ep_2 \times e'c_2 + e'c_2 \times A_{32} \\ &= ep_2 \times A_{32} + e'c_2 \times A_{32} \\ &= es_2 \times A_{32}, \end{aligned}$$

where we have used the relations

$$ep_4 = op_4, \qquad ep_2 = op_2,$$
$$e'c_2 + oc_2 = A_{32}, \quad \text{and} \quad es_2 = ep_2 + e'c_2.$$

Thus, with this choice of ec_4, the only 1's in es_4 are where both es_2 and A_{32} have 1's, so they all coincide with 1's of es_2, thus ensuring that there are no coordinates ± 4. To ec_4 as thus chosen, we may optionally add one or both of the tetrads which make up es_2, which admits only this division into tetrads ($\times 4$). This has now exhausted our freedom of choice; the rows $c_8, c_{16}, \ldots$ have to be such that the rows $s_8, s_{16}, \ldots$ are all identical with s_4. Thus there are $5.2^{15} = 163{,}840$ sum points of this class.

We conclude then that each sphere of this packing touches a further $9.2^{15} = 294{,}912$ spheres, bringing the total number of spheres each touches to 9,988,992. In proportion to the total, this increase seems remarkably small, considering that the density has been doubled.

1.72. As observed in §1.63, in $[2^m]$ for $m > 6$ the introduced spheres do not touch the original spheres in the new packings as there given. Clearly, however, the set of introduced spheres can be moved so as to bring them into contact with the original spheres. This will bring about a small increase in the number of spheres touched by each, proportionately much smaller than in [64] (§1.71). (At first sight, an increase of 2^m is clearly possible, and this is capable of improvement.) In combination with other results of these notes, however, this serves to show that the number

$$(2 + 2)(2 + 2^2) \ldots (2 + 2^m)$$

of spheres, obtained in §1.7, which may touch any one, has been surpassed for all $m \geqslant 5$.

1.73. There are two small slips near the beginning of §1.7. On p. 664, line 6, the expression $2k = r$ should read $k = m - 2r$, and on line 7, the expression $2k = r + 1$ should read $k = m - 2r - 1$. No use was made of these wrong values, the correct values having been inferred from §1.6, so the subsequent work is correct.

2. Packings in up to 24 dimensions

2.11. I remarked in §2.1 on the lack of convenient integer coordinates for the packing K_{12} of Coxeter and Todd **(3)**. Such coordinates can now be given, based on the work of Barnes **(1)**. The coordinates are 18-dimensional and fall naturally into triads, so a double suffix notation is convenient; denote the coordinates by x_{ik}, with $i = 1, 2, 3, 4, 5, 6$ and $k = 1, 2, 3$. The lattice is defined to be the set of points with integer coordinates satisfying the following relations:

(1) $x_{i1} + x_{i2} + x_{i3} = 0 \qquad i = 1, \ldots, 6,$

(2) $x_{i1} - x_{j1} \equiv x_{i2} - x_{j2} \equiv x_{i3} - x_{j3} \pmod{3}, \qquad i, j = 1, \ldots, 6,$

(3) $x_{1k} + x_{2k} + x_{3k} + x_{4k} + x_{5k} + x_{6k} \equiv 0 \pmod{3}, \qquad k = 1, 2, 3.$

(These symmetrical statements are in fact redundant. The equations (1) and the congruence of any two of the terms of each of the congruences (2) implies their congruence to the third, and these congruences and any one of the congruences (3) imply the others.)

The centres nearest the origin have coordinates which may be expressed as follows. First, the triads x_{i1}, x_{i2}, x_{i3} may be of the forms

$$\alpha: 0, 1, -1, \qquad \beta: -1, 0, 1, \qquad \gamma: 1, -1, 0;$$

we can choose five of the triads arbitrarily from these, and the sixth has then to be that which is consistent with the congruences (3). There are thus $3^5 = 243$ such centres. Next, we may change the sign of every coordinate, obtaining a further 243 centres. Lastly we may choose one of the triads to be one of the set

$$\delta: -2, 1, 1, \qquad \epsilon: 1, -2, 1, \qquad \zeta: 1, 1, -2,$$

another of the triads to be the negatives of one of this set, and the remaining four triads to be all 0, 0, 0. This gives a further 270 centres, bringing the total to 756, as given by Coxeter and Todd **(3)**. This packing has a centre density of $3^{-3} = 0.037037\ldots.$

Sections of this packing are considered in §2.42 below.

2.31. Further investigation has led to the discovery of a new lattice packing in [24] with centre density 1, twice that of the packing of §2.3, which occurs as a sublattice. It is convenient to express the coordinates in the binary scale, as in Part 1. The centres are those points whose coordinates satisfy the following conditions. The ones digits are either all 0 or all 1, i.e. the coordinates are either all even (the even centres) or all odd (the odd centres). The twos digits form rows that are congruent modulo 2 to sums of rows of the matrix **C** of §2.3. The fours digits form rows that have even parity for the even centres but lack it for the odd centres. More significant digits are unrestricted. The lattice of §2.3 comprises just the even centres, the coordinates of §2.3 being the halves of the present coordinates.

The nearest even centres to the origin are those with two coordinates ± 4

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or eight coordinates ± 2, other coordinates being all 0 in each case; there are 98,256 of these, as in §2.3. The odd centres have their coordinates congruent to ± 1 modulo 4, with the signs in these congruences forming rows with the same arrangement as 0's and 1's in sums modulo 2 of rows of **C**. Now these rows all have even parity (since the rows of **C** have even parity and this is conserved by addition modulo 2), while the fours digits form rows having odd parity, so there is at least one coordinate in which the twos and fours digits are different. Hence the coordinates cannot all be ± 1, this one being not smaller than ∓ 3, so the nearest such centres to the origin are those which have 23 coordinates ± 1 and one coordinate ∓ 3. These centres are thus at a distance $\sqrt{(23.1 + 9)} = 4\sqrt{2}$ from the origin, which is the same as the distance of the nearest even centres.

There are 2^{12} possible sign combinations in the congruences to ± 1, and 24 places for the exceptional coordinate ∓ 3, so the number of odd centres at the minimum distance from the origin is $24.2^{12} = 98{,}304$. The total number of spheres touched by each in this packing is thus 196,560.

2.41. The densest sections found of the new packing in [24] are verbally the same as those of the packing in [8] as given in §2.4. Thus in [23] we equate any two coordinates to each other; the centre density is 2^{-1}, and each sphere touches 93,150 others. In [22] we equate any three coordinates to each other; the centre density is $2^{-1}3^{-\frac{1}{2}}$, and each sphere touches 49,896 others. In [21] we equate any four coordinates to each other, or any three to 0; the centre density is $2^{-2\frac{1}{2}}$, and each sphere touches 27,720 others. This last section, and sections in spaces of fewer dimensions, are as given in §2.4.

2.42. As shown by Barnes **(1)**, the packing K_{12} has a dense section in [11]. In the coordinates of §2.11 this is obtained by putting

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61} = 0.$$

The centre density of this packing is $2^{-1}3^{-2\frac{1}{2}} = 0.032075\ldots$, which is greater than those of sections of J_{12} and L_{12} given in §2.4. In contrast, however, the number of spheres touched by each is 432, which is smaller than the numbers touched by each in the other sections. The figures for the density and for the number of spheres touched by each are very close to one another for these packings, as shown in the following table, though they are not close to Rogers' and Coxeter's bounds for these respective quantities. This closeness to one another for unrelated packings tempts one to conjecture that the density and the number of spheres touched are not maximal for any of these packings.

	Centre density	Number of spheres touched
Bound	$0.06136\ldots$	1035
K_{11}	$2^{-1}3^{-2\frac{1}{2}} = 0.03207\ldots$	432
L_{11}	$2^{-14\frac{1}{2}}3^{6} = 0.03146\ldots$	440
J_{11}	$2^{-5} = 0.03125$	438

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There is a misprint in Rogers' figure (**5**, p. 3) for the density of K_{11}: the denominator should read 187, 110 $\sqrt{3}$.

The densest section of K_{12} in [10] found is obtained by putting

$$x_{1k} + x_{2k} + x_{3k} + x_{4k} + x_{5k} + x_{6k} = 0, \qquad k = 1, 2, 3$$

(actually any two of these imply the third). The centre density is again

$$2^{-1}2^{-2\frac{1}{2}} = 0.03207\ldots,$$

and the number of spheres each touches is 270. These figures are substantially inferior to those for the densest known section of J_{12}, namely

$$2^{-4}3^{-\frac{1}{2}} = 0.03608\ldots$$

and 336 respectively.

2.43. A proof of the statement (§2.4) that the packing of §1.2 in [16] is a section of that of §2.3 in [24] (and hence of that of §2.31 above) is given here, as this result is not obvious and is of some combinatorial interest. The proof is based on the observation that among linear combinations modulo 2 of rows of the matrix **C** (§2.3) are the 759 octads of the Steiner system $S(5, 8, 24)$, and vice versa, since the last eleven rows of **C** are octads and the first row is the sum modulo 2 of any other row and an octad. (Octads, and tetrads below, are here identified with the rows of 24 binary digits with 1's in the selected positions and 0's elsewhere.)

Let us choose any four of the coordinates. Then this choice induces a partition of the remaining 20 coordinates into five tetrads,* any one of which forms an octad with the chosen four coordinates. Since no two octads have five coordinates in common, these tetrads are disjoint. Further, because of the method of generation by addition modulo 2 of rows of **C**, we see that any two of these tetrads form an octad, so that our chosen set of four coordinates is not distinguished; we have a set of six tetrads, determined by any one of them, such that any two of them together form an octad.

Now suppose for convenience that the columns of **C** are rearranged so that the first eight coordinates form an octad. If we now choose any four of these first eight coordinates as a tetrad, then the remaining four also form a tetrad, and this partition into two tetrads induces a partition of the last 16 coordinates into four tetrads. (We notice in passing that the 140 tetrads so formed from the last 16 coordinates by the 35 partitions of the first eight coordinates into two tetrads form a Steiner system $S(3, 4, 16)$ (cf. §1.5), but we do not use this property.) Each set of four tetrads may be grouped into two octads in three ways, and we have now to determine how many different pairs of octads are

*Although rows of 24 binary digits which are linear combinations of rows of **C** are somewhat analogous to 3-parity rows, since they differ from one another in at least eight places, there is no analogue of 2-parity, and tetrads are here arbitrary sets of four coordinates; octads, however, are Steiner sets.

so formed. Any octad of one such pair has exactly four coordinates in common with each octad of any other pair, so any two pairs of octads divide each other into four tetrads, corresponding to a partition of the first eight coordinates into two tetrads. There are 35 such partitions, and each of these is determined by any two of the three pairs of octads which may be formed from the four tetrads in the last 16 coordinates. Thus two divisions into pairs of octads may be chosen in 105 ways, which implies that there are 15 pairs of octads from which to choose.

We now define a matrix $\mathbf{A}_{16}$ with its first row all 0's and each of its other 15 rows having eight 0's and eight 1's, the 0's and 1's of each row being arranged according to each of the 15 divisions of the 16 coordinates above into two octads, the 0's being in that octad which includes the first element of the row. Then this matrix has the same properties as the matrices of the same name in §1.2 and Note 2 of **(4)**, since the sum modulo 2 of any two rows is another row. Since the packings of §1.2 in [16] and of §2.3 in [24] are defined by congruence modulo 2 to sums of rows of the matrices $\mathbf{A}_{16}$, $\bar{\mathbf{A}}_{16}$, and the matrix $\mathbf{C}$, respectively, with the sum of the coordinates divisible by 4, we see that the former packing is a section of the latter.

3. Bounds and limits

3.11. Mr. G. R. Lang has kindly sent me a copy of tables of the Schläfli functions $f_n(x)$, which he has computed for $n \leqslant 32$. From these I have extended the range of n for which I have calculated Rogers' bound for the density of packings and Coxeter's bound for the number of spheres which may touch any one. These figures for $n \leqslant 24$ are given in Table I, including for convenience the values already given in **(4)** (correcting an end-figure error in the line for $n = 11$), so this table may be regarded as superseding those of **(4)**. Note that the figures for the densest known packing and the greatest numbers of spheres touched in [11] do not refer to the same packing (see §2.42 above), and that the new packings of §§2.31, 2.41 are included. As in the tables of **(4)**, decimal quantities are truncated; the last digit given has not been increased where the amount truncated exceeds half a unit of this digit. In particular, the bounds for the numbers of spheres that may touch any one are given to the integer below, which is thus the greatest number not known to be impossible (if Coxeter's bound can be proved valid), except in [3] where it is known to be impossible for a sphere to touch 13 others equal to it. The ratios, however, were evaluated using the untruncated values, so they do not match these integers exactly.

3.12. The last two paragraphs of §3.1 need revision to take account of the new packings of §§2.31, 2.41. The regular pattern in the densities remarked on there is now seen to extend to 24 dimensions. As stated, further extension is incompatible with Rogers' results on the maximum possible densities for 28 and more dimensions, so it cannot be expected beyond 24 dimensions. Figures

TABLE I

n	Centre density			Number of spheres touched		
	Bound	Best achieved	Ratio	Bound	Best achieved	Ratio
1	0.5	0.5	1.0	2	2	1.0
2	0.28867	0.28867	1.0	6	6	1.0
3	0.18612	0.17677	0.9497	13	12	0.8957
4	0.13127	0.125	0.9521	26	24	0.9077
5	0.09987	0.08838	0.8850	48	40	0.8213
6	0.08112	0.07216	0.8895	85	72	0.8390
7	0.06981	0.0625	0.8952	146	126	0.8596
8	0.06326	0.0625	0.9878	244	240	0.9811
9	0.06007	0.04419	0.7356	401	272	0.6782
10	0.05953	0.03608	0.6060	648	336	0.5184
11	0.06136	0.03207	0.5226	1,035	440	0.4249
12	0.06559	0.03703	0.5646	1,637	756	0.4615
13	0.07253	0.03125	0.4308	2,569	906	0.3525
14	0.08278	0.03608	0.4358	4,003	1,422	0.3552
15	0.09735	0.04419	0.4539	6,198	2,340	0.3775
16	0.11774	0.0625	0.5308	9,544	4,320	0.4526
17	0.14624	0.0625	0.4273	14,628	5,346	0.3654
18	0.18629	0.07216	0.3873	22,324	7,398	0.3313
19	0.24308	0.08838	0.3636	33,940	10,668	0.3143
20	0.32454	0.125	0.3851	51,421	17,400	0.3383
21	0.44289	0.17677	0.3991	77,664	27,720	0.3569
22	0.61722	0.28867	0.4676	116,965	49,896	0.4265
23	0.87767	0.5	0.5696	175,696	93,150	0.5301
24	1.27241	1.0	0.7859	263,285	196,560	0.7465

1 and 2 of **(4)** may be revised by raising the ringed points for [22] by 0.035 in., for [23] by 0.083 in., and for [24] by 0.165 in., the amounts being almost identical for the two figures. It will be seen from these revised figures, or from Table I of this paper, that the packing in [24] is closer to Rogers' and Coxeter's bounds than any other known packing in more than eight dimensions, and one can hardly doubt that it is the densest possible packing in [24].

3.21. The density of the packings in $[n]$, for $n = 2^m$, as $m \to \infty$ (§3.2), is of a smaller order of magnitude than the lower bound $cn/2^n$ obtainable from refinements of the Minkowski–Hlawka theorem **(5)**. Thus these packings are certainly not the densest possible for large n; nor are those given or conjectured in §1.63, which are proportionately only slightly better. However, I cannot trace any denser packings that have been given explicitly; their existence is inferred by averaging.

3.22. I take this opportunity of correcting a printing mishap in **(4)** near the

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foot of p. 677. The last sentence should begin: "Of these we accept a proportion

$$\exp_2 - \left(\binom{m}{3} + \binom{m}{4} + \ldots + \binom{m}{m}\right)$$

because . . ."; the formula is missing in the printed text.

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