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THE INTEGRATION OF EXACT PEANO DERIVATIVES

BY

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ABSTRACT. It is well known that the Riemann-complete integral (or equivalently the Perron integral) integrates an everywhere finite ordinary first derivative (which may be thought of as a Peano derivative of order one). It is also known that the Cesàro–Perron integral of order $(n - 1)$ integrates an everywhere finite Peano derivative of order n. The present work concerns itself with necessary and sufficient conditions for the Riemann-complete integrability of an exact Peano derivative of order n. It is shown that when the integral exists, it can be expressed as the 'Henstock' limit of the sum of a particular kind of interval function. All functions considered will be real valued.

1. **The Peano derivative**. Let $f(x)$ be a continuous function defined on $[a, b]$. If there exist (finite) constants $f_1(x_0), f_2(x_0), \ldots, f_n(x_0)$, such that $P_n(f, x_0 + h, x_0) \equiv f(x_0 + h) - f(x_0) - hf_1(x_0) - \ldots - (h^n)/(n!) f_n(x_0) = o(h^n)$, as $h \to 0$, then $f_n(x_0)$ is called the nth Peano derivative of f at x_0. If f has an nth Peano derivative $f_n(x_0)$ at x_0, then f has also a kth Peano derivative at x_0, $f_k(x_0)$, $k = 1, 2, \ldots, n - 1$, where $f_1(x_0) = f'(x_0)$. If the ordinary nth derivative of f exists at x_0, then so does the nth Peano derivative and $f^n(x_0) = f_n(x_0)$. That the converse does not hold is demonstrated by the following example: $f(x) = x^3 \sin x^{-1}$ for $x \neq 0$ and $f(0) = 0$. It is easy to see that $f_2(0) = 0$, while $f''(0)$ does not exist.

If $f_n(x_0)$ exists for every $x_0 \in [a, b]$, f is said to have an exact Peano derivative f_n [7].

2. **The Riemann complete integral**. A tagged division of $[a, b]$ is a set of non-overlapping subintervals $[x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$, where $a = x_0 < x_1 < \ldots < x_n = b$, together with a tag z_k in each interval $[x_{k-1}, x_k]$. A tagged division is said to be *compatible* with a function $\delta(x) > 0$, defined on $[a, b]$, if $x_k - x_{k-1} < \delta(z_k)$, $k = 1, 2, \ldots, n$. It is shown in [5] (page 83) that if $\delta(z)$ is an arbitrary positive function in $[a, b]$ there is a division of $[a, b]$ compatible with $\delta(z)$.

Suppose $f(x)$ is defined on $[a, b]$. If there exists a number I and a function $\delta(x, \epsilon) > 0$, defined for $x \in [a, b]$ and $\epsilon > 0$, such that $| I - \Sigma f(z_j)(x_j - x_{j-1}) | < \epsilon$ for all sums over tagged divisions compatible with $\delta(x, \epsilon)$, then f is Riemann-complete integrable (RC-integrable) on $[a, b]$ with integral I ([4] and [5]).

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It is easy to see that if $f'(x)$ exists (finitely) everywhere in $[a, b]$, then

$$\text{RC} \int_a^b f'(x)\mathrm{d}x = f(b) - f(a).$$

It is known that the RC-integral properly includes the Lebesgue integral ([3] and [5]) and that the RC-integral is equivalent to the Perron and the special Denjoy integrals ([4], pp. 124-126, and [8]).

3. **The CP-scale of integration**. J. C. Burkill [2] constructed a scale of (Cesàro–Perron) integrals, the C_kP-integrals, $k = 0, 1, 2, \ldots, n, \ldots$, in which the C_0P-integral is the Perron integral (or by the above, the RC-integral):

Suppose that for $0 \leq k \leq n - 1$ the C_kP-integral has been defined. A function $M(x)$ defined on $[a, b]$ is said to be C_n-continuous on $[a, b]$ if it is C_{n-1}P-integrable over $[a, b]$ and if

$$C_n(M, x, x + h) \equiv \left(\frac{n}{h^n}\right) C_{n-1}\text{P} \int_x^{x+h} (x + h - t)^{n-1} M(t)\, \mathrm{d}t \rightarrow M(x),$$

as $h \rightarrow 0$, for every x in $[a, b]$. Let

$$C_n\bar{D}M(x) \equiv \overline{\lim_{h\rightarrow 0}} \left(\frac{C_n(M, x, x + h) - M(x)}{h/n + 1}\right)$$

and define $C_n\underline{D}M(x)$ in the obvious way. If $C_n\bar{D}M(x) = C_n\underline{D}M(x)$, then the common value is taken to be the C_n-derivative, $C_nDM(x)$.

The functions $M(x)$ and $m(x)$ are called C_nP-major and minor functions, respectively, of $f(x)$ over $[a, b]$ if

(3.1) $M(x)$ and $m(x)$ are C_n-continuous in $[a, b]$;
(3.2) $M(a) = m(a) = 0$;
(3.3) $C_n\underline{D}M(x) \geq f(x) \geq C_n\bar{D}m(x), \quad x \in [a, b]$;
(3.4) $C_n\underline{D}M(x) \neq -\infty$, $C_n\bar{D}m(x) \neq +\infty$.

If for every $\epsilon > 0$, there is a pair $M(x)$, $m(x)$ satisfying the conditions (3.1), (3.2), (3.3) and (3.4) above and such that $|M(b) - m(b)| < \epsilon$, then $f(x)$ is said to be C_nP-integrable in $[a, b]$ and $C_n\text{P} \int_a^b f(t)\, \mathrm{d}t = \inf M(b) = \sup m(b)$, where the inf and sup are taken over all major and minor functions, respectively.

It follows easily that the C_nP-integral integrates an everywhere finite C_n-derivative, and $g(b) - g(a) = C_n\text{P} \int_a^b C_nDg(x)\, \mathrm{d}x$. Moreover, if $f(x)$ is C_{n-1}P-integrable on $[a, b]$, then $f(x)$ is C_nP-integrable on $[a, b]$, and the integrals agree. In particular, if f is RC-integrable, then it is C_kP-integrable for $k = 1, 2, 3, \ldots$, and the integrals all have the same value.

THEOREM 3.1. (cf. Lemma 6.2 [1]) *If f has an exact Peano derivative f_n in $[a, b]$, then* $C_{n-1}Df_{n-1}(x) = f_n(x)$, *for each* $x \in [a, b]$.

PROOF. Since $f_1(x) = f'(x)$ is finite in $[a, b]$, it is Riemann complete or C_0P-integrable over $[a, b]$, and

$$C_1Df_1(x) = \lim_{h\to 0} (2/h)\left\{\frac{1}{h}\int_x^{x+h} f_1(t)\,dt - f_1(x)\right\}$$
$$= \lim_{h\to 0} (2/h^2)(f(x+h) - f(x) - hf_1(x)) = f_2(x).$$

This shows that $f_2(x)$ is an exact C_1D-derivative and therefore is C_1P-integrable. Similarly,

$$C_2Df_2(x) = \lim_{h\to 0}\left(\frac{3}{h}\right)(C_2(f_2, x, x+h) - f_2(x))$$
$$= \lim_{h\to 0}\frac{3}{h}\left(\frac{2}{h^2}\right) C_1\mathrm{P}\int_x^{x+h}(x + h - t) f_2(t)\,dt - f_2(x)\Big)$$
$$= \lim \frac{6}{h^3}\left[-hf_1(x) + f(x+h) - f(x) - \frac{h^2}{2}f_2(x)\right] = f_3(x)$$

and, after a finite number of steps, the proof is complete.

COROLLARY. *If f has an exact Peano derivative $f_n(x)$, then $C_{n-1}\mathrm{P}\int_a^b f_n(t)\,dt = f_{n-1}(b) - f_{n-1}(a)$.*

4. **Variational equivalence of step functions**

DEFINITION 4.1. The pair of finite interval functions $h^* = (h_\ell^*, h_r^*)$ is variationally equivalent to the pair of finite interval functions $h = (h_\ell, h_r)$ if there exists a $\delta(x) > 0$ and a positive finitely superadditive interval function χ_1 such that $\chi_1[a, b) < \epsilon$ and

$$(4.1)\qquad \begin{aligned} |h_\ell^*(t,x) - h_\ell(t,x)| &\le \chi_1(t,x), & x - \delta(x)) \le t < x,\\ |h_r^*(x,u) - h_r(x,u)| &\le \chi_1(x,u), & x < u < x + \delta(x), \end{aligned}$$

where $[x, u)$ and $[t, x)$ are contained in $[a, b]$. (See [4], p. 39.)

We now introduce several pairs of interval functions that play a crucial role in the statement and proof of the main result of this paper. We shall write $h(n, u, v) = \{h_\ell(n, u, v), h_r(n, u, v)\}$ where, for $u < v$

$$h_\ell(n, u, v) = \frac{(n)!\,P_{n-1}(f, u, v)}{(-1)^n (v - u)^{n-1}},$$

$$h_r(n, u, v) = \frac{(n)!\,P_{n-1}(f, v, u)}{(v - u)^{n-1}},$$

$$I(n,u,v) = I_\ell(n,u,v) = I_r(n,u,v) = f_{n-1}(v) - f_{n-1}(u),$$

$$F(n,u,v) = \{F_\ell(n,u,v),\ F_r(n,u,v),$$

where

$$F_\ell(n,u,v) = f_n(v)\,(v-u)$$

and

$$F_r(n,u,v) = f_n(u)\,(v-u)\}.$$

It is easy to see that if f has an exact Peano derivative f_n, then $h(n,u,v)$ is variationally equivalent to $F(n,u,v)$ on $[a,b]$. Indeed corresponding to $\epsilon > 0$ their exists $\delta_1(x) > 0$ such that

$$\text{(4.2)}\qquad \begin{aligned} &|\, h_\ell(n,t,x) - f_n(x)\,(x-t)\,| \le \epsilon(x-t) \equiv \chi_2(t,x), \\ &\qquad x - \delta_1(x) \le t < x \\ &|\, h_r(n,x,u) - f_n(x)\,(u-x)\,| \le \epsilon(u-x) \equiv \chi_2(x,u), \\ &\qquad x < u \le x + \delta_1(x) \end{aligned}$$

for each $[x,u)$ and $[t,x)$ contained in $[a,b]$.

5. **Integrability of Peano derivatives**. As we have seen, the exact Peano derivative f_1 is RC-integrable. In the next two sections we consider the question of the RC-integrability of $f_n(x)$.

It is clear that if the exact Peano derivative $f_n(x)$, for some fixed n, is RC-integrable, then

$$\text{(5.1)}\qquad \mathrm{RC}\int_a^b f_n(t)\,dt = C_{n-1}\,\mathrm{P}\int_a^b f_n(t)\,dt = f_{n-1}(b) - f_{n-1}(a).$$

Moreover if the relationship $f_n(x) = (f_{n-1}(x))'$ holds in $[a,b]$, then (5.1) holds. Since this is the case if $f_n(x)$ is bounded above or below in $[a,b]$ [7], we can state the following:

THEOREM 5.1. *If f has an exact Peano derivative f_n which is bounded above or below in* $[a,b]$, then $f_n(x)$ *is RC-integrable and* (5.1) *holds.*

On the other hand if (6.1) is valid, then

$$\text{(5.2)}\qquad \mathrm{RC}\int_a^x f_n(t)\,dt = f_{n-1}(x) - f_{n-1}(a), \qquad x \in [a,b]$$

and therefore $f_{n-1}(x)$ is continuous on $[a,b]$. It follows that $f^{(n-1)}(x)$ exists and equals $f_{n-1}(x)$ in $[a,b]$ and also that $f_k(x)$ is RC-integrable for $0 \le k \le n-1$.

It is now easy to construct an example of an exact Peano derivative which is not RC-integrable. Consider the function

$$F(x) = x^3 \sin x^{-4}, \qquad x \neq 0,$$
$$F(0) = 0.$$

Then

$$F'(x) = -4x^{-2} \cos x^{-4} + 3x^2 \sin x^{-4}, \qquad x \neq 0,$$
$$F'(0) = 0,$$

and

$$F_2(x) = F''(x), \qquad x \neq 0$$
$$F_2(0) = 0.$$

If $F_2(x)$ were RC-integrable on $[-1, 1]$, then by the above, $F'(x)$ would be continuous in $[-1, 1]$, but $F'(x)$ is unbounded in every neighbourhood of the origin.

6. **Main result**.

THEOREM 6.1. *If f has an exact Peano derivative f_n on $[a, b]$, then f_n is RC-integrable on $[a, b]$ if and only if $I(n, u, v)$ is variationally equivalent to $h(n, u, v)$.*

PROOF. We first assume that $I(n, u, v)$ is variationally equivalent to $h(n, u, v)$. In that case given $\epsilon > 0$ there exist $\delta_2(x) > 0$ and a finite positive superadditive interval function χ_1 such that $\chi_1[a, b] < \epsilon$ and

$$(6.1) \qquad \begin{aligned} &| h_\ell(n, t, x) - I(n, t, x) | \leq \chi_1(t, x), && x - \delta_2(x) \leq t < x, \\ &| h_r(n, x, u) - I(n, x, u) | \leq \chi_1(x, u), && x < u \leq x + \delta_2(x). \end{aligned}$$

It is clear that there exists $\delta(x)$ such that (4.2) and (6.1) both hold with $\delta_1(x) = \delta_2(x) = \delta(x)$.

Consider any tagged division $\mathfrak{D}$ of $[a, b]$ compatible with $\delta(x)$ and sums

$$\begin{aligned} S &= (\mathfrak{D}) \Sigma f_n(z_k)(x_k - x_{k-1}) = (\mathfrak{D}) \Sigma [f_n(z_k)(x_k - z_k) + \Sigma f_n(z_k)(z_k - x_{k-1})] \\ &= (\mathfrak{D}) \Sigma F_r(n, z_k, x_k) + (\mathfrak{D}) \Sigma F_\ell(n, x_{k-1}, z_k) \\ &= S_r + S_\ell. \end{aligned}$$

Then

$$\begin{aligned} | I(n, a, b) - S | &= | \Sigma I(n, z_k, x_k) - S_r + \Sigma I(n, x_{k-1}, z_k) - S_\ell | \\ &\leq \Sigma | I(n, z_k, x_k) - h_r(n, z_k, x_k) | + \Sigma | h_r(n, z_k, x_k) - F_r(n, z_k, x_k) | \\ &\quad + \Sigma | I(n, x_{k-1}, z_k) - h_\ell(n, x_{k-1}, z_k) | + \Sigma | h_\ell(n, x_{k-1}, z_k) - F_\ell(n, x_{k-1}, z_k) | \\ &\leq \chi_1[a, b] + \chi_2[a, b] < \epsilon + \epsilon(b - a). \end{aligned}$$

This proves that $f_n(x)$ is RC-integrable over $[a, b]$.

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Conversely if $f_n(x)$ is RC-integrable, we have

$$|I(n,u,v) - h_s(n,u,v)|$$
$$\leq |I(n,u,v) - F_s(n,u,v)(v-u)| + |F_s(n,u,v)(v-u) - h_s(n,u,v))|.$$

We have seen before that $h(n,u,v)$ is variationally equivalent to $F(n,u,v)$ and it follows from [4] (pages 40–41) that $I(n,u,v)$ is variationally equivalent to $F(n,u,v)$. Thus $I(n,u,v)$ is variationally equivalent to $h(n,u,v)$.

7. **Concluding remarks**.

(1) The result of the preceding section shows that when the Riemann-complete integral of $f_n(x)$ exists, we can write RC $\int_a^b f_n(t)\,dt = \lim \Sigma h_s(n,u,v)$ where the expression $h_s(n,u,v)$ involves derivatives up to order $(n-1)$ and the limit is with respect to divisions in the Henstock sense. In the absence of any knowledge of the existence of $f_n(x)$ the limit on the right hand side may still exist. Moreover if $I(n,u,v)$ is variationally equivalent to $h(n,u,v)$, then in Henstock's terminology ([4], p. 39), I is the variational integral of h.

(2) In the case $n = 2$, the condition that I is variationally equivalent to h takes a form that is interesting in its own right. It is easy to see that

$$I_r(2,x,u) - h_r(2,x,u) = f_1(u) - f_1(x) - \left[\frac{f(u) - f(x) - (u-x)f_1(x)}{(u-x)/2}\right]$$
$$= 2\left[\left(\frac{f_1(u) + f_1(x)}{2}\right) - \left(\frac{f(u) - f(x)}{u-x}\right)\right],$$

and

$$I_\ell(2,t,x) - h_\ell(2,t,x) = f_1(x) - f_1(t) - \left[\frac{f(t) - f(x) - (t-x)f_1(x)}{-(x-t)/2}\right]$$
$$= -2\left[\left(\frac{f_1(x) + f_1(t)}{2}\right) - \left(\frac{f(x) - f(t)}{x-t}\right)\right].$$

By the Darboux property of the derivative and the mean value theorem, the differences above are of the form $f_1(\alpha_s) - f_1(\beta_s)$, $s = r, \ell$, where α_s and β_s are points in the appropriate interval.

To say that I is variationally equivalent to h on $[a,b]$ therefore implies that given $\epsilon > 0$ there exists $\delta(x) > 0$ such that for all tagged divisions $\mathfrak{D}$ of $[a,b]$ compatible with $\delta(x)$ we have $(\mathfrak{D})\ |\Sigma f_1(\alpha_s) - f_1(\beta_s)| < \epsilon$.

(3) It is easy to see, of course, that if $f^{(n)}(x)$ exists everywhere, then $I(n,u,v)$ is variationally equivalent to $h(n,u,v)$. This follows since we may write (for n odd or even):

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$$\frac{I_r(n, x, u) - h_r(n,x,u)}{u - x} = \left(\frac{f^{(n-1)}(u) - f^{(n-1)}(x)}{u - x}\right) - \left(\frac{f(u) - f(x) - (u-x)f'(x) - \ldots - (u-x)^{n-1}f^{(n-1)}(x)}{(u-x)^n/n!}\right)$$
$$\to f^{(n)}(x) - f^{(n)}(x) = 0, \qquad \text{as } u \to x+,$$

and

$$\frac{I_\ell(n,t,x) - h_\ell(n,t,x)}{x - t} = \left(\frac{f^{(n-1)}(x) - f^{(n-1)}(t)}{x - t}\right) - \left(\frac{f(t) - f(x) - (t-x)f'(x) - \ldots - (t-x)^{n-1}f^{(n-1)}(x)}{(x-t)^n/n!}\right)$$
$$\to f^{(n)}(x) - f^{(n)}(x) = 0, \qquad \text{as } t \to x-.$$

(4) If f_n is RC integrable on $[a,b]$, then

$$f^{(n-1)}(x) - f_{(n-1)}(a) = \text{RC} \int_a^x f_n(t)\,dt, \; x \in [a,b].$$

It then follows from the property of the integral that $(f^{(n-1)}(x))' = f^{(n)}(x) = f_n(x)$, a.e.

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