Can. J. Math., Vol. XXIV, No. 5, 1972, pp. 989–992

THE CONTINUITY OF THE VISIBILITY FUNCTION ON A STARSHAPED SET

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1. Introduction.

Definition. The visibility function assigns to each point x of a fixed measurable set E in a Euclidean space E_n the Lebesgue outer measure of $S(x)$, the set $\{y : rx + (1 - r)y \in E$ for every r in $[0, 1]\}$.

The purpose of this paper is to determine sufficient conditions for the continuity of the function on the interor of a starshaped set.

2. Preliminaries. We basically use the same terminology as in [**1**], where the reader may find a more general investigation of the continuity properties of the visibility function. Lebesgue measure in E_n is denoted by m or m_n (if more than 1 measure is under discussion). The convex kernel of E, $\{x \in E : S(x) = E\}$ is expressed as conv ker E, and the convex hull of E is denoted by conv E. The open r-ball about a point x is given by $B_r(x)$. The interior of E relative to the smallest flat containing E is given by intv E. Finally, xy will denote the line segment joining x to y, $L(x, y)$ will denote the line determined by x and y, and $\langle W \rangle$ will denote the flat generated by the set of vectors W.

In the sequel, we must draw upon 3 facts established in [**1**], which we state as theorems. As in [**1**], we will designate the visibility function for a fixed set by v.

THEOREM 1. *If $O \subset E_n$ is open, then v is lower semicontinuous on O.*

THEOREM 2. *If $K \subset E_n$ is compact, then v is upper semicontinuous on K.*

THEOREM 3. *Let E be a compact set in E_n. If $x \in E$, the set of endpoints of all maximal segments in $S(x)$ with one endpoint x forms a measurable set and has measure zero.*

It is easy to see that the visibility function may be discontinuous on the interior of a compact starshaped set in E_n, if the dimension of the convex kernel does not exceed $n - 2$. For example, let K be a Cantor set of positive measure in $[0, 2\pi]$ and let E be the following planar starshaped set: $\{(r, \theta) : r \leqq 1\} \cup \{(r, \theta) : 1 < r \leqq 2, \theta \in K\}$. Let $q \in E \cap \{(r, \theta) : r \leqq 1\}$. Since $E \cap \{(r, \theta) : 1 < r \leqq 2\}$ is nowhere dense, and E is starshaped with respect to 0, $S(q) \cap E \cap \{(r, \theta) : 1 < r \leqq 2\} \subset L(q, 0)$, so that the visibility function for E is discontinuous at the origin. Using "Cantor cylinders", we may construct analogous examples in E_n for any n.

Received October 7, 1971 and in revised form, May 9, 1972.

3. Results. In establishing our main theorem, we use induction and a basic property of generalized cylindrical coordinates. Specifically given any flat F of dimension $n - 2$ in E_n, we can find a collection of hyperplanes $\{H_\theta\}$, $\theta \in [0, \pi)$, such that $F = H_{\theta_1} \cap H_{\theta_2} (\theta_1 \neq \theta_2)$, $\bigcup H_\theta = E_n$, and if K is an arbitrary Borel set satisfying $m_{n-1}(H_\theta \cap K) = 0$ for almost every θ, then $m(K) = 0$.

THEOREM 4. *Let E be a compact starshaped set in E_n such that* int $E \neq \emptyset$. *Suppose* dim conv ker $E \geqq n - 1$. *Then the visibility function v is continuous on* int E.

Proof. We first establish our theorem in the case $n = 2$. Let x be an arbitrary point of int E different from some point in conv ker E and let $\{x_n\} \to x$. The lower semicontinuity of v at x follows if we can show $S = \{y : y \in S(x), y \notin \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S(x_n)\}$ has measure zero. If we denote the set of points which x sees via E but not via int E by M, then clearly $S \subset M$. First, it can be seen that any ray with endpoint x intersects M in one point or an interval. Excluding the ray on the one possible line which might contain all the points of conv ker E, if such a ray R contains an interval in M, we associate a rational point in E_2 with it. Fix $p \in$ conv ker E, $p \neq x$. $L(p, x)$ divides the plane into two open half planes, H_1 and H_2. Suppose without loss of generality $R \subset H_1$ and $y_1y_2 \subset M \cap R$. Clearly there exists a point z such that $z \in$ intv y_1y_2 and pz passes through a point r_R in H_1 with rational coordinates. We claim the assignment $R \to r_R$ is 1-1. Suppose there were another ray R' such that $R' \subset H_1$ and R' were also assigned r_R. Then there exists z' on $R' \cap M$ such that r_R is in intv pz' and we may harmlessly suppose $z \in$ intv $z'p$. Since conv$(p \cup z' \cup x) \subset E$, it follows that $z \notin M$, a contradiction.

The remaining points of M not contained in these intervals must be endpoints of maximal segments in $S(x)$ with one endpoint x. But these points have measure zero by Theorem 3. Hence, $m(S(x)) \leqq m(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S(x_n)) \leqq \liminf m(S(x_k))$ and the lower semicontinuity of v at x follows in the case $n = 2$.

For general n, we must distinguish two cases for an arbitrary point $x \in$ int E:

(1) there exists n independent points $\{y_1, \ldots, y_n\} \subset$ conv ker E such that $x \notin \langle y_1, \ldots, y_n \rangle$, and

(2) there exists n independent points $\{y_1, y_2, \ldots, y_n\} \subset$ conv ker E such that $x \in \langle y_1, \ldots, y_n \rangle$.

If dim conv ker $E = n$, both conditions are satisfied for every x, and if dim conv ker $E = n - 1$, then exactly one is satisfied by each x in E. (See Valentine [**3**] for a thorough discussion of flats, convex kernels and convex hulls).

In case (1) we first establish by induction that if p is any point in intv conv$(\{y_1, \ldots, y_n\})$ where $\{y_1, \ldots, y_n\}$ are as above, then if $\{x_n\} \to x$ on $L(x, p)$ we have $m(S(x)/\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S(x_n)) = 0$. This, of course, has been shown when $n = 2$. Assume it is true if $n = k$, and now suppose $n = k + 1$. Let $\{y_1, \ldots, y_{k+1}\}$ be independent points in conv ker E satisfying $x \notin \langle y_1, \ldots, y_{k+1} \rangle$. Let $p \in$ intv conv$(\{y_1, \ldots, y_{k+1}\})$. Clearly there exists a set of hyperplanes

$\{H_\theta\}$, $\theta \in [0, \pi)$, as in the previous discussion such that $L(x, p) \subset H_\theta$ for every θ, dim conv ker $(H_\theta \cap E) \geqq k - 1$, $\bigcup H_\theta = E_{k+1}$ and for every θ, there exists independent points $\{y_1{}^\theta, \ldots, y_k{}^\theta\}$ contained in conv ker$(H_\theta \cap E)$ such that $p \in$ intv conv$(\{y_1{}^\theta, \ldots, y_k{}^\theta\})$ and $x \notin \langle y_1{}^\theta, \ldots, y_k{}^\theta \rangle$.

Now let $\{x_n\}$ be an arbitrary sequence of points on $L(x, p)$ converging to x. By the induction hypothesis we have $m_k(S(x) \cap H_\theta / \bigcup_{k=1}^\infty \bigcap_{n=k}^\infty S(x_n) \cap H_\theta) = 0$. Hence, by our previous remarks we have $m_{k+1}(S(x)/\bigcup_{k=1}^\infty \bigcap_{n=k}^\infty S(x_n)) = 0$. Hence our proposition is true in E_n for every n.

This all of course implies that given any point $p \in$ intv conv$(\{y_1, \ldots, y_n\})$, v is continuous on $L(x, p)$. Therefore there exists a point x_0 in E such that $x \in$ intv px_0 and $v(x_0) > v(x) - \epsilon$. Since p was chosen in intv conv$(\{y_1, \ldots, y_n\})$, it follows that conv$\{x_0, y_1, \ldots, y_n\}$ will contain a neighborhood N of x and since $y_i \in$ conv ker E, $i = 1, 2, \ldots, n$, we conclude that $v(y) > v(x) - \epsilon$ for every $y \in N$ so that v is lower semicontinuous at x.

In case (2) we establish by induction that the set M of points which x sees via E but not via int E has measure zero, which is enough to establish the continuity of the visibility function at such points as we have noted before. We have seen this to be true when x is any interior point of E if $n = 2$. Now suppose the assertion has been established for $n = k$. If $n = k + 1$ we again rotate a hyperplane to sweep out $k + 1$ space such that at each stage H_θ, $\theta \in [0, \pi)$, $E \cap H_\theta$ satisfies the induction hypothesis. Let H denote a hyperplane containing x and a subset of conv ker(E) of dimension k. There exists a flat $F \subset H$, dim $F = k - 1$, such that $x \in F$ and dim conv ker$(F \cap E) = k - 1$. Let $\{H_\theta\}$, $\theta \in [0, \pi)$, denote the set of hyperplanes generated by rotating $H = H_0$ about F. Then for all $\theta \in [0, \pi)$, we have dim conv ker$(H_\theta \cap E) \geqq k - 1$, and x is located on a hyperplane in H_θ (namely F) for every θ containing a subset of conv ker$(H_\theta \cap E)$ of dimension $k - 1$. By the induction hypothesis the set $M_\theta =$ those points of $E \cap H_\theta$ which x sees via $E \cap H_\theta$ but not via intv $E \cap H_\theta$ has k dimensional measure zero. By our earlier remarks, $\bigcup_{\theta \in [0,\pi)} M_\theta$ has $k + 1$ dimensional measure zero.

We claim that $M/M \cap H_0 \subset \bigcup_{\theta \in [0,\pi)} M_\theta$. Suppose z is an interior point of $E \cap H_\theta/F$ relative to H_θ where $\theta \neq 0$. Let N be an H_θ neighborhood of z contained in $E \cap H_\theta/F$. Then $\dim(N \cup \text{conv ker } E) = \text{k} + 1$, and $z \in$ int conv$(N \cup$ conv ker $E) \subset$ int E. Hence boundary points of E on H_θ, $\theta \neq 0$, are boundary points of $E \cap H_\theta$ relative to H_θ. Thus, $M/M \cap H_\theta \subset \bigcup M_\theta$ so that $m(M) = 0$, and the continuity of v at such points x follows.

Some observations are now in order. Clearly, the converse of Theorem 4 fails. If E is a compact starshaped set in E_n the dimension of whose convex kernel exceeds $n - 2$, then the boundary of E has measure zero. Thus, the reader might guess that Theorem 4 is a special case of the following more general theorem: if $E \subset E_n$ is a compact set whose boundary is of measure zero, then the visibility function is continuous on int E. However, the above proposition is false. In the standard Cantor set of measure π in $[0, 2\pi]$ derived by tossing out a sequence of

https://doi.org/10.4153/CJM-1972-100-7 Published online by Cambridge University Press

open sets $\{O_n\}$ from $[0, 2\pi]$ in the usual way, let our residual closed set after the nth deletion be called K_n. Letting

$$r_n = 3 \sup_{x \in K_n} \inf_{y \in [0, 2\pi]/K_n} |y - x|,$$

a counterexample is seen to be

$$\{(r, \theta) : r \leqq 2\} / \bigcup_{n=1}^{\infty} \{(r, \theta) : \theta \in O_n, 1 < r < 1 + \sqrt{r_n}\}.$$

For details, see [**2**].

For the case when E is a bounded open starshaped set, we are only able to give the following planar result.

THEOREM 5. *Let O be any bounded open starshaped set in the plane. Then the visibility function is continuous on O.*

Proof. Let x be an arbitrary element of O and let $\{x_n\} \to x$. If $x \in \text{conv ker } O$, $v(x_n) \to v(x)$, so we may assume $x \notin \text{conv ker } O$. We show that $S = \{y : y \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S(x_n)/S(x)\}$ has measure zero. Fix a point p in conv ker O and consider any ray R emanating from x. We claim that $R \cap S$ is either empty or contains a line segment. If the line L determined by R contains p, then $R \cap S = \emptyset$. If not, and $R \cap S \neq \emptyset$, then all but finitely many of the $\{x_n\}$ which see a fixed point y of $R \cap S$ must lie on the p side of L, or else we have $\text{conv}(x \cup y \cup p) \subset O$. Since $\text{int conv}(y \cup p \cup x) \subset O$, there exists an open rectangle in O with one edge xv containing y in its relative interior where $yv \subset O$. It is clear that all but finitely many members of the range of $\{x_n\}$ which could see y can also see yv, and since $yv \cap S(x) = \emptyset$, $R \cap S$ contains yv, an interval.

We now proceed in the same manner as in the compact case: to each ray R containing an interval in S we associate a rational point r_R.

This point corresponds uniquely to R, for suppose that r_R lies on both pw and pw' where $w \in R \cap S$, $w' \in R' \cap S$ and $w \in \text{intv } w'p$, say. Since $\text{int conv}(w' \cup x \cup p) \subset 0$, we have $w \in S(x)$, a contradiction. The upper semicontinuity of v now follows in the obvious way.

In addition to establishing more general results for open sets, the following conjecture is of interest: let E be a compact starshaped set in E_n whose convex kernel is of dimension $n - 2$. If $x \in \text{int } E$ is a point of discontinuity of the visibility function, then x is a point on the smallest flat containing the convex kernel of E.

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