Determinantal Systems of Apolar Triads in a Conic.

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The discovery of the configuration known as a Determinantal System of Points was made by Dr W. P. Milne while investigating the problem of the "Generation of a cubic curve by apolar pencils of lines." In the *Proc. L. M. S.*, Ser. 2, Vol. 15, Part 4, he obtained a complete solution, but in unsymmetrical form. He therefore suggested to me that as he was up to that time unable to find a symmetrical solution for the general cubic I should investigate Determinantal Systems for the case of rational curves. The results of my investigations are given in this and the ensuing paper, and have enabled Dr Milne to solve the general problem in symmetrical form in a paper which will appear in the *Proc. L. M. S.* at an early date. I commence from the result given in the paper by Dr Milne, entitled Determinantal Systems of Points, in the *Proc. E. M. S.*, Vol. XXXIV., Part 2.

1. The present paper discusses the Det. System of nine points when the nine points lie on a conic and the initial triad is apolar to a given triad.

Construction of the system.

$P_1 Q_2 R_3$ is a triad of points on a conic apolar to a given triad ABC.

Let Z and X (real points) be the hessian points of ABC, an imaginary triad on the conic.

Y, T are the poles of ZX, $Q_2 R_3$. $P_1 X$ meets YT in V, and VQ_2 and VR_3 meet the conic again in Q_3 and R_2.

By the isolation of P_1 two new points Q_3 and R_2 appear, which with P_1 form a triad $P_1 Q_3 R_2$. Similarly the isolation of Q_2 gives a triad $Q_2 R_1 P_3$, and the isolation of R_3 gives $R_3 P_2 Q_1$.

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Using Dr Milne's determinantal arrangement

$$\begin{matrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{matrix}$$

we prove that the nine points can be reached from whatever term of the determinant we start.

The choice of the initial five points is unrestricted. Any five will serve, but these being chosen ABC is determined uniquely.

2. *First characteristic.*

If $P_1 Q_2 R_3$ is apolar to ABC then all the other triads which are terms of the determinantal configuration are apolar to ABC.

The polar of the intersection of $Q_2 R_3$ and ZX is YT which is also the polar of the intersection of $Q_2 R_3$ and $Q_3 R_2$.

Therefore ZX, $Q_2 R_3$, $Q_3 R_2$ are concurrent; ....(1)

and $Q_2 R_2$, $Q_3 R_3$, YT are concurrent. ....(2)

Let the equation of the conic referred to XYZ be $y^2 = zx$, and let the parameters of the nine points be

$$\begin{matrix} l_1 & m_1 & n_1 \\ n_2 & l_2 & m_2 \\ m_3 & n_3 & l_3 \end{matrix} \quad \text{corr. to} \quad \begin{matrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{matrix}$$

$[P_1 X Q_2 R_3] = [P_1 X R_2 Q_3]$ By construction,

i.e.
$$\frac{l_1 - l_2}{l_1 - l_3} = \frac{l_1 - n_3}{l_1 - m_2}. \qquad \text{............(3)}$$

By (1) $[ZXQ_2 Q_3] = [ZXR_2 R_3]$

i.e.
$$\frac{l_2}{m_2} = \frac{n_3}{l_3}. \qquad \text{............(4)}$$

From (3) and (4)

$$\frac{l_1 - l_2}{l_1 - l_3} = \frac{l_1 - n_3}{l_1 - m_2} = \frac{-l_2}{m_2} = \frac{-n_3}{l_3}. \qquad \text{............(5)}$$

(5) follows from the isolation of P_1. The isolation of Q_2 and R_3 give (6) and (7)

$$\frac{l_2 - l_3}{l_2 - l_1} = \frac{l_2 - n_1}{l_2 - m_3} = \frac{-l_3}{m_3} = \frac{-n_1}{l_1}. \qquad \text{............(6)}$$

$$\frac{l_3 - l_1}{l_3 - l_2} = \frac{l_3 - n_2}{l_3 - m_1} = \frac{-l_1}{m_1} = \frac{-n_2}{l_2}. \qquad \text{............(7)}$$

Multiply
(5), (6) and (7), and $l_1 l_2 l_3 = m_1 m_2 m_3 = n_1 n_2 n_3 = l_1 m_2 n_3 = \dots$, so that if $P_1 Q_2 R_3$ is apolar to ABC so are all the other triads.

Further, $l_1 + n_2 + m_3 = m_1 + l_2 + n_3 = \text{etc.} = 0$ for $l_1 + n_2 + m_3$

$$= l_1 + l_2 \frac{l_3 - l_1}{l_2 - l_3} + l_3 \frac{l_1 - l_2}{l_2 - l_3} = 0. \qquad \dots\dots\dots\dots(8)$$

We require this equation (8) for the evaluation of the parameters of the hessian points of the six triads.

3. *The system of nine points is a symmetrical or closed system.* It is a matter of indifference from which triad we start.

Suppose we start from $P_2 Q_3 R_1$ which will lead to $P_2 R_3 Q_1$ if

$$\frac{m_1 - m_2}{m_1 - m_3} = \frac{m_1 - l_3}{m_1 - n_2} = \frac{-m_2}{n_2} = \frac{-l_3}{m_3}. \qquad \dots\dots\dots\dots(9)$$

(9) follows readily from (5), (6) and (7), and thus we can begin from $P_2 Q_3 R_1$ or any other triad and reach the same system of nine points as when we begin from $P_1 Q_2 R_3$.

4. *Second characteristic.*

The hessian lines of the six triads and of the triad to which they are apolar are concurrent.

Consider $P_1 Q_2 R_3$ and $P_1 R_2 Q_3$. Let H_1, H_2 be the hessian points of the one, and K_1, K_2 of the other.

$$[H_1 H_2 P_1 Q_2 R_3 X] = [K_1 K_2 P_1 R_2 Q_3 X].$$

Therefore $H_1 K_2$ and $H_2 K_1$ meet on $P_1 X$. Similarly $H_1 K_1$ and $H_2 K_2$ meet on $P_1 Y$, and hence $H_1 H_2$ and $K_1 K_2$ meet on XY.

5. Evaluation of the parameters of the hessian points of the six triads.

Let H_1', H_2' be the hessian points of $P_1 Q_2 R_3$
H_1'', H_2'' „ „ „ $R_1 P_2 Q_3$
H_1''', H_2''' „ „ „ $Q_1 R_2 P_3$

and K_1, K_2 with the requisite marks the hessian points of the other triads.

The parameter of H is h.

$$[H_1' P_1 Q_2 R_3] = \frac{h_1' - l_2}{h_1' - l_3} \cdot \frac{l_1 - l_3}{l_1 - l_2} = -\omega.$$

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Therefore since $\dfrac{l_1 - l_3}{l_1 - l_2} = \dfrac{-l_3}{n_3}$

$$h_1' = \frac{l_2 l_3 - \omega n_3 l_3}{l_3 - \omega n_3}$$

$$[H_1'' R_1 P_2 Q_3] = \frac{h_1'' - m_1}{h_1'' - m_2} \cdot \frac{m_3 - m_2}{m_3 - m_1} = -\omega$$

$$= \frac{h_1'' - m_1}{h_1'' - m_2} \cdot \frac{l_1 - l_3}{l_1 - l_2} \text{ since } \frac{m_3 - m_2}{m_3 - m_1} = \frac{l_1 - l_3}{l_1 - l_2}.$$

Therefore
$$h_1'' = \frac{m_1 l_3 - \omega l_2 l_3}{l_3 - \omega n_3} = \frac{l_3(-l_2 - n_3) - \omega l_2 l_3}{l_3 - \omega n_3}$$

$$= \frac{\omega^2 l_2 l_3 - n_3 l_3}{l_3 - \omega n_3}.$$

Therefore $\qquad h_1'' = \omega^2 h_1'$. So $h_2'' = \omega h_2'$.

The parameters of the hessian points may now be written

$$(h_1, h_2); \; (\omega^2 h_1, \omega h_2); \; (\omega h_1, \omega^2 h_2)$$

with corresponding expressions in k and further $h_1 h_2 = k_1 k_2$.

6. *Derivative Systems.*

Since $\omega h_1 . \omega h_2 = h_1 . \omega^2 h_2 = \omega^2 h_1 . \omega h_2 = \omega k_1 . \omega k_2 = \ldots$, there are other six lines intersecting on ZX. These are the hessian lines of six triads, the parameters of the nine points of which are ωl_1, ωl_2, ωl_3; ωm_1, ωm_2, ωm_3; etc.

There are also six triads with parameters $\omega^2 l_1$, $\omega^2 l_2$, $\omega^2 l_3$; etc.

These two (imaginary) sets of six triads have the two characteristics of the original six.

7. It can be shown that if nine points on a conic are arranged in six triads apolar to a given triad on the conic, each of the nine points occurring in two triads and, the hessian lines of these six triads are concurrent, then the point of concurrence must lie on the hessian line of the given triad to which they are all apolar.

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