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HOMOGENEOUS COMPLEX MANIFOLDS WITH MORE THAN ONE END

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1. Introduction. For homogeneous spaces of a (real) Lie group one of the fundamental results concerning *ends* (in the sense of Freudenthal [**8**]) is due to A. Borel [**6**]. He showed that if $X = G/H$ is the homogeneous space of a connected Lie group G by a closed connected subgroup H, then X has at most two ends. And if X does have two ends, then it is diffeomorphic to the product of **R** with the orbit of a maximal compact subgroup of G.

In the setting of homogeneous complex manifolds the basic idea should be to find conditions which imply that the space has at most two ends and then, when the space has exactly two ends, to display the ends via bundles involving $\mathbf{C}^*$ and compact homogeneous complex manifolds. An analytic condition which ensures that a homogeneous complex manifold X has at most two ends is that X have non-constant holomorphic functions and the structure of such a space with exactly two ends is determined, namely, it fibers over an affine homogeneous cone with its vertex removed with the fiber being compact [**9**], [**13**].

The problem of classifying those homogeneous complex manifolds X with more than one end for which the field of meromorphic functions separates the points of X was the starting point for this paper. Related to this is the setting where the set of *analytic hypersurfaces* in X, i.e., the set of pure 1-codimensional analytic subsets of X, separates the points of X. Such a space is called *hypersurface separable*. (For the proofs it suffices to have locally hypersurface separable; for the definition and the corresponding properties, see [**18**].) While it is not true, in general, that hypersurface separable implies meromorphically separable for homogeneous complex manifolds, see e.g. [**24**], it turns out that, due to the classification, this is true if the homogeneous space has more than one end.

Now what we actually do is classify homogeneous complex manifolds which have more than one end and are Kähler. The connection is that it has now been proved that the smoothing of (p, p)-currents is possible in the homogeneous setting and this allows, in particular, the passage from the assumption of hypersurface separable to the assumption of Kähler, see [**23**] and the Appendix.

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Our aim is to show that the only homogeneous spaces which have more than one end and which are Kähler are certain bundles which are composed of $\mathbf{C}^*$, tori and homogeneous rational manifolds, where these latter two kinds of spaces are denoted by T and Q respectively. A particular example of such a space is a Cousin group with two ends, where by a *Cousin group* we mean a connected complex Lie group without non-constant holomorphic functions. Such a Cousin group with two ends fibers as a $\mathbf{C}^*$-bundle over a complex torus and forms one of the spaces in the classification of complex Lie groups with two ends, see [**10**, Theorem 1]. It is not surprising that, along with homogeneous cones minus their vertices, these Cousin groups again occur in the Kähler (resp. hypersurface separable) setting. The precise result that we prove in this paper is the following:

MAIN THEOREM. *Suppose $X = G/H$ is a complex manifold homogeneous under the holomorphic action of a real Lie group G. Assume X has more than one end. Then X is Kähler if and only if X is one of the following*:

(a) $X \overset{\mathbf{C}^*}{\to} Q \times T$,

where the fiber over $\{q\} \times T$ is a Cousin group isomorphic to $\mathbf{C}^k/\Gamma_{2k-1}$ for every $q \in Q$,

(b) *a finite unramified covering of X is a product*

$$(F \overset{\mathbf{C}^*}{\to} Q) \times T.$$

Moreover, X is hypersurface separable, if and only if the tori (resp. Cousin groups) which occur are abelian (resp. quasi-abelian) varieties and then X is quasi-projective. In either case, X has an equivariant compactification and there is a complex Lie group which acts transitively on X.

Note that the two cases in the theorem correspond to whether X has only constant holomorphic functions or not. Also note that the "if" parts of the theorem are straightforward. For, if X is as given in (a) or (b), one can see that X is Kähler by considering the associated $\mathbf{P}_1$-bundle, see [**5**]. If the base is algebraic, then similar reasoning implies that X is algebraic, this time by a result of K. Kodaira [**15**]. This $\mathbf{P}_1$-bundle is also the equivariant compactification of X and the stabilizer of the complex submanifolds comprising the zero and infinity sections of this bundle is a complex Lie group which acts transitively on X.

The proof of the "only if" parts of the main theorem proceeds by applying various homogeneous fibrations to the given space. The fibration lemma [**9**, p. 549] then ensures that either the fiber or the base has more than one end, while the other is compact. These components are further analyzed using various recent results [**4**], [**20**], [**21**], [**22**] etc. This allows us to conclude that the only possibilities are the spaces given in the statement of the theorem. These results are proved under the Kähler assumption, except for the algebraic remark about the tori and Cousin groups which occur in the hypersurface separable case (see the proof of the Main Theorem). In the case of Cousin groups this depends on the fact that a hypersurface separable Cousin group is quasi-abelian, see [**23**], [**1**] and [**2**].

The paper is organized as follows. We first present a splitting result which is useful in the subsequent proofs. Next we handle the case of discrete isotropy in the third section. The fourth section consists of some technical results needed for the proof of the Main Theorem and the proof of the Main Theorem itself is given in the fifth section. Throughout the proofs use the existence of an exhaustion of the space by open subsets together with a sequence of positive semidefinite forms which are positive definite on these open sets, see the statement of Lemma 4. The appendix contains a remark of F. Berteloot which shows that for homogeneous spaces this is equivalent to the space being Kähler.

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2. Splitting certain torus bundles. The first result concerns the almost triviality of homogeneous torus bundles over $\mathbf{C}^*$ in the Kähler setting.

PROPOSITION 1. *Suppose* $G/H \xrightarrow{T} G/J$ *is a homogeneous torus bundle over* $\mathbf{C}^*$ *with* G *acting almost effectively. Then* G/H *is Kähler if and only if a finite covering of* G/H *is a product* $T \times \mathbf{C}^*$. *Furthermore,* G/H *is hypersurface separable if and only if* T *is an abelian variety.*

Proof. First it is only necessary to look at the case where a complex Lie group is acting. In order to see this assume that G is real and let $G/H \to G/I$ denote the $\mathfrak{g}$-anticanonical fibration of G/H. Since T is a torus, one has $J^\circ \subset I$. Thus either $I = G$ and then G is complex, or else $I^\circ = J^\circ$ and the base of the $\mathfrak{g}$-anticanonical fibration is one dimensional. But then it is clear from [**13**, Theorem 10] that this base must be $\mathbf{C}^*$. Thus we may apply [**19**, Lemma 1] to see that there is a complex Lie group acting transitively on the space G/H.

Clearly the quotient map $\alpha: G \to G/J$ is a homomorphism. Also since $(J^\circ)'$ is normal in G and is contained in H°, it follows that $(J^\circ)' = \{e\}$ by the almost effectivity assumption. Thus J° is a normal abelian subgroup of G, a fact which we use later.

In the following methods from the paper [**20**] are applied. Let $G_{\mathbf{R}} := \alpha^{-1}(S^1)$, where $S^1 \subset \mathbf{C}^*$. Then $G_{\mathbf{R}}$ is connected, because one has the bundle $G/J \to G/G_{\mathbf{R}} \simeq \mathbf{R}^{>0}$ and $\pi_1(\mathbf{R}^{>0}) = 1$. Obviously

$$G_{\mathbf{R}}/H \xrightarrow{T} G_{\mathbf{R}}/J = S^1$$

and so the $G_{\mathbf{R}}$-orbit of the point corresponding to H is compact. Let ω be a positive semidefinite (1,1)-form on G/H, which is positive definite on a neighborhood of $G_{\mathbf{R}}/H$. Let $\pi:G \to G/H$ denote the projection and $\omega' = \pi^*\omega$.

Since $G_{\mathbf{R}}$ is solvable, $G_{\mathbf{R}}/H$ admits a $G_{\mathbf{R}}$-invariant finite measure μ. For $v, w \in T_pG$ the function

$$G_{\mathbf{R}} \ni g \mapsto r_g^*\omega_p'(v, w) = \omega_{p\cdot g}'(dr_g v, dr_g w) =: f(g)$$

is right H-invariant. Thus we may define

$$\tilde{\omega}_p(v, w) := \int_{G_{\mathbf{R}}/H} f d\mu.$$

The form $\tilde{\omega}$ is a closed (1, 1)-form on G which is positive semidefinite and is right $G_{\mathbf{R}}$-invariant.

We claim that the form $\tilde{\omega}$ is positive definite near $G_{\mathbf{R}}$. In order to see this let j denote the complex structure on T_eG. Then

$$\tilde{\omega}(v, jv) = \int_{G_{\mathbf{R}}/H} \omega_g'(dr_g v, jdr_g v) = 0$$

if and only if $d\pi_g dr_g v = 0$ for every $g \in G_{\mathbf{R}}$. Let X_v denote the vector field on G/H induced by $v \in T_eG$. Then

$$X_v(g \cdot H) = d\pi_g dr_g v.$$

Therefore $\tilde{\omega}(v, jv) = 0$ if and only if X_v vanishes along $G_{\mathbf{R}}/H$. Since X_v is the real part of a holomorphic vector field and $G_{\mathbf{R}}/H$ is a real hypersurface in G/H, this implies $X_v \equiv 0$ and thus $v \equiv 0$ by almost effectivity.

Now let $\mathfrak{g}_{\mathbf{R}}$ denote the Lie algebra of right invariant vector field on $G_{\mathbf{R}}$. Since $\tilde{\omega}|G_{\mathbf{R}}$ is right invariant and $d\tilde{\omega} = 0$, we have

$$\tilde{\omega}(X, [Y, Z]) + \tilde{\omega}(Z, [X, Y]) + \tilde{\omega}(Y, [Z, X]) = 0$$

for $X, Y, Z \in \mathfrak{g}_{\mathbf{R}}$. Let $\mathfrak{m} := Lie(J^\circ)$ and let $\mathfrak{a} \subset \mathfrak{g}_{\mathbf{R}}$ be defined by

$$\mathfrak{a} := \{X \in \mathfrak{g}_{\mathbf{R}}:\tilde{\omega}(X, \mathfrak{m}) = 0\}.$$

Then $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}_{\mathbf{R}}$ and all eigenvalues of the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}_{\mathbf{R}}$ restricted to $\mathfrak{m}$ are imaginary. (One sees this by making appropriate changes in the proof of Theorem 2 in [**20**, pp. 406-7].) In order to show that the ad$\mathfrak{a}$-action on $\mathfrak{m}$ is diagonalizable, we perform computations similar to those in [**20**, 3.2′]. For this let $\tilde{\omega}$ be an eigenvector of ad_X for $X \in \mathfrak{a}$ and $w^\perp := \{v \in \mathfrak{m}:\tilde{\omega}(v, \langle w\rangle_{\mathbf{C}}) = 0\}$. Then for $v \in w^\perp$ we have

$$\tilde{\omega}(X, [v, w]) + \tilde{\omega}(w, [X, v]) + \tilde{\omega}(v, [w, X]) = 0.$$

Since J is abelian, $[v, w] \in \mathfrak{m}' = \mathrm{Lie}(J^\circ)' = 0$ and $[w, X] = \lambda w$ for $\lambda \in \mathbf{C}$

by definition. Thus $\tilde{\omega}(v, [w, X]) = \tilde{\omega}(v, \lambda w) = 0$. Hence $[X, v] \in w^{\perp}$. So by induction it follows that the complex linear action of $\mathrm{ad}\,\mathfrak{a}$ restricted to $\mathfrak{m}$ is completely reducible, i.e., $\mathrm{ad}\,\mathfrak{a}$ is diagonalizable.

Let A denote the real Lie subgroup of $G_{\mathbf{R}}$ generated by $\mathfrak{a}$. Since J° is a normal abelian subgroup of $G_{\mathbf{R}}$, the adjoint action of $G_{\mathbf{R}}$ on $\mathfrak{m}$ factors through $G_{\mathbf{R}}/J^{\circ} \simeq A$, i.e., it is diagonalizable. Because $\mathrm{ad}\,\mathfrak{a}$ has imaginary spectrum on $\mathfrak{m}$, we have that $\mathrm{Ad}_{G_{\mathbf{R}}} \subset GL_{\mathbf{C}}(\mathfrak{m})$ is contained in a compact real torus of $GL_{\mathbf{C}}(\mathfrak{m})$. On the other hand $\mathrm{Ad}_H \subset GL_{\mathbf{C}}(\mathfrak{m})$ stabilizes the lattice $H \cap J^{\circ}$, where we have identified $J^{\circ} \simeq \mathfrak{m}$. Hence $\mathrm{Ad}_H \subset GL_{\mathbf{C}}(\mathfrak{m})$ is discrete and is contained in a compact subgroup of $GL_{\mathbf{C}}(\mathfrak{m})$. Therefore $\mathrm{Ad}_H \subset GL_{\mathbf{C}}(\mathfrak{m})$ is finite. Let $\tilde{H}$ denote the kernel of the adjoint representation of H on $\mathfrak{m}$. Thus $G/\tilde{H} \to G/H$ is a finite covering. Let

(1) $\quad \tilde{J} := J^{\circ} \cdot \tilde{H}$.

Then we have

$$\begin{array}{ccc} G/\tilde{H} & \longrightarrow & G/H \\ T\big\downarrow & & \big\downarrow T \\ \mathbf{C}^* = G/\tilde{J} & \longrightarrow & G/J = \mathbf{C}^* \end{array}$$

where each of the horizontal arrows is a finite covering. Since J° centralizes $\tilde{H}$, it follows that $\tilde{H} \lhd \tilde{J}$. Thus $G/\tilde{H} \to G/\tilde{J}$ is a principal torus bundle. But this is easily seen to be trivial as follows. In general, the bundle $G/H \to G/J$ is associated to the bundle $\mathbf{C} = G/J^{\circ} \to G/J = \mathbf{C}^*$, i.e., if we let $\rho: J/J^{\circ} \to \mathrm{Aut}(T)$ be the adjoint representation, then $G/H = \mathbf{C} \times_{\rho} T$. Taking $\tilde{H}$ instead of H and $\tilde{J}$ instead of J we have

$$\rho(\tilde{J}/J^{\circ}) = \rho(\tilde{H}/\tilde{H} \cap J^{\circ})$$

by (1). But this last is equal to $\{e\}$.

3. The discrete case. In this section we look at the case where the group G is complex and the isotropy subgroup $H =: \Gamma$ is discrete and show that one has the following.

PROPOSITION 2. *Assume G/Γ is a Kähler homogeneous space with more than one end, where the isotropy subgroup Γ is discrete and G is complex. Then G is solvable and X is either*

1. *a torus bundle over $\mathbf{C}^*$, where a finite covering splits as a product, or*
2. *a Cousin group $\mathbf{C}^n/\Gamma_{2n-1}$ and G is abelian.*

Proof. First we assume G is solvable. If X has nonconstant holomorphic functions, then by [**9**, Theorem 6] one has that X fibers with compact connected fiber over $\mathbf{C}^*$. The fiber is a compact solvmanifold and hence a torus by [**7**]. Thus it is clear by Proposition 1 that a finite covering of X

splits as a product. If $\mathcal{O}(X) \simeq \mathbf{C}$, then using [**20**] we have that G is abelian and hence X is a Cousin group with two ends.

Next we exclude the possibility that G is semisimple. For, if it were, then since X is Kähler, Γ would be algebraic and hence finite [**4**]. But then $X = G/\Gamma$ would be Stein. Since G is semisimple and Γ is discrete one has $\dim X > 1$. But this would then contradict the fact that a Stein homogeneous space with more than one end is biholomorphic to $\mathbf{C}^*$, see [**10**, p. 186].

Finally we assume $G = R \rtimes S$ with $\dim R > 0$ and $\dim S > 0$, where R is the radical and S is a maximal semisimple subgroup of G and show that this also leads to a contradiction. Suppose first that the R-orbits in X are not closed. Letting $I := \overline{\Gamma R}$ and proceeding as in [**9**, Theorem 2] one gets a proper connected closed solvable complex subgroup H in G containing I°. Letting $N := N_G(H)$ one has $\Gamma R \subset N$ and $N \neq G$, since $H \neq R$ by assumption. This yields a proper homogeneous fibration $G/\Gamma \to G/N$ and by passing to an (*a posteriori* finite) covering one may assume that its fibers are connected. If G/N is compact and thus homogeneous rational, then Γ is contained in a maximal parabolic subgroup P of G. Because G/P is compact and simply connected, by means of the fibration $G/\Gamma \to G/P$ one sees that P/Γ has at least two ends [**9**]. In particular, it follows by induction that P is solvable. But this is only possible if $S = SL_2(\mathbf{C})$ and $P \cap S$ is a Borel subgroup of S with $G/P \simeq \mathbf{P}_1$. Since P is *not* abelian, it follows from what was observed above that the holomorphic reduction $P/\Gamma \to P/J$ has base $\mathbf{C}^*$ and a compact torus as fiber. Thus we get

$$G/\Gamma \overset{T}{\to} G/J \overset{\mathbf{C}^*}{\to} G/P \simeq \mathbf{P}_1.$$

Since G/J is a finite quotient of $\mathbf{C}^2\backslash 0$, we see that J has only a finite number of connected components. Modulo a finite covering G/Γ is therefore a torus principal bundle over G/J. Let K be a maximal compact subgroup of $SL_2(\mathbf{C})$. Then the K-orbits in G/J have finite isotropy and the inverse image of any of these K-orbits is a homogeneous CR-hypersurface in the Kähler G/Γ. But this is impossible [**22**].

If G/N is not compact, then it is an orbit in some $\mathbf{P}_k$ with two ends and N/Γ is compact and thus a torus. By [**9**, Theorem 7] there is a parabolic subgroup P in G which contains N. As in the last paragraph this again gives a contradiction.

Next we assume that the R-orbits are closed. Then there is a fibration

$$G/\Gamma \to G/R \cdot \Gamma = S/\Lambda,$$

where $\Lambda := S \cap R \cdot \Gamma$ is a discrete subgroup of S. If the fiber $R\Gamma/\Gamma$ is compact, then the base is Kähler, e.g. see [**5**], and thus Λ is algebraic and hence finite [**4**]. But then S/Λ is Stein and hence has one end which is the desired contradiction.

So we suppose that S/Λ is compact and $R/R \cap \Gamma$ has at least two ends. We again consider the holomorphic reduction $R\Gamma/\Gamma \to R\Gamma/L =: Y$ of the fiber $R/R \cap \Gamma = R\Gamma/\Gamma$. If $\dim Y > 0$, then $Y = \mathbf{C}^*$ and L/Γ is a torus T. There are two cases to consider here. First if $\dim L = 0$, then $R\Gamma/\Gamma \simeq \mathbf{C}^*$. Thus

$$G/\Gamma \overset{\mathbf{C}^*}{\to} G/R \cdot \Gamma$$

and so $\dim R = 1$ and G is a product $R \times S$. But this contradicts the fact that the Ahiezer conjecture holds for a complex Lie group G which is a product. In particular, the group has no semisimple factor, c.f. [**21**, pp. 116-117]. On the other hand if $\dim L > 0$, then we consider

$$G/\Gamma \to G/L \to G/N_G(L^\circ).$$

If $G/N_G(L^\circ)$ is not $\mathbf{C}^*$, then there is a proper parabolic subgroup P in G with $\Gamma \subset N(L^\circ) \subset P$ and one again has a contradiction. Otherwise, $N(L^\circ)^\circ$ is solvable by induction and therefore G is solvable which is also a contradiction.

So finally we assume $\mathcal{O}(R/R \cap \Gamma) \simeq \mathbf{C}$. Thus R is abelian, see [**20**], and since $R \simeq \mathbf{C}^n$, the representation ρ of S on R is given by

$$\rho: S \to GL(R) \simeq GL_n(\mathbf{C}).$$

LEMMA 1. *Under the assumptions of Proposition* 2 *we have* $\dim \ker \rho = 0$.

Proof. We assume that $\dim \ker \rho > 0$ and let $H = \ker \rho$. Note that H° is a semisimple, normal subgroup of S. The groups H and $R \cdot \Gamma$ are both contained in $N_G(\Gamma \cap R)$. In fact,

$$(N_G(\Gamma \cap R))^\circ = H^\circ \times R,$$

since $\Gamma \cap R$ contains a basis of R. Thus we have a fibration

$$\begin{array}{ccc} G/\Gamma & \longrightarrow & G/R \cdot \Gamma \\ & \searrow \quad \swarrow & \\ & G/N_G(\Gamma \cap R) & \end{array}$$

The manifold $Y := N_G(\Gamma \cap R)/\Gamma$ has two ends, is Kähler and satisfies $\mathcal{O}(Y) \simeq \mathbf{C}$. As well, its group is a product group $H^\circ \times R$. By the solution of the Ahiezer conjecture in the product case [**21**], it follows that $H^\circ = \{e\}$, contrary to the assumption. Hence $\dim \ker \rho = 0$.

LEMMA 2. *One has* $S = \{e\}$.

Proof. Assume the contrary. Then since S/Λ is compact by assumption, Λ contains a semisimple element λ. Let $A := (\mathbf{C}^*)^k = \overline{(\lambda)}_{\mathbf{Z}}$, where

$\lambda = \pi_S(\gamma)$ for some $\gamma \in \Gamma$ and $\pi_S:G \to S$ is the projection map. Define $\Gamma := (\Gamma \cap R) \cdot (\gamma)_{\mathbf{Z}}$ and $\widetilde{G} := R \rtimes A$. Then we have

$$\begin{array}{ccc} \widetilde{G}/\widetilde{\Gamma} & \longrightarrow & A/(\lambda)_{\mathbf{Z}} = \widetilde{G}/R \cdot \widetilde{\Gamma} \\ & {\scriptstyle \pi} \searrow \quad \swarrow & \\ & \widetilde{G}/\widetilde{H} & \end{array}$$

where π is the holomorphic reduction of $\widetilde{G}/\widetilde{\Gamma}$. Since $A/(\lambda)_{\mathbf{Z}}$ is a Cousin group and $\mathcal{O}(R/R \cap \Gamma) \simeq \mathbf{C}$, we have $R \subset \widetilde{H}$. Therefore $\widetilde{G} = \widetilde{H}$, i.e., $\mathcal{O}(\widetilde{G}/\widetilde{\Gamma}) \simeq \mathbf{C}$. But then $\widetilde{G}$ is solvable, $\widetilde{G}/\widetilde{\Gamma}$ is Kähler and $\mathcal{O}(\widetilde{G}/\widetilde{\Gamma}) \simeq \mathbf{C}$. It follows that $\widetilde{G}$ is abelian [**20**] and hence $A \subset \ker \rho$ which is a contradiction to Lemma 1.

This contradiction to our assumption that G is a nontrivial semidirect product completes the proof of Proposition 2.

Remark. This proof depends on the fibers of the radical fibration being Cousin groups and the isotropy subgroup of the base being Zariski dense. While one could exploit this to prove a slightly more general version of Ahiezer's conjecture [**3**], this would not contribute to understanding the essential difficulties suggested by the complex analogues of the examples of Margulis, see [**16**]. In these examples the radical meets the discrete isotropy only at the identity and the projection of the isotropy into any semisimple factor contains no semisimple elements.

4. Kählerian bundles with more than one end. Our first goal in this section is to prove part (b) of the main theorem for a homogeneous torus bundle over a cone.

PROPOSITION 3. *Let Y be an affine cone with its vertex removed. Suppose $X = G/H \to Y$ is a torus bundle. If X is Kähler, then a finite covering $\widetilde{X}$ of X is a product*

(2) $\quad \widetilde{X} = T \times \widetilde{Y},$

where T is a torus and $\widetilde{Y}$ is a finite covering of Y.

Proof. Since the fibers of the bundle $X \to Y$ are compact and connected, the bundle projection is G-equivariant, i.e., $Y = G/J$, where $H \subset J$. If $\dim Y \leqq 1$, then the result follows from Proposition 1. Thus we may assume that G contains a maximal compact semisimple subgroup K. It is clear that K has real hypersurface orbits in Y. Since $\pi_1(Y)$ is finite, the group J/J° is finite and we have the torus bundle

(3) $\quad \pi:\widetilde{X} = G/H \cap J^{\circ} \xrightarrow{T} G/J^{\circ} = \widetilde{Y}.$

Note that the induced representation of J° in $\mathrm{Aut}_{\mathcal{O}}(T)$ is contained in T. Thus $\tilde{X} \to \tilde{Y}$ is a holomorphic principal bundle. The K-orbits in $\tilde{Y}$ are also real hypersurfaces.

In order to finish the proof of the proposition we need the following lemma and corollary.

LEMMA 3. ([**22**], see also [**14**]) *Let $K \cdot y$ be a K-orbit in $\tilde{Y}$ and A be the inverse image of $K \cdot y$ in $\tilde{X}$. If A admits a Kählerian neighborhood, then for every $x \in A$ the map*

(4) $\quad \pi : K \cdot x \to K \cdot y$

is a CR-isomorphism.

Since $\tilde{X}$ is Kähler, the conditions of this lemma are satisfied for every K-orbit in $\tilde{Y}$. Thus we obtain the following.

COROLLARY 1. *For $x \in \tilde{X}$ let H_x denote the complex subspace of $T_x\tilde{X}$ generated by the vector fields which are induced by the K-action on $\tilde{X}$. Then the collection $\{H_x : x \in \tilde{X}\}$ defines a complex connection for the principal bundle $\tilde{X} \to \tilde{Y}$.*

Since the connection given by the corollary is clearly integrable and since $\tilde{Y}$ is simply connected, this concludes the proof of the proposition.

Remark 1. The article [**22**] classifies all compact homogeneous CR-hypersurfaces which have a Kählerian neighborhood. One parameter which occurs in the case where a compact group K is acting transitively is the degeneration of the Kähler form along the K-orbits. This is analyzed in detail using the Lie algebra cohomology of K. The situation in Proposition 3 occurs exactly when the Kähler form is non degenerate along the orbit of the center of the group $K \times T$.

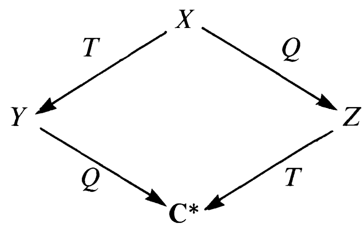
Remark 2. If G is complex, then for the conclusion of Proposition 3 it is enough to have the condition of the lemma for one $y \in Y$.

Now in preparation for the proof of the Main Theorem we consider Kähler homogeneous manifolds with more than one end. Their structure is determined in the following two propositions, where two cases are distinguished according to whether the space has non-constant holomorphic functions or not.

PROPOSITION 4. *Suppose $G/H \to G/J$ is the holomorphic reduction of the Kähler homogeneous space G/H having more than one end and assume* dim $G/J > 0$, *i.e.*, $\mathcal{O}(X) \not\cong \mathbf{C}$. *Then X is of the form* (b) *in the Main Theorem.*

Proof. First if dim $G/J > 1$, then G/J is a nontrivial $\mathbf{C}^*$-bundle over a homogeneous rational manifold, i.e., an affine homogeneous cone with its vertex removed. After passing to a finite covering, if necessary, we see that by Proposition 3 the result is clear in this case.

Now if $\dim G/J = 1$, then it follows that G/J has two ends and is biholomorphic to $\mathbf{C}^*$. Note that under the assumptions of the proposition $J/H = Q \times T$, see [7], and we have the following intermediate fibrations

(5) 

The first step is to show that there is an equivariant fibration

$$Y \xrightarrow{\mathbf{C}^*} Q.$$

To see this note that the normalizer fibration $G/H \rightarrow G/N$ factors through Y. If $G/N = Q$, then this is clear. Otherwise, we can argue as follows. Since any non-trivial $\mathbf{C}^*$-bundle over Q is holomorphically separable and Y is not, $Y = Q \times \mathbf{C}^*$. Thus we may consider $\hat{G}$, which is defined to be the smallest complex Lie subgroup of $\mathrm{Aut}_{\mathcal{O}}(Y)$ containing the representation of G. It is clear that $\hat{G}$ is not semisimple. Let $\hat{R}$ denote the radical of $\hat{G}$ and let $\hat{\rho}:\hat{G} \rightarrow \mathbf{C}^*$ denote the representation of $\hat{G}$ on $\mathbf{C}^*$ given by the fibration $Y \rightarrow \mathbf{C}^*$. Then $\ker(\hat{\rho}|\hat{R})$ has codimension one in $\hat{R}$ and since the semisimple part of $\hat{G}$ acts trivially on $\mathbf{C}^*$

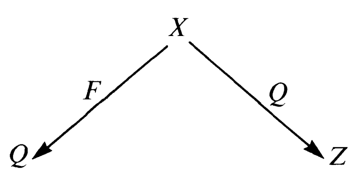
$$\ker \hat{\rho}|\hat{R} = \ker \hat{\rho} \cap \hat{R} \lhd \hat{G}.$$

Since $(\ker \hat{\rho}|\hat{R})^\circ$ has a fixed point in Q it follows that

$$\dim(\ker \hat{\rho}|\hat{R}) = 0.$$

Hence $\dim \hat{R} = 1$. Thus $\hat{R}$ is just the identity component of the center of $\hat{G}$ and the $\hat{G}$-normalizer fibration has Q as its base and its typical fiber is 1-dimensional and has two ends, i.e., it is $\mathbf{C}^*$.

Hence we have the equivariant diagram



where F is a torus bundle over $\mathbf{C}^*$. We want to show that $X = Q \times Z$. In order to see this consider a maximal compact semisimple subgroup K of G and the K-orbit $K \cdot x_0$ through a point $x_0 \in X$. Since any orbit $K \cdot x_0$ is contained in $Q \times T$ (see the left hand side of diagram 5), clearly the map $K \cdot x_0 \to Q$ is biholomorphic. Since K acts trivially on Z, it is also clear that the map $R \cdot x_0 \to Z$, where R denotes the radical of G, is surjective. Suppose this map is not injective, i.e., suppose there exists $x_1 \neq x_0$ contained in $K \cdot x_0 \cap R \cdot x_0$. Let $\pi:X \to Q$ denote the projection. Then $\pi(x_0) = \pi(x_1)$, since $\pi(R \cdot x_0) = \pi(x_0)$. Thus $x_0 = x_1$ because $\pi|K \cdot x_0$ is one-to-one. So

$$K \cdot x_0 \cap R \cdot x_0 = x_0 \quad \text{for every } x_0 \in X.$$

Thus the map $X \to Q \times Z$ is biholomorphic.

PROPOSITION 5. *Suppose $X = G/H$ is a Kähler homogeneous manifold with more than one end and $\mathcal{O}(X) \simeq \mathbf{C}$. Then X can be realized as a homogeneous $\mathbf{C}^*$-bundle over the product of a torus with a homogeneous rational manifold, i.e., X is of the form* (a) *in the Main Theorem.*

Proof. First we let $G/H \to G/J$ be the $\mathfrak{g}$-anticanonical fibration of X (resp. the normalizer fibration of X, if G is complex) and we note that $\mathcal{O}(J/H) \simeq \mathbf{C}$. If not, then by the discrete case (see the third section) one would have a fibration

$$J/H \xrightarrow{T} J/I = \mathbf{C}^*.$$

But then

$$G/I \xrightarrow{\mathbf{C}^*} G/J$$

would be a $\mathbf{C}^*$-bundle over a homogeneous rational Q. Since every $\mathbf{C}^*$-bundle over Q has nonconstant holomorphic functions, one would then get the contradiction $\mathcal{O}(G/H) \simeq \mathcal{O}(G/I) \not\simeq \mathbf{C}$. Thus the fibration $G/H \to G/J$ has base Q a homogeneous rational and by induction the fiber $J/H = A$ is a Cousin group. Let K_S denote a maximal compact semisimple subgroup of G. Then K_S acts transitively on Q and $Q = K_S/L$. We rename and let $G = A \times K_S$. The bundle $X \xrightarrow{A} Q$ is K_S homogeneous, i.e., it is given by a representation

$$\rho:L \to (\mathrm{Aut}\, A)^\circ = A,$$

since L is connected. Note that A contains a unique compact subgroup K_A (which is also a real hypersurface in A) and thus $\rho(L) \subset K_A$.

Now we have $X = G/H$, where

$$(6) \qquad H = \{ (\rho(l), l\} \in A \times K_S : l \in L\} \subset K_A \times K_S.$$

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Letting $K = K_A \times K_S$, we see by (6) that $H = H \cap K$. Then by [**22**, Proposition 2.4.9, Proposition 2.4.10], see also [**14**, Theorem 2, p. 174 and Theorem 3, p. 175], there are two possibilities which can occur. First the K_S-orbit through the point $H \in G/H = X$ can be complex and thus a section of the bundle $X \xrightarrow{A} Q$. In this case $X = A \times Q$ and ρ is trivial. Or there can exist a closed subgroup $I \subset K$ such that

(7) $K/H \to K/I$

is an S^1-CR-principal bundle over the Kähler manifold $K/I = T \times Q$, [**17**] or [**7**].

More precisely, let $\rho_1 : \mathfrak{k} = \mathfrak{k}_A \oplus \mathfrak{k}_S \to \mathfrak{k}_A$ denote the projection. Then

(8) $I = \exp(\rho_1(\mathfrak{h})) \cdot H$

and $I/H \simeq S^1$. Since $H \cap A = \{e\}$, it follows that $\dim \rho_1(\mathfrak{h}) = 1$. But by the description of H above we have $\exp(\rho_1(\mathfrak{h})) = \rho(L)$. Thus $\rho(L) \simeq S^1$. Moreover, by [**22**] we have $\sqrt{-1} \cdot \rho(L) \cap K_A = \{e\}$, which means that $\rho(L)^{\mathbf{C}} \subset A$ is a closed complex subgroup isomorphic to $\mathbf{C}^*$.

So finally we obtain the diagram

$$\begin{array}{ccc} K/H & \hookrightarrow & G/H \\ S^1 \downarrow & & \downarrow \mathbf{C}^* \\ K/I & \xrightarrow{\sim} & G/\rho(L)^{\mathbf{C}} \cdot H = T \times Q \end{array}$$

and the result is clear from this.

5. Proof of the Main Theorem. Assume that X is Kähler and has more than one end. First let $G/H \to G/J$ be the $\mathfrak{g}$-anticanonical fibration if G is real [**13**] or else the normalizer fibration if G is a complex Lie group [**7**]. In either case if $G = J$, then G is a complex Lie group and H is discrete, so the result follows from Proposition 2. Otherwise, in the real case one has $\dim J/H > 0$ and either $G/J = Q$ is a homogeneous rational manifold and the fiber is complex parallelizable, or else the fiber is compact complex parallelizable and thus a torus and the base is a $\mathbf{C}^*$-bundle over a homogeneous rational, see [**13**, Theorem 10, pp. 78-79]. If G is complex, then we see by [**9**, Theorem 7 and Theorem 8] that G/H fibers either as a torus bundle over a $\mathbf{C}^*$-bundle over a rational or as a complex parallelizable homogeneous manifold over a rational.

We are now in the situation where either X has no nonconstant holomorphic functions and the result follows from Proposition 5 or else it does have nonconstant holomorphic functions and then Proposition 4 handles all the possibilities.

Now assume X is hypersurface separable. It is clear that the tori which appear in (b) are algebraic. That the Cousin groups which occur in (a) are quasi-abelian follows from [**23**, Theorem 2, Section 6] and can also be deduced from the results in [**2**]. And that the tori which can occur in (a) are also algebraic can be seen in the following way.

Let $x \in X = G/H$ be a point. Take n hypersurfaces F_i in X, where $n = \dim_{\mathbf{C}} X$, which are smooth at x and are such that if locally $F_i = \{f_i = 0\}$, then $df_1 \wedge \ldots \wedge df_n(x) \neq 0$, i.e., $(f_i)_{i=1}^n$ is a local coordinate system near x. Using the smoothing process [**23**] we obtain a Kähler form ω which is positive at x and which represents the first Chern class of the divisor $\cup_{i=1}^n F_i$. We may assume that ω is invariant under the principal action of the maximal compact subgroup K of $A: = \mathbf{C}^k/\Gamma_{2k-1}$. Then ω is a Kähler form in an open neighborhood of the K-orbit of x. We are in the same setting as in Proposition 5 and we maintain the same notation, i.e., we have $\rho(L)^{\mathbf{C}} \subset A$. The orbit of $\rho(L)^{\mathbf{C}}$ through x is closed and is isomorphic to $\mathbf{C}^*$. By [**22**] this complex direction is orthogonal to the maximal complex subgroup $\mathbf{C}_\Gamma$ in K. With the help of [**23**, Theorem 2, Section 6] or [**2**] it is easy to deduce that the projection of Γ_{2k-1} by the map $A \to T$, given by moding out by the $\rho(L)^{\mathbf{C}}$-orbits, is an algebraic lattice in $\mathbf{C}^{k-1}$. For, there exists a hermitian form on A given by that theorem and this projects to a positive definite hermitian form on T. Thus T is an abelian variety.

Remark. In particular, if G is semisimple, then $\mathcal{O}(X) \not\simeq \mathbf{C}$ and X is a cone. But this can also be seen more easily by other means.

6. Appendix. First we recall that it was proved in [**23**] that a hypersurface separable homogeneous complex manifold satisfies the condition expressed in the next lemma. The purpose of this short appendix is to note that this implies that the homogeneous space is Kähler.

LEMMA 4. (Berteloot) *Suppose $X = G/H$ is a homogeneous complex manifold admitting an exhaustion U_n by relatively compact sets so that there are positive semidefinite closed forms ω_n on X which are positive definite on U_n. Then X is Kähler.*

Proof. Let V_k denote a locally finite covering of X by contractible, relatively compact, open sets. Then for each k we can find a positive semidefinite, closed (1, 1)-form ω_k on X which is positive definite on V_k. Clearly we may write

$$(9) \quad \omega_k | V_k = \partial\bar{\partial}\phi_{kk}$$

with ϕ_{kk} being a smooth strictly plurisubharmonic function on V_k. Furthermore, for each $l \in \mathbf{N}$ there is a smooth plurisubharmonic function ϕ_{kl} on V_l such that

$$(10) \quad \omega_k | V_l = \partial\bar{\partial}\phi_{kl}.$$

Therefore the function $\phi_{kl} - \phi_{kj}$ is pluriharmonic on $V_l \cap V_j$ if this set is not empty. Let $\{W_k\}_{k \in \mathbf{N}}$ be a countable covering of X with the property $W_k \subset\subset V_k$ for all $k \in \mathbf{N}$. By a diagonal process one can find a sequence $\{\lambda_k\}_{k \in \mathbf{N}}$ with $\lambda_k > 0$ such that $\sum_k \lambda_k \phi_{kl}$ is normally convergent in the C^2-norm on W_l for each $l \in \mathbf{N}$. Denote the sum $\sum_k \lambda_k \phi_{kl}$ by ψ_l for all $l \in \mathbf{N}$. Then $(\psi_k)_{k \in \mathbf{N}}$ is a C^2 Kähler cocycle, i.e., $\psi_k - \psi_l$ is pluriharmonic for all $k, l \in \mathbf{N}$. Define $\omega = \partial\bar{\partial}\psi_k$. Then ω is a continuous positive definite (1, 1)-form on X. By smoothing the form ω (see **[23]**) we obtain that X is Kähler.

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