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ON EIGENSOLUTIONS OF THE ONE-SPEED NEUTRON TRANSPORT EQUATION IN PLANE GEOMETRY

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Abstract

We revisit the singular eigensolution to the steady state one-speed transport equation for an isotropically scattering and multiplying heterogeneous slab. It is proved that this solution is a sum of Stieltjes integrals over the resolvent set of only the operator of multiplication by the angular variable.

1. Introduction

Consider the one-speed, variable coefficient and source-free stationary neutron transport equation [2], with isotropic scattering and multiplication in plane geometry

$$\mu\frac{\partial}{\partial x}\psi(x,\mu)+\Sigma(x)\psi(x,\mu)=\frac{c}{2}\Sigma(x)\int_{-1}^{1}\psi(x,\mu')\,d\mu' \qquad x\in[0,L],\ \ \mu\in[-1,1]. \tag{1.1}$$

In this equation

$$\Sigma(x)=\Sigma_s(x)+\Sigma_f(x)+\Sigma_c(x) \tag{1.2}$$

is the total neutronic cross section of scattering, fission and capture, and

$$c=[\Sigma_s(x)+\nu\Sigma_f(x)]/\Sigma(x) \tag{1.3}$$

is the yield (neutron per neutron), in which ν is the average number of neutrons per fission; the reader is referred to (for example) [2, 7] for other details.

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Recently [7] it was argued that for many situations of practical interest, c may be assumed to be a constant, independent of x. An optical path transformation may then be performed with x, namely

$$z = b^{-1} \int_0^x \Sigma(\xi)\, d\xi \tag{1.4}$$

$$b = \int_0^L \Sigma(\xi)\, d\xi \tag{1.5}$$

to put (1.1) into the equivalent constant-coefficient form

$$\mu \frac{\partial}{\partial z} \phi(z, \mu) + b\phi(z, \mu) = \frac{c}{2} b \int_{-1}^{1} \phi(z, \mu')\, d\mu',$$
$$\phi(z, \mu) = \psi(x(z), \mu), \qquad z \in [0, 1], \ \mu \in [-1, 1]. \tag{1.6}$$

Equation (1.6) was solved exactly in 1960 by the singular eigenfunction technique introduced by Case [4]. Other analytical methods for solving this equation include the application of Lie series [11], summation operators [3], and double Legendre transforms [7]. In the present work, we intend to revisit only the singular eigenfunction method, in which the solution is sought in the form

$$\phi(z, \mu) = h(z) f(\mu) = hf. \tag{1.7}$$

Substitution then of this ansatz in (1.6) leads to

$$\frac{1}{h}\frac{dh}{dz} = \frac{b}{\mu}\left[\frac{1}{f}\frac{c}{2}\int_{-1}^{1} f(\mu')\, d\mu' - 1\right] = \kappa \tag{1.8}$$

for which

$$h(z) = e^{\kappa z} \tag{1.9}$$

and $f(\mu)$ satisfies the eigenvalue equation

$$(b + \kappa\mu) f(\mu) = \frac{c}{2} b \int_{-1}^{1} f(\mu')\, d\mu'. \tag{1.10}$$

The explicit form of $f(\mu)$ and its properties have been studied by several authors [4, 8, 12]. To date, this essentially analytical technique has extensively been developed to treat a wide variety of neutron transport problems [10, 14], and it has been extended even to analyse anisotropic scattering of multigroup neutron transport [6] in multilayer slabs. In particular, we aim to further explore the basic nature of the eigensolutions of (1.1) by application of the theory of Stieltjes integrals [13].

2. Analysis

Let us rewrite (1.10) as

$$\mathcal{F} f = \frac{c}{2}\frac{b}{\mu}\int_{-1}^{1} f(\mu')\, d\mu' - \frac{b}{\mu} f = \kappa f \tag{2.1}$$

in which it is apparent that κ is an eigenvalue of the nonselfadjoint singular integral operator:

$$\mathcal{F} = \frac{c}{2}\frac{b}{\mu}\int_{-1}^{1} d\mu - \frac{b}{\mu} \tag{2.2}$$

whose spectrum must, according to the Weyl-von Neumann theorem [16], contain a continuous part. Such spectra may satisfactorily be analysed either variationally [9] or via the technique of singular integral equations [4]. Here we shall follow a rather indirect route, which enables us to utilise the theory of selfadjoint operators to investigate the κ-eigenvalue spectrum (denoted as $\mathbb{K}$) of $\mathcal{F}$, with the aim of trying to add another dimension to these methods.

Proposition 1. *If the result of applying a generalised operator $\mathcal{H}$ to the eigenvalue problem of* (2.1) *is another parametric equation*

$$G(f, \kappa) = 0 \tag{2.3}$$

then $\mathbb{K}$ may possibly remain unaltered, whereas if it is a Fredholm problem

$$\mathcal{L}X = \lambda X + Y \tag{2.4}$$

with $\mathbb{L}$ being the λ-resolvent set of the corresponding $\mathcal{L}$-operator, then $\mathbb{K}$ can only be a subset of $(\mathbb{R} - \mathbb{L})$ *where* $\mathbb{R} = (-\infty, \infty)$.

Proof. By virtue of the definition of the resolvent [5] of an operator and as a consequence of Weinstein's theorem on finite-dimensional perturbation [15].

2.1. Double differentiation with respect to μ

Since (1.10) is an integral equation of the second kind, then the simplest $\mathcal{H}$-operator one can think of in this connection is $\mathcal{H} = d^2/d\mu^2$. Indeed

$$\frac{d^2}{d\mu^2}\{\mathcal{F} f\} = \frac{d^2}{d\mu^2}\{\kappa f\} \tag{2.5}$$

satisfies (2.3) because it equals

$$\mathcal{F}\left\{\frac{d^2 f}{d\mu^2}\right\} + 2\frac{d\mathcal{F}}{d\mu}\left\{\frac{df}{d\mu}\right\} + \frac{d^2\mathcal{F}}{d\mu^2}\{f\} - \kappa\frac{d^2 f}{d\mu^2} = 0. \tag{2.6}$$

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Let us rewrite (1.10) as

$$f(\mu) = \frac{c}{2}\frac{b}{(b+\kappa\mu)}\left[\int_{-1}^{\mu} f(\mu')\,d\mu' - \int_{1}^{\mu} f(\mu')\,d\mu'\right], \tag{2.7}$$

and differentiate it throughout twice with respect to μ to obtain the equivalent Cauchy-type homogeneous differential eigenvalue equation

$$\frac{d^2 f}{d\mu^2} - \frac{2\kappa^2}{(b+\kappa\mu)} f = G(f, \kappa) = 0. \tag{2.8}$$

The general solution of this equation is of the form

$$f(\mu) = A(b+\kappa\mu)^{-1} + B(b+\kappa\mu)^2 \tag{2.9}$$

where A and B are arbitrary constants.

Seeking the eigensolutions $f(\mu)$ in the class of bounded, $\|f\| < M$, functions over $-\infty < \kappa < \infty$ demands that $B = 0$. Further normalisation of $f(\mu)$ as

$$\|f\| = \int_{-1}^{1} f(\mu')\,d\mu' = 1 \tag{2.10}$$

and substitution of (1.4), (1.5), (1.8) and (1.7) back into (1.6) provides for the transcendental generalised dispersion formula,

$$\Lambda = 1 - \frac{c}{2\kappa}\left[\int_0^L \Sigma(\xi)\,d\xi\right] \ln\left|\frac{\int_0^L \Sigma(\xi)\,d\xi + \kappa}{\int_0^L \Sigma(\xi)\,d\xi - \kappa}\right| = 0 \tag{2.11}$$

whose roots define the discrete eigenvalue spectrum for (1.1). A graphical solution of (2.11) illustrates that when $c < 1$ only two distinct discrete eigenvalues $\pm\kappa_0$ exist in the range

$$|\kappa| < \int_0^L \Sigma(\xi)\,d\xi = b,$$

and that as $\Sigma(x)$ or L is increased without bound $(b \to \infty)$, then $|\kappa_0| \to \infty$. Here the associated eigenfunctions

$$\psi_{\kappa_0}(x, \mu) = \frac{c}{2}\frac{\int_0^L \Sigma(\xi)\,d\xi}{\left[\int_0^L \Sigma(\xi)\,d\xi + \kappa_0\mu\right]} \exp\left[\kappa_0 \frac{\int_0^x \Sigma(\xi)\,d\xi}{\int_0^L \Sigma(\xi)\,d\xi}\right] \tag{2.12}$$

are defined only for $\kappa \neq -b/\mu$. For $\kappa = -b/\mu$, however, the dispersion function becomes irrelevant, and $|\kappa|$ can take any value over the interval (b, ∞), depending on the value of μ. The range $|\kappa| > b$ represents the continuum of the κ-spectrum and it embodies also two other pseudodiscrete [16] eigenvalues $\pm\kappa_1$, which lie in the vicinity of the singular $\kappa = \pm b$ points.

2.2. Weighted integration with respect to μ

Expansion of the solution of (2.1) in Legendre or other orthogonal polynomials has frequently been employed to examine the associated eigenvalue spectrum. It is useful to note that this approach is equivalent to a weighted integration over μ, for example, application of

$$\mathcal{H} = \int_{-1}^{1} P_n(\mu)\, d\mu . \tag{2.13}$$

Moreover, since $\mathcal{H}\{\mathcal{F}f\} = \eta\mathcal{H}f$, which may be rewritten as

$$\mathcal{M}u = \eta u \tag{2.14}$$

with $u = \mathcal{H}f$, cannot be reduced for $\mathcal{F}$ of (2.2) to any of the forms stated in Proposition 1, the spectrum of the η-eigenvalues is not expected to elucidate much about the κ-spectrum. This follows also from the simple fact that the η-eigenvalue spectrum is purely discrete, whereas the κ-spectrum is a mixed discrete-continuous one.

2.3. Integration with respect to α

Consider first the normalisation condition of (2.10) in (1.10) to reduce it to the equivalent Fredholm problem

$$(\mathcal{L} - \lambda)X = Y \tag{2.15}$$

where use has been made of the substitutions

$$\mathcal{L} = \mu, \quad \lambda = -b/\kappa, \quad X = f(\mu), \quad Y = -c\lambda/2 . \tag{2.16}$$

Here the operator $\mathcal{L} = \mu$ is selfadjoint in the Hilbert L^2 space and the solution of (2.13) is given [5] by

$$X = R_\lambda Y = (\mathcal{L} - \lambda)^{-1} Y \tag{2.17}$$

with R_λ being the resolvent of $\mathcal{L}$. The eigenvalue spectrum of this operator is defined by

$$\mathcal{L}v = \mu v = \alpha v \tag{2.18}$$

where

$$v = v_\alpha(\mu) = v(\mu, \alpha) . \tag{2.19}$$

As (2.18) cannot be satisfied by any normal type of eigenfunctions (since we would have to assume v to be zero for all values of μ except $\mu = \alpha$), we shall seek its solution as a generalised function. Since X and Y are independent of α, integration of (2.15) over α (i.e. $\mathcal{H} = \int_{\alpha_0}^{\alpha} d\alpha$) does not affect this equation. The situation is entirely different, however, as far as (2.18) is concerned. Let us therefore integrate (2.18) over the closed interval $[\alpha_0, \alpha]$,

$$\mu \int_{\alpha_0}^{\alpha} v(\mu, \alpha)\, d\alpha = \int_{\alpha_0}^{\alpha} \alpha v(\mu, \alpha)\, d\alpha . \tag{2.20}$$

Substitution of

$$\varepsilon_\alpha(\mu) = \int_{\alpha_0}^{\alpha} v(\mu, \alpha)\, d\alpha \tag{2.21}$$

in the preceding relation leads to

$$\mu \int_{\alpha_0}^{\alpha} d\varepsilon_\alpha(\mu) = \int_{\alpha_0}^{\alpha} \alpha\, d\varepsilon_\alpha(\mu) \tag{2.22}$$

where $d\varepsilon_\alpha(\mu) = d_\alpha \varepsilon_\alpha(\mu)$.

It turns out here that if we assume $\varepsilon_\alpha(\mu)$ to be the discontinuous shifted step function

$$\varepsilon_\alpha(\mu) = \begin{cases} 1 & \alpha \geq \mu \\ 0 & \alpha < \mu \end{cases} \tag{2.23}$$

with a specified semi-continuity on the right with respect to α, that is,

$$\lim_{\alpha \to \mu + 0} \varepsilon_\alpha(\mu) = \varepsilon_\mu(\mu), \tag{2.24}$$

then

$$d\varepsilon_\alpha(\mu) = \delta(\alpha - \mu)\, d\alpha, \tag{2.25}$$

with $\delta(\alpha)$ being the Dirac delta function for which $\int_{\alpha_0}^{\alpha} d\varepsilon_\alpha(\mu) = 1$, and (2.22) reduces therefore to

$$\mu = \int_{\alpha_0}^{\alpha} \alpha\, d\varepsilon_\alpha(\mu). \tag{2.26}$$

By virtue of the sifting property of the Dirac delta function, it is obvious that (2.26) is an identity and the generalised function $\varepsilon_\alpha(\mu)$ satisfies (2.20). Since this $\varepsilon_\alpha(\mu)$ does not diminish when α increases, that is,

$$\varepsilon_\beta(\mu) \geq \varepsilon_\alpha(\mu); \qquad \beta > \alpha, \tag{2.27}$$

then it is essentially a projection operator [1] or "expansion unit" [13] and the integral to the right of (2.26) is a Stieltjes integral.

Note however that (2.26) is nothing but a special case of the basic general formula, derived from the Riesz representation theorem [1, 13], for the whole theory of selfadjoint operators:

$$\mathcal{L} = \int_{\alpha_0}^{\alpha} \alpha\, d\varepsilon_\alpha. \tag{2.28}$$

It is possible to utilise further the theorem on the resolvent of selfadjoint operators [1, 13],

$$R_\lambda = \int_{\alpha_0}^{\alpha} \frac{\alpha}{(\alpha - \lambda)} d\varepsilon_\alpha(\mu) \tag{2.29}$$

where λ represents only the resolvent set of $\mathcal{L}$, in (1.7), (1.8), (2.16) and (2.17) together with (1.4)–(1.8) to finish the proof of the theorem that follows.

THEOREM 1. *The solution of the integrodifferential equation* (1.1) *is a sum, over the resolvent set of the* $\mathcal{L} = \mu$ *operator,*

$$\psi(x, \mu) = \frac{c}{2} \sum_{\lambda} A(\lambda) \exp\left[-\lambda^{-1} \int_0^x \Sigma(\xi)\, d\xi\right] \int_{\alpha_0}^{\alpha} \frac{\lambda\alpha}{(\lambda - \alpha)}\, d\varepsilon_{\alpha}(\mu) \qquad (2.30)$$

where the expansion unit in the Stieltjes integrals corresponds to this operator and $A(\lambda)$ *are coefficients satisfying appropriate orthogonality conditions.*

Substitute finally (2.23) in (2.28) to reduce it to

$$\psi(x, \mu) = \frac{c}{2} \sum_{\lambda} A(\lambda) \exp\left[-\lambda^{-1} \int_0^x \Sigma(\xi)\, d\xi\right] \frac{\lambda\mu}{(\lambda - \mu)}. \qquad (2.31)$$

This relation illustrates clearly how the angular flux, which is proportional to $c/2$, could be related, on one hand, to discrete values of the resolvent set (of the operator $\mathcal{L} = \mu$) only for directions μ different from λ. For $\mu = \lambda$, on the other hand, $\psi(x, \mu)$ is determined by the continuum part of the resolvent λ set, and a part of the sum in (2.31) should accordingly be replaced by an integral.

In conclusion, regardless of the rather new formulation for the eigensolutions of (1.1) that Theorem 1 provides, no new results about the associated eigenvalue spectrum appear to emerge from it. This theorem proposes nevertheless an additional inlet for further research on this subject.

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References

[1] R. G. Bartle, *The elements of real analysis* (Wiley, New York, 1964).
[2] G. I. Bell and S. Glasstone, *Nuclear reactor theory* (Van Nostrand Reinhold, New York, 1970).
[3] V. I. Bilenko, "Applying summator operators to the computer solution of integrodifferential equation of radiation transfer" *Sov. J. Atom. Energy* **53** (1982) 800–803.
[4] K. M. Case and P. F. Zweifel, *Linear transport theory* (Addison-Wesley, Massachusetts, 1967).
[5] P. R. Garabedian, *Partial differential equations* (Wiley, New York, 1964).
[6] R. D. M. Garcia and C. E. Siewert, "Multislab multigroup transport theory with Lth order anisotropic scattering", *J. Comput. Phys.* **50** (1983) 181–192.
[7] N. H. S. Haidar, "A coordinate transform method for one-speed neutron transport in composite slabs", *J. Phys. D: Appl. Phys* **21** (1988) 1543–1549.

[8] H. G. Kaper, C. G. Lekkerkerker and J. Hejtmanek, *Spectral methods in linear transport theory* (Birkhauser Verlag, Basel, 1982).
[9] T. Kikuta, "Extensions of variational methods, I. Super-stationary variational method", *Prog. Theor. Phys.* **14** (1955) 457–472.
[10] J. T. Kriese, C. E. Siewert and Y. Yener, "Two-group critical problems for slabs and spheres in neutron-transport theory", *Nucl. Sci. Eng.* **50** (1973) 3–9.
[11] M. Lemanska, "On the analytic solutions of the monoenergetic neutron transport equation for slab geometry", *J. Appl. Math. Phys.* **30** (1979) 655–661.
[12] S. Pahor and J. K. Shultis, "Half-space general multigroup transport theory", *J. Nucl. Energy* **23** (1969) 477–493.
[13] V. I. Smirnov, *A course of higher mathematics*, Vol. V (Pergamon Press, Oxford, 1964).
[14] C. E. Siewert and P. Benoit, "Multigroup transport theory, I. Basic analysis", *Nucl. Sci. Eng.* **78** (1981) 311–314.
[15] H. F. Weinberger, *Variational methods for eigenvalue approximation* (SIAM, Philadelphia, 1974).
[16] M. M. R. Williams, *Mathematical methods in particle transport theory* (Wiley Interscience, London, 1971).