A NOTE ON THE GENERALIZED (POSITIVE) CAUCHY DISTRIBUTION

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In this note, a certain generalization of the Cauchy distribution is obtained, using the result of Malik [2].

1. Introduction. The generalized gamma distribution having the density

$$(1) \qquad f(x, a, d, p) = \frac{p}{\Gamma(d/p)\, a^d}\, x^{d-1}\, e^{-(x/a)^p}, \quad x > 0;\ a, d, p > 0$$

is introduced by Stacy [1], who studied some of its properties. As remarked by Stacy [1], the familiar gamma, chi, chi-squared, exponential and Weibull distributions are special cases of (1), as are the distributions of certain functions of a normal variable - viz., its positive even powers, its modulus, and all positive powers of its modulus.

Malik [2] obtained the distribution of the ratio $W = X/Y$ where X and Y are independent random variables distributed according to (1) with parameters (a_1, d_1, p) and (a_2, d_2, p). The density of W is (see [2, Eq. 2.6])*

$$(2) \qquad g(w) = \frac{p\left(\frac{a_1}{a_2}\right)^{d_2}}{B\left(\frac{d_1}{p}, \frac{d_2}{p}\right)} \frac{w^{-d_2-1}}{\left[1 + \left(\frac{a_2}{a_1}\right)^{-p} w^{-p}\right]^{\frac{d_1+d_2}{p}}}, \quad w > 0 .$$

It is the purpose of this note to examine Stacy's remark as applied to the distribution of W and to obtain a generalized Cauchy distribution. Incidentally Malik's result (2) may be called a generalization of the beta distribution of the second kind, to which it will reduce if we specialize to $p = 1$ and $a_1 = a_2$.

2. The generalized Cauchy and other allied distributions. It is well known that the ratio of two independent normal random variables has a Cauchy distribution. Further, the ratio of the moduli of two independent normal random variables $U = |X| / |Y|$ has the (positive) Cauchy distribution with the density

*Malik's Eq. 2.6 is slightly incorrect. The factor $\log\left(\frac{a_1}{a_2}\right)$ in the exponent of the denominator is to be multiplied by p.

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(3) $$p(u) = \frac{2}{\pi} \frac{1}{(1+u^2)}, \quad u \geq 0.$$

To pursue Stacy's remark, let X be distributed according to N(0, 1). Then the distribution of $Z = |X|^{\ell}$ for $\ell > 0$ is

(4) $$g(z) = \frac{2}{\ell\sqrt{(2\pi)}} z^{\frac{1}{\ell}-1} \left\{\exp -\tfrac{1}{2} z^{2/\ell}\right\} \quad z \geq 0$$

which is in Stacy's form $f\left(z,\ 2^{\frac{1}{2}\ell},\ \frac{1}{\ell},\ \frac{2}{\ell}\right)$.

Now let X_1 and X_2 be independent standard normal variables and write

$$Y = |X_1|^{\ell} \text{ and } Z = |X_2|^{\ell}, \quad \ell > 0.$$

Then Y and Z are independently and identically distributed according to Stacy's form $f\left(\cdot,\ 2^{\frac{1}{2}\ell},\ \frac{1}{\ell},\ \frac{2}{\ell}\right)$. If we now define

$$W = \frac{Y}{Z} = \left\{\frac{|X_1|}{|X_2|}\right\}^{\ell}$$

the density of W is obtained from Malik's equation 2.6 (our equation (2)) by choosing in particular

$$a_1 = a_2 = 2^{\frac{1}{2}\ell}, \quad d_1 = d_2 = \frac{1}{\ell}, \quad p = \frac{2}{\ell}.$$

Thus the density of W is

(5) $$g(w) = \frac{2}{\ell\ B(\frac{1}{2}, \frac{1}{2})} \cdot \frac{w^{\frac{1}{\ell}-1}}{\left[1 + w^{2/\ell}\right]}, \quad w > 0.$$

For $\ell = 1$, this reduces to equation (3). Thus (5) may be called the generalized (positive) Cauchy distribution.

Next, a generalization of the Beta distribution of the first kind, analogous to (2), may be obtained as follows. Let X and Y be independently distributed according to (1) with parameters (a_1, d_1, p) and (a_2, d_2, p). If we define

$$V = \frac{X}{X+Y} = \frac{1}{1+\frac{Y}{X}} = \frac{1}{1+W'} \quad \text{with } W' = \frac{Y}{X},$$

the distribution of W' is obtained from (2) by interchanging the suffixes 1 and 2 as

$$(2') \qquad g(w') = \frac{p\left(\frac{a_2}{a_1}\right)^{d_1}}{B\left(\frac{d_1}{p}, \frac{d_2}{p}\right)} \frac{(w')^{-d_1-1}}{\left[1+\left(\frac{a_1}{a_2}\right)^{-p} w'^{-p}\right]^{\frac{d_1+d_2}{p}}}, \qquad w' > 0$$

so that the density of V is

$$(6) \qquad p(v) = \frac{p\left(\frac{a_2}{a_1}\right)^{d_1}}{B\left(\frac{d_1}{p}, \frac{d_2}{p}\right)} \frac{v^{d_1-1}(1-v)^{-d_1-1}}{\left[1+\left(\frac{a_2}{a_1}\right)^{p}\left(\frac{v}{1-v}\right)^{p}\right]^{\frac{d_1+d_2}{p}}}, \quad 0 \le v \le 1 .$$

For $a_1 = a_2$ and $p = 1$, (6) reduces to the beta distribution of the first kind.

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