THE DOUBLE SIX OF LINES AND A THEOREM IN EUCLIDEAN PLANE GEOMETRY

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The object of the present paper is to establish the equivalence of the well-known theorem of the double-six of lines in projective space of three dimensions and a certain theorem in Euclidean plane geometry. The latter theorem is of considerable interest in itself for two reasons. In the first place, it is a natural extension of Euler's classical theorem connecting the radii of the circumscribed and the inscribed (or the escribed) circles of a triangle with the distance between their centres. Secondly, it gives in a geometrical form the invariant relation between the circle circumscribed to a triangle and a conic inscribed in the triangle. For a statement of the theorem, see § 13 (4).

1. The configuration of the double-six of lines was discovered by the Swiss mathematician Schläfli in connection with the theory of the twenty-seven lines on a cubic surface.

Schläfli showed that from the twenty-seven lines, two sets of six lines, say

$$a, b, c, d, e, f;$$
$$a', b', c', d', e', f'$$

could be selected (and that in thirty-six ways), with a corresponding to a', b to b', and so on, so that any line of either set of the double-six is met by the five which do not correspond to it in the other set; that is, a' meets b, c, d, e, f; *a meets* b', c', d', e', f'; and so on.

2. Dropping all reference to a cubic surface, we have, in the above double-six, twelve lines and twelve examples of one line meeting five.

But only one *theorem* is involved; for the configuration may be built up as follows. Begin with any line f and take any five lines meeting it, but with no pair of these meeting each other, viz., a', b', c', d', e'. Any four of these is intersected by a second line besides f; we thus obtain a, b, c, d, e. The *theorem* is that a, b, c, d, e are met by a twelfth line f'; in other words, the two sets form a double-six.*

From the datum that no two of a', b', c', d', e' intersect, it is easy to prove that no two of a, b, c, d, e, f intersect; and that no two of a', b', c', d', e', f' intersect.

3. Any line in projective space of three dimensions can be specified by means of six homogeneous coordinates p_{ij}, $(i, j=1, 2, 3, 4;\ i\neq j)$ defined in terms of the homogeneous coordinates (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) of any two points in it by the equations

$$p_{ij}=x_iy_j-x_jy_i.$$

These six coordinates p_{ij} are connected by the homogeneous equation of the second degree

$$p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}=0.$$

It follows that the lines of projective space of three dimensions may be represented by the points of a quadric Q in projective space of five dimensions.

Our main concern is with the condition that two lines should intersect. This condition is in effect that the two points on Q which represent the lines are conjugate with respect to Q.

* Salmon, *Analytic Geometry of Three Dimensions*, Vol. II (5th edition) §§ 534, 536a. H. F. Baker, *Principles of Geometry*, Vol. III, p. 159; Vol. IV, pp. 58-64.

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In fivefold space, instead of a tangent plane to a surface we have a tangent fourfold and the condition that two points on Q are conjugate with respect to Q is that the tangent fourfold at either passes through the other ; or again, that the line joining the points lies entirely on Q. Thus, using the same symbols to denote the lines of § 1 and the points which represent them on Q, the fourfold *bcdef* is the tangent fourfold at a' ; *acdef* is the tangent fourfold at b' ; and so on.

The theorem of the double-six can now be stated in the equivalent form :

If a, b, c, d, e, f are six points on Q, and if *bcdef*, *acdef*, *abdef*, *abcef*, *abcdf* are tangent fourfolds, then *abcde* is also a tangent fourfold.

4. By means of the change of coordinates

$$p_{12}=x+iw,\ p_{13}=y+iu,\ p_{14}=z+it,$$
$$p_{34}=x-iw,\ p_{42}=y-iu,\ p_{23}=z-it,$$

in the five-dimensional space in which Q lies, the equation of Q is brought to the form

$$x^2+y^2+z^2+w^2+t^2+u^2=0, \quad \ldots\ldots(1)$$

By homogeneous linear transformations which leave the quadratic form

$$x^2+y^2+z^2+w^2+t^2+u^2$$

invariant, other coordinate systems can be found in which the equation of Q takes the same form. Among these there are systems for which any given fourfold is one of the coordinate fourfolds, say the fourfold $u=0$. Furthermore, since there exist fourfolds which do not pass through any one of a particular set of six given points, it is possible to choose a coordinate system such that Q has the form (1) and the coordinate fourfold $u=0$ does not pass through any of the six points $a, b, \ldots, f$. Let us choose such a system of coordinates and then introduce non-homogeneous coordinates by replacing the ratios of x, y, z, w, t to u by ix, iy, iz, iw and it respectively, so that, in the non-homogeneous coordinate system, Q has equation

$$x^2+y^2+z^2+w^2+t^2=1.$$

From now on we shall only use, in the five-dimensional space in which Q lies, coordinate systems obtained from this system by orthogonal transformations. We shall introduce the terminology of metric geometry into this space by calling the function

$$(x_1-x_2)^2+(y_1-y_2)^2+\ \ldots\ +(t_1-t_2)^2,$$

in any one of these coordinate systems, and therefore in all of them, the square of the distance between the points $(x_1, y_1, \ldots, t_1)$ and $(x_2, y_2, \ldots, t_2)$ and referring to the coordinate systems as rectangular cartesian coordinate systems.

When this has been done the quadric Q becomes a hypersphere, or as we shall call it to indicate its dimensions, a five-sphere in the, now complex Euclidean, five-dimensional space. It is the five-sphere with centre at the origin and radius unity.

5. The point f (§ 1) can be taken to be the point (0, 0, 0, 0, 1). For if it is not initially this point, a new rectangular cartesian coordinate system can be introduced in which it is. (Compare with the ordinary sphere $x^2+y^2+z^2=a^2$).

The equation of the tangent fourfold to Q at the point $(x_1, y_1, z_1, w_1, t_1)$ is

$$xx_1+yy_1+zz_1+ww_1+tt_1=1. \quad \ldots\ldots(1)$$

Let this pass through f ; then $t_1=1$. But

$$x_1{}^2+y_1{}^2+z_1{}^2+w_1{}^2+t_1{}^2=1\ ;$$

therefore

$$x_1{}^2+y_1{}^2+z_1{}^2+w_1{}^2=0, \quad \ldots\ldots(2)$$

a relation which is fundamental for what follows.

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6. As a first step towards reaching a plane figure, we now project the figure on Q from f onto the fourfold $t=t_0$ where t_0 is a constant. This fourfold we shall refer to as S_4.

The five relations given and the one to be proved, as stated in § 3, have now to be transferred to the figure in S_4.

Let a, b, c, d, e project into A, B, C, D, E. The points B, C, D, E lie on two fourfolds, viz., on the tangent fourfold at a' $(bcdef)$ and on $t=t_0$. Hence B, C, D, E lie on a threefold in the fourfold $t=t_0$; in this fourfold the variable coordinates are x, y, z, w.

If a' is the point $(x_1, y_1, z_1, w_1, t_1)$, the equation of this threefold in terms of these coordinates is, from § 5(1), since $t=t_0$ and $t_1=1$,

$$xx_1+yy_1+zz_1+ww_1=1-t_0, \quad\quad (1)$$

where, by § 5(2),

$$x_1^2+y_1^2+z_1^2+w_1^2=0. \quad\quad (2)$$

Any threefold (in any S_4, i.e., in any fourfold whose equation in some rectangular cartesian coordinate system is $t=t_0$) whose equation is of the form

$$lx+my+nz+pw=q, \quad\quad (3)$$

where

$$l^2+m^2+n^2+p^2=0, \quad\quad (4)$$

we shall call a *special* threefold.

Thus, in virtue of (1) and (2), $BCDE$ is a special threefold; and so also, similarly, are the threefolds $ACDE$, $ABDE$, $ABCE$ and $ABCD$.

7. To make certain that the definition is consistent, it is necessary to verify that, if the condition for a special threefold is satisfied in one rectangular cartesian system, it is true in all those for which the S_4 has equation $t=t_0$. For this and other reasons we consider the effect of the orthogonal transformation relating two such coordinate systems. This transformation is given by the scheme:

	x	y	z	w
X	l_1	m_1	n_1	p_1
Y	l_2	m_2	n_2	p_2
Z	l_3	m_3	n_3	p_3
W	l_4	m_4	n_4	p_4

signifying that

$$X=l_1x+m_1y+n_1z+p_1w,$$
$$x=l_1X+l_2Y+l_3Z+l_4P$$

and so on; where, since

$$x^2+y^2+z^2+w^2=X^2+Y^2+Z^2+W^2,$$
$$l_1^2+m_1^2+n_1^2+p_1^2=1, \quad\quad (1)$$
$$l_1^2+l_2^2+l_3^2+l_4^2=1, \quad\quad (2)$$
$$l_1l_2+m_1m_2+n_1n_2+p_1p_2=0, \quad\quad (3)$$
$$l_1m_1+l_2m_2+l_3m_3+l_4m_4=0, \quad\quad (4)$$

and so on.

We note that if

$$lx+my+nz+pw=LX+MY+NZ+PW,$$

then

$$l^2+m^2+n^2+p^2=L^2+M^2+N^2+P^2,$$

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so that $l^2+m^2+n^2+p^2$, like $x^2+y^2+z^2+w^2$, is invariant under these changes of coordinates. It follows that the definition of special threefolds is consistent.

A comparison of (1) here and § 6 (4) shows that a special threefold cannot be used as a coordinate threefold.

We note also that for two points or two threefolds we have the following invariants:

$$x_1x_2+y_1y_2+z_1z_2+w_1w_2,$$
$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+(w_1-w_2)^2,$$
$$l_1l_2+m_1m_2+n_1n_2+p_1p_2.$$

8. In S_4, the locus with equation

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(w-w_0)^2=C^2 \qquad \text{.........................(1)}$$

is the four-sphere with centre (x_0, y_0, z_0, w_0) and radius C.

The case where $C=0$ is of particular importance; the locus is then at once the four-sphere of radius zero with centre (x_0, y_0, z_0, w_0) and a four-cone, the asymptotic cone of the sphere (1).

The tangent fourfold to Q at any point g on it meets Q in the four-sphere with centre g and radius zero.

To prove this we need only, as we may, introduce in the five-dimensional space, a rectangular cartesian coordinate system in which g has the coordinates given to f in § 5. The tangent fourfold at g is then $t=1$, which meets Q in points at which $x^2+y^2+z^2+w^2=0$, as required.

9. In the figure $ABCDE$ in S_4 we now have five data, viz., that the five threefolds $BCDE$, $ACDE$, $ABDE$, $ABCE$ and $ABCD$ are all special (§ 6). These five data represent in S_4 the five *data* in the original five-dimensional space stated at the end of § 3. We have now to consider the transform to S_4 of the *conclusion* from these data, stated at the same place, viz., that the fourfold $abcde$ is a tangent fourfold to Q.

We assume that this conclusion is correct; that is, we assume that the double-six theorem is true, and deduce from the tangency of $abcde$ that A, B, C, D, E lie on a four-sphere of radius zero.

We first form the equation of the four-cone with vertex at f and base the section of Q by the fourfold $abcde$. Let the equation of this fourfold be

$$lx+my+nz+pw+qt=r. \qquad \text{..(1)}$$

If the origin is changed to f by writing $t=1+\tau$, this equation becomes

$$lx+my+nz+pw+q\tau=r-q; \qquad \text{...................................(2)}$$

and the equation of Q becomes

$$x^2+y^2+z^2+w^2+\tau^2+2\tau=0. \qquad \text{.....................................(3)}$$

Hence, using the method familiar in three-dimensional space, the four-cone has equation

$$x^2+y^2+z^2+w^2+\tau^2+2\tau(lx+my+nz+pw+q\tau)/(r-q)=0,$$

and this meets $t=t_0$, i.e., $\tau=t_0-1$, or $\tau=h$, say, in the four-sphere

$$x^2+y^2+z^2+w^2+h^2+2h(lx+my+nz+pw+qh)/(r-q)=0.$$

The square of its radius is

$$\frac{h^2}{(r-q)^2}(l^2+m^2+n^2+p^2)-h^2-\frac{2qh^2}{(r-q)},$$

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that is,

$$\{h^2/(r-q)^2\}\{l^2+m^2+n^2+p^2-(r-q)^2-2q(r-q)\}. \quad \text{.......................}(4)$$

The second factor of (4) is

$$l^2+m^2+n^2+p^2+q^2-r^2,$$

which is zero, since (1) is a tangent fourfold to Q.

It has therefore been proved, from the double-six theorem, that the four-sphere through A, B, C, D, E has radius zero.

Thus, to return to the beginning of § 9, it has been shown that the equivalent in S_4 of the double-six theorem is that, if the five threefolds $BCDE$, $ACDE$, $ABDE$, $ABCE$ and $ABCD$ are special (end of § 6), then the four-sphere $ABCDE$ has radius zero, i.e., is a four-cone.

10. We now confine our attention to the four-dimensional space S_4 and particularly to the triangle ABC.

We choose, as origin of coordinates in S_4, the centre of the circle ABC and axes such that the plane ABC is the x-y plane having equations $z=0$, $w=0$. This can be done in the following way. Take as Ox and Oy any two perpendicular lines through the origin O and in the plane ABC; for Oz take any line in S_4 perpendicular to the plane ABC and for Ow take either direction on the line in S_4 perpendicular to the lines Ox, Oy, Oz.

In the coordinate system so chosen, let D be the point (x_4, y_4, z_4, w_4) and E the point (x_5, y_5, z_5, w_5), and let D' be the point $(x_4, y_4, 0, 0)$ and E' the point $(x_5, y_5, 0, 0)$. If the coordinate system is changed so that the plane ABC remains the plane $z=0$, $w=0$, the x and y coordinates of D' will still be equal to those of D, and similarly with E' and E.

11. In the type of coordinate system just chosen, the threefold $ABCD$ has equation of the form

$$\lambda z+\mu w=0,$$

for it passes through the plane $z=0$, $w=0$. Since it is a special threefold,

$$\lambda^2+\mu^2=0.$$

Hence $ABCD$ is either $z+iw=0$ or $z-iw=0$, and $ABCE$ is the other. It is merely a matter of notation to take $z+iw=0$ for $ABCD$ and $z-iw=0$ for $ABCE$. Then D is the point (x_4, y_4, z_4, iz_4) and E is the point $(x_5, y_5, z_5, -iz_5)$ (§ 10).

12. The remaining special threefolds $BCDE$, $ACDE$ and $ABDE$ all pass through D and E.

Consider first $BCDE$. Let us choose the x- and y-axes in the plane ABC so that BC has equation $y-y'=0$. It may be remarked that this would not be possible if b, c intersected; but it has been observed in § 2 that they do not. The equation of $BCDE$ has then the form

$$\lambda(y-y')+\mu z+\nu w=0,$$

for it passes through the line $y-y'=0$, $z=0$, $w=0$. Since the threefold is special,

$$\lambda^2+\mu^2+\nu^2=0.$$

Since it passes through D and E,

$$\lambda(y_4-y')+(\mu+i\nu)z_4=0$$

and

$$\lambda(y_5-y')+(\mu-i\nu)z_5=0;$$

hence $\lambda^2(y_4-y')(y_5-y')=(\mu^2+\nu^2)z_4z_5$, or, since $\lambda^2+\mu^2+\nu^2=0$,

$$(y_4-y')(y_5-y')=-z_4z_5. \quad \text{.......................................}(1)$$

If $(\alpha_4, \beta_4, \gamma_4)$ and $(\alpha_5, \beta_5, \gamma_5)$ are the ordinary trilinear coordinates of D' and E', (1) gives

$$\alpha_4\alpha_5=-z_4z_5.$$

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Similarly, $\beta_4\beta_5 = -z_4z_5$

and $\gamma_4\gamma_5 = -z_4z_5$.

D' and E' are therefore isogonal points of the triangle ABC. This property represents the *data* of the double-six theorem (§ 9).

If k^2 is the common value of $\alpha_4\ \alpha_5$, $\beta_4\ \beta_5$ and $\gamma_4\ \gamma_5$, then

$$z_4z_5 = -k^2. \qquad \text{.........(2)}$$

13. Next for the *conclusion* (§ 9) of that theorem, which was shown to be represented by the property that the four-sphere $ABCDE$ has zero radius.

What does this give in the triangle ABC?

The equation of the sphere $ABCDE$ has the form

$$(x^2+y^2-R^2)+z^2+w^2+2pz+2qw=0, \qquad \text{.........(1)}$$

for it contains the circle whose equations are

$$x^2+y^2-R^2=0,\ z=0,\ w=0,$$

R^2 being the square of the radius of the circumcircle of the triangle ABC.

(i) The sphere (1) is to pass through the points $D(x_4, y_4, z_4, iz_4)$ and $E(x_5, y_5, z_5, -iz_5)$. For both of these points $z^2+w^2=0$. Hence, from (1),

$$x_4{}^2+y_4{}^2-R^2+2z_4(p+iq)=0$$

and

$$x_5{}^2+y_5{}^2-R^2+2z_5(p-iq)=0.$$

Thus

$$(x_4{}^2+y_4{}^2-R^2)(x_5{}^2+y_5{}^2-R^2)=4z_4z_5(p^2+q^2)$$
$$= -4k^2(p^2+q^2), \qquad \text{.........(2)}$$

from § 12 (2).

(ii) The radius of (1) is zero ; i.e.,

$$p^2+q^2+R^2=0. \qquad \text{.........(3)}$$

From (2) and (3),

$$(x_4{}^2+y_4{}^2-R^2)(x_5{}^2+y_5{}^2-R^2)=4k^2R^2, \qquad \text{.........(4)}$$

that is to say, we have the theorem :

In a plane triangle, the product of the powers, with respect to the circumcircle, of two isogonal points is equal to $4k^2R^2$; where, if (α, β, γ) and $(\alpha', \beta', \gamma')$ are the trilinear coordinates of the isogonal points, k^2 is the common value of $\alpha\alpha'$, $\beta\beta'$ and $\gamma\gamma'$, and R^2 is the square of the radius of the circumcircle.

The preceding analysis can be regarded as a deduction of this theorem from the theorem of the double-six (§ 3, last sentence). But the analysis could be reversed, so the two theorems are equivalent.

Further, instead of thinking of D' and E' as isogonal points of the triangle, we may, in virtue of a fundamental focal property of conics, consider them to be the foci of a conic inscribed in the triangle, in which case k^2 is b^2, the square of the minor semi-axis.

If the conic is a circle, the theorem (4) at once reduces to the familiar theorem, due to Euler :

$$(R^2-\delta^2)^2=4R^2\rho^2,$$

where δ^2 is the square of the distance between the circumcentre and the centre of the inscribed or of an escribed circle, and ρ is the radius of the inscribed or escribed circle.

14. An algebraic proof of the theorem of § 13 (4) may be derived from the invariant theory of two conics.*

If the conics are

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,$$
$$S' \equiv (x - \alpha)^2 + (y - \beta)^2 - r^2,$$

the four invariants are :

$$\Delta = -\frac{1}{a^2b^2}, \quad \Theta = \frac{1}{a^2b^2}(\alpha^2 + \beta^2 - a^2 - b^2 - r^2),$$
$$\Theta' = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1 - r^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right), \ \Delta' = -r^2.$$

The origin being at the centre of the ellipse, the coordinates of the foci are $\pm ae$, 0 (where $a^2e^2 = a^2 - b^2$), and those of the centre of the circle are α, β.

The condition that a triangle can be circumscribed to S and inscribed in S' is :

$$\Theta^2 = 4\Delta\Theta'. \quad \text{..........(1)}$$

The expression of this formula in terms of a, b, α, β, r does not lend itself to immediate geometrical interpretation. We know, however, from the analysis in this paper that it must be equivalent to § 13 (4), with $k^2 = b^2$; that is, to

$$\{(\alpha - ae)^2 + \beta^2 - r^2\}\{(\alpha + ae)^2 + \beta^2 - r^2\} = 4b^2r^2. \quad \text{..........(2)}$$

The proof that (1) and (2) are the same is a matter of the simplest algebra.

* Salmon, *Conic Sections*, Chapter on *Invariants and Covariants of Systems of Conics.*

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