TOPOLOGICAL H-SURFACES

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1. Introduction. An H-space is a topological space T for which it is possible to define a continuous binary composition

$$f : T \times T \rightarrow T$$

with the following properties: there exists a homotopy unit, i.e. an element $u \in T$ such that

(a) $f(u, u) = u$ (u is an idempotent), and

(b) the two continuous maps $T \rightarrow T$ defined by $l_u(z) = f(u, z)$ and $r_u(z) = f(z, u)$ are homotopic with the identity map $I(z)$ of T relative to u.

If T is arcwise connected, it follows that the maps $l_a(z) = f(a, z)$ and $r_a(z) = f(z, a)$ are homotopic to $I(z)$ for all $a \in T$.

The main examples of H-spaces are topological groups and loop spaces. Their definition is due to J.-P. Serre **(5)** and is based on the concept of Γ-manifold introduced by H. Hopf in **(2)**, with a generalization to Γ-complexes by S. Lefschetz **(4)**.

If, in addition to the above conditions, there exists a continuous map $\xi(z) : T \rightarrow T$ such that the combined map $z \rightarrow f[z, \xi(z)]$ is homotopic to a constant map in T, we speak of an H-structure with homotopy inversion.

By a (topological) surface we shall mean a connected 2-dimensional manifold with a countable basis for its open sets. We shall show that there are exactly four non-homeomorphic types of H-surfaces S. Imposing differentiability conditions (such as S belonging to a class $C^{(k)}$, C^{∞}, or C^{ω}) does not alter this situation, since all our explicit binary compositions, as well as the connecting homotopies can be chosen so as to belong to C^{ω}, and since a topological type of surface can carry at most one differentiable structure of class C^{ω}. A change does occur, however, if one assumes S to be complex analytic, with appropriate restrictions on the multiplications and homotopies so as to make this concept of "analytic H-surface" conformally invariant. We have discussed this modification in **(1)**.

Although neither associativity nor commutativity are used in the derivation of these results, it turns out that all H-surfaces can carry compositions with these properties. Of course, none of the multiplications is unique for a given surface.

2. The fundamental groups. An arcwise connected H-space S possesses an abelian fundamental group $\pi_1(S)$. This is a special case of the fact that an

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H-space is n-simple for every $n > 0$; cf. **(3)**. Hence all H-surfaces must be found among the six known types of surfaces with abelian fundamental groups, viz.: Cartesian plane, circular cylinder, sphere, torus, projective plane, Moebius strip. We examine each of these cases separately.

3. Enumeration of H-surfaces.

(a) The plane carries, for example, the topological group of the complex numbers under addition. Similarly, the circular cylinder can be realized as the Euclidean plane punctured at the origin, and is therefore capable of carrying the group of the complex numbers under multiplication.

(b) The 2-sphere is not an H-surface, for only odd-dimensional spheres are Γ-manifolds **(2)**.

(c) The torus can be represented as the topological product $S^1 \times S^1$ of two unit circles $|z| = 1$. It can therefore be given the group structure of the direct product into itself of the multiplicative group of complex numbers of modulus one.

(d) If X is an H-space with composition f and Y a covering space of X with projection mapping $p : Y \to X$, it is possible to define an H-structure g in Y by lifting f; i.e. such that in the diagram (in general non-commutative)

$$\begin{array}{ccc} & g & \\ Y \times Y & \rightarrow & Y \\ p \times p \downarrow & & \downarrow p \\ X \times X & \rightarrow & X \\ & f & \end{array}$$

the two possible maps of $Y \times Y$ into X are homotopic. Applying this remark to X = projective plane and Y = sphere, we conclude that the projective plane cannot carry an H-structure either.

(e) Finally the Moebius strip M cannot be made into a topological group because of its non-orientability. We are going to show, however, that it can carry an H-structure all the same, even with a homotopy inversion.

Let M be represented as quotient space of the z-plane $\dot{E}$ punctured at the origin with respect to the group G of covering transformations consisting of

$$I(z) = z \quad \text{and} \quad T(z) = (-1)/\bar{z} \quad (\bar{z} = \text{conjugate complex to } z).$$

The associative and commutative composition defined in $\dot{E}$ by

$$f(z_1, z_2) = z_1 z_2 |z_1 z_2|^{-1}$$

satisfies $f \neq 0$, $f(\pm 1, \pm 1) = \pm 1$, and

$$f[T(z_1), z_2] = f[z_1, T(z_2)] = T[f(z_1, z_2)].$$

Hence it covers a continuous composition in M with idempotent 1. A homotopy in $\dot{E}$ carrying $I(z)$ into $f(1, z)$ relative to $u = 1$ is defined by

$$h_t(z) = |z|^{-t} z, \qquad 0 \leqslant t \leqslant 1.$$

It is automorphic with respect to the group G, i.e.

$$h_t[T(z)] = T[h_t(z)] \qquad \text{for all } z \neq 0 \text{ and } 0 \leqslant t \leqslant 1.$$

Therefore $h_t(z)$ covers an analogous homotopy in M.

Letting

$$\xi(z) = \bar{z}\,|z|^{-1}, \qquad z \neq 0,$$

the following identities in z hold:

$$\xi[T(z)] = T[\xi(z)] \quad \text{and} \quad f[z, \xi(z)] = 1;$$

we conclude that $\xi(z)$ covers a homotopy inversion in M.

Summing up, we have proved:

THEOREM. *Every H-surface is homeomorphic to one of the following four types: plane, circular cylinder, torus, Moebius strip. The first three types can be given abelian topological group structures; the Moebius strip can be made into a commutative topological semigroup with homotopy unit and homotopy inversion, but not into a topological group.*

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