A CHARACTERIZATION OF THE DEFINITENESS OF A HERMITIAN MATRIX

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1. Introduction. We denote by F the field R of real numbers, the field C of complex numbers or the skew-field H of real quaternions, and by F^n an n-dimensional left vector space over F. If A is a matrix with elements in F, we denote by A^* its conjugate transpose. In all three cases of F, an $n \times n$ matrix A is said to be *hermitian* (*unitary* resp.) if $A = A^*$ ($AA^* =$ identity matrix resp.). An $n \times n$ hermitian matrix A is said to be *definite* (*semidefinite* resp.) if $uAu^*vAv^* > 0$ ($uAu^*vAv^* \geqq 0$ resp.) for all nonzero u and v in F^n. If A and B are $n \times n$ hermitian matrices, then we say that A and B can be diagonalized simultaneously into blocks of size less than or equal to m (abbreviated to d.s. $\leqq m$) if there exists a nonsingular matrix U with elements in F such that $UAU^* = \text{diag}\{A_1, \ldots, A_k\}$ and $UBU^* = \text{diag}\{B_1, \ldots, B_k\}$, where, for each $i = 1, \ldots, k$, A_i and B_i are of the same size and the size is $\leqq m$. In particular, if $m = 1$, then we say A and B can be diagonalized simultaneously (abbreviated to d.s.).

The purpose of this note is to give a characterization of the definiteness of a hermitian matrix in terms of simultaneous diagonalization. (For other characterizations see, for example, [3].)

2. Characterization of the definiteness of a hermitian matrix. In the following we shall use A, B and X to denote $n \times n$ hermitian matrices. We now give a characterization of the definiteness of a hermitian matrix.

THEOREM 1. *Let $A \neq O$. Then A is definite if and only if A and X can be diagonalized simultaneously for all X.*

Proof. Since any hermitian matrix can be diagonalized by a unitary matrix (for $F = R$ or C, this is well known, for $F = H$, see [4] or [6]), if A is definite, then A and X can be d.s. Conversely, if A is not definite, then we may assume that

$$A = \text{diag}\{A_1, A_2\}, \quad \text{where} \quad A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now take

$$X = \text{diag}\{X_1, O\}, \quad \text{where} \quad X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then A_1 and X_1 cannot be d.s. and hence (see [1] or [2]) A and X cannot be d.s.

In order to give a characterization of the semidefiniteness of a hermitian matrix, we need the following lemmas.

LEMMA 1. *Let*

$$A_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and

A

$$A_2=\begin{pmatrix}1&&\\&1&\\&&-1\end{pmatrix},\qquad X_2=\begin{pmatrix}1&-1&1\\-1&3&-2\\1&-2&1\end{pmatrix}.$$

Then A_i and X_i cannot be d.s. $\leqq 2$ for $i=1,2$.

Proof. Suppose that A_i and X_i can be d.s. $\leqq 2$ by a nonsingular matrix $U_i=\begin{pmatrix}u_i\\v_i\\w_i\end{pmatrix}$, where $i=1$ or 2, and u_i, v_i, w_i form a basis of F^3. Then at least one of them, say u_i, possesses the property that

$$u_iA_iv_i^*=u_iA_iw_i^*=u_iX_iv_i^*=u_iX_iw_i^*=0.$$

Hence the subspace $L\{u_iA_i,u_iX_i\}$ spanned by u_iA_i and u_iX_i is of dimension less than or equal to 1 and v_i and w_i are in the orthogonal complement of this subspace. But, by direct calculation, it can be easily seen that such a basis cannot exist. This completes the proof.

The following lemma is obvious.

LEMMA 2. *Let $A=\operatorname{diag}\{A_1,O\}$ and $B=\operatorname{diag}\{B_1,O\}$, where A_1 and B_1 are of the same size. If A and B can be d.s. $\leqq 2$, then so can A_1 and B_1.*

LEMMA 3. *Let $A=\operatorname{diag}\{A_1,A_2\}$ and $B=\operatorname{diag}\{B_1,O\}$, where A_1 and B_1 are of size k, and let B_1 be nonsingular. If A and B can be d.s. $\leqq 2$, then so can A_1 and B_1.*

Proof. By Lemma 2, without loss of generality, we may assume that A_2 is nonsingular. Suppose that A and B can be d.s. $\leqq 2$ by the matrix U. Let u_i $(i=1,\ldots,n)$ be the ith row of U. Then $\{u_1,\ldots,u_n\}$ is a basis of F^n. Now put $u_i=(x_i,y_i)$, where $x_i\in F^k$ and $y_i\in F^{n-k}$. Let $\{x_{i_1},\ldots,x_{i_k}\}$ be a basis of F^k and let M be the $k\times k$ matrix with rows $x_{i_1},\ldots,x_{i_k}$; then M is nonsingular. We shall show that A_1 and B_1 are d.s. $\leqq 2$ by M.

Let $\{j_1,\ldots,j_{n-k}\}=\{1,\ldots,n\}\backslash\{i_1,\ldots,i_k\}$. We first show that U can be chosen so that it shall have $x_{j_t}=O$ for $t=1,\ldots,n-k$. To prove this we consider the matrix UBU^*, noting that its (j,h)th entry is $x_jB_1x_h^*$. From the " diagonalized $\leqq 2$ " form of UBU^* we see that, for any particular j_t, either (i) $x_{j_t}B_1x_h^*=0$ for all $h\in\{1,\ldots,n\}\backslash\{j_t\}$ or (ii) there is an l, with $l=j_t-1$ or $l=j_t+1$, such that $x_{j_t}B_1x_h^*=0$ for all $h\in\{1,\ldots,n\}\backslash\{j_t,l\}$.

If (i) holds or if (ii) holds with $l\in\{j_1,\ldots,j_{n-k}\}$, then $x_{j_t}B_1x_h^*=0$ for all $h\in\{i_1,\ldots,i_k\}$ and hence, since B_1 is nonsingular and the k columns $x_{i_1}^*,\ldots,x_{i_k}^*$ are linearly independent, we have $x_{j_t}=O$.

If (ii) holds with $l\in\{i_1,\ldots,i_k\}$, then $x_{j_t}B_1x_h^*=0$ and $x_lB_1x_h^*=0$ for all $h\in\{i_1,\ldots,i_k\}\backslash\{l\}$ and hence, since B_1 is nonsingular and the $k-1$ columns x_h^* with $h\in\{i_1,\ldots,i_k\}\backslash\{l\}$ are linearly independent, x_{j_t} and x_l are linearly dependent, so that there is a $\lambda\in F$ such that $x_{j_t}=\lambda x_l$. Let $\tilde{u}_{j_t}=u_{j_t}-\lambda u_l=(O,\tilde{y}_{j_t})$. Then A and B are d.s. $\leqq 2$ by the matrix with rows $u_1,\ldots,u_{j_t-1}$, $\tilde{u}_{j_t}$, $u_{j_t+1},\ldots,u_n$. This result, applied in turn to each of the rows u_{j_t} of U for which $x_{j_t}\neq O$, establishes the existence of a matrix, which we shall again call U, in which the rows u_{j_t} $(t=1,\ldots,n-k)$ all have $x_{j_t}=O$ and the rows $u_{i_1},\ldots,u_{i_k}$ are as in the original matrix U, and which is such that A and B are d.s. $\leqq 2$ by U.

We assume, then, that U has each u_{j_t} of the form (O, y_{j_t}) and note that, since $u_{j_1}, \ldots, u_{j_{n-k}}$ are linearly independent, $\{y_{j_1}, \ldots, y_{j_{n-k}}\}$ is a basis of F^{n-k}.

We prove that A_1 and B_1 are d.s. $\leqq 2$ by M, by showing that, if $r \neq s$ and

$$u_{i_r} A u_{i_s}^* = u_{i_r} B u_{i_s}^* = 0,$$

then

$$x_{i_r} A_1 x_{i_s}^* = x_{i_r} B_1 x_{i_s}^* = 0.$$

For each i_r, we have two cases:

Case 1. $u_{i_r} A u_{j_t}^* = 0$ for all $t = 1, \ldots, n-k$.
In this case, we have

$$y_{i_r} A_2 y_{j_t}^* = 0 \quad \text{for all} \quad t = 1, \ldots, n-k,$$

and evidently $y_{i_r} = O$ as A_2 is nonsingular. Therefore

$$x_{i_r} A_1 x_{i_s}^* = x_{i_r} B_1 x_{i_s}^* = 0 \quad \text{if} \quad u_{i_r} A u_{i_s}^* = u_{i_r} B u_{i_s}^* = 0.$$

Case 2. There is some j_t such that $u_{i_r} A u_{j_t}^* \neq 0$.
In this case j_t is either $i_r - 1$ or $i_r + 1$, and the subspace spanned by $u_{i_r} A$, $u_{i_r} B$, $u_{j_t} A$, $u_{j_t} B$ is of dimension 2 with a basis $\{u_{i_r} B, u_{j_t} A\}$. Hence there exist a and b in F such that

$$u_{i_r} A = a u_{i_r} B + b u_{j_t} A,$$

and, by using the fact that A_2 is nonsingular, we have $y_{i_r} = b y_{j_t}$. If $u_{i_r} A u_{i_s}^* = u_{i_r} B u_{i_s}^* = 0$, then

$$u_{j_t} A u_{i_s}^* = u_{j_t} B u_{i_s}^* = 0 \text{ as } r \neq s,$$

and hence

$$(u_{i_r} - b u_{j_t}) A u_{i_s}^* = (u_{i_r} - b u_{j_t}) B u_{i_s}^* = 0.$$

Consequently, we have

$$x_{i_r} A_1 x_{i_s}^* = x_{i_r} B_1 x_{i_s}^* = 0.$$

THEOREM 2. (a) *If A is semidefinite, then A and X can be diagonalized simultaneously into blocks of size $\leqq 2$ for all X.* (b) *If $n \geqq 3$ and A and X can be diagonalized simultaneously into blocks of size $\leqq 2$ for all X, then A is semidefinite.*

Proof. (a) If $n \leqq 2$, then the first statement is obviously true. We now suppose it holds for all $k < n$ with $n > 2$. Let A be semidefinite and of size n. Then, without loss of generality, we may assume that

$$A = \operatorname{diag}\{I_m, O\} \quad \text{and} \quad X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix $(0 < m < n)$ and X_{11} is of size m. If $X_{22} \neq O$, then obviously we can reduce the problem to the case for $n-1$, and hence, by our assumption, statement (a) is true. We now assume that

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & O \end{pmatrix}$$

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with $X_{12} \neq O$. Let v be a nonzero vector in F^{n-m} such that $vX_{12}^* \neq O$, and let $u_1 = (O, v)$ and $u_2 = (vX_{12}^*, O)$ be vectors in F^n. Then

$$u_1 A = O,\ u_1 X = u_2 A = (vX_{12}^*, O) \text{ and } u_2 X = (vX_{12}^* X_{11}, vX_{12}^* X_{12}).$$

Since $vX_{12}^* \neq O$, we have $vX_{12}^* X_{12} v^* = vX_{12}^*(vX_{12}^*)^* \neq 0$. Hence $vX_{12}^* X_{12} \neq O$ and the vectors (vX_{12}^*, O) and $(vX_{12}^* X_{11}, vX_{12}^* X_{12})$ are linearly independent. Therefore the subspace L spanned by $u_1 A$, $u_1 X$, $u_2 A$ and $u_2 X$ is of dimension 2. If we denote by $L^\perp$ the orthogonal complement of L and take a basis $\{u_3, \ldots, u_n\}$ in $L^\perp$, then it can easily be seen that $\{u_1, \ldots, u_n\}$ is a basis in F^n. Let U be the matrix whose ith row is u_i ($i = 1, \ldots, n$). Then U is nonsingular with elements in F, and $UAU^* = \operatorname{diag}\{M_1, M_2\}$, $UXU^* = \operatorname{diag}\{N_1, N_2\}$, where M_1 and N_1 are 2×2 matrices and M_2 is semidefinite. By the induction assumption M_2 and N_2 can be d.s. $\leqq 2$, and therefore so can A and X.

(b) If $n \geqq 3$ and A is not semidefinite, then we may assume that

$$A = \operatorname{diag}\{A_1, A_2\},$$

where

$$A_1 = A_1^{(1)} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad \text{or} \quad A_1^{(2)} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

Take

$$X = \operatorname{diag}\{X_1, O\},$$

where

$$X_1 = X_1^{(1)} = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{or} \quad X_1^{(2)} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 1 \end{pmatrix}.$$

By Lemma 1, $A_1^{(k)}$ and $X_1^{(k)}$ cannot be d.s. $\leqq 2$ for $k = 1, 2$. Hence by Lemma 3, A and X cannot be d.s. $\leqq 2$. Thus the theorem is proved.

Since it is obvious that, if A and B are semidefinite and of size 2, then A and B are d.s., the following theorem, which is already known (see [2] or [5]), follows immediately from Theorem 2, Part (a).

THEOREM 3. *If A and B are semidefinite, then A and B can be diagonalized simultaneously.*

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