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SPACES OF SPECIAL QUADRILATERALS

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Abstract

We describe the parameter spaces of some families of quadrilaterals, such as parallelograms, rectangles, rhombuses, cyclic quadrilaterals and trapezoids. For this purpose, we prove that the closed n-disc $\mathbb{D}^n$ is the unique topological n-manifold (with boundary) whose boundary and interior are homeomorphic to $\mathbb{S}^{n-1}$ and $\mathbb{R}^n$, respectively. Roughly speaking, our main result states that the natural compactifications of the parameter spaces of cyclic quadrilaterals and of trapezoids, modulo similarity, are both homeomorphic to $\mathbb{D}^3$.

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1. Introduction

Let $\overline{pq} \subset \mathbb{C}$ be the segment between $p, q \in \mathbb{C}$. Associate the point $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4$ to the set $\overline{z_1 z_2} \cup \overline{z_2 z_3} \cup \overline{z_3 z_4} \cup \overline{z_4 z_1}$. Thus, the set of *quadrilaterals* contained in the plane inherits a natural topology from $\mathbb{C}^4$.

Let $\mathcal{A}_{\mathbb{C}} := \{f(z) = az + b \mid a \in \mathbb{C}^*, b \in \mathbb{C}\}$ be the complex affine group (where as usual $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$). Consider the $\mathcal{A}_{\mathbb{C}}$-action in $\mathbb{C}^4$ given by

$$(f, (z_1, z_2, z_3, z_4)) \mapsto (az_1 + b, az_2 + b, az_3 + b, az_4 + b).$$

The quotient

$$P(4) := (\mathbb{C}^4 \setminus \{(z, z, z, z) \mid z \in \mathbb{C}\})/\mathcal{A}_{\mathbb{C}} \tag{1.1}$$

will be called *the space of shapes of quadrilaterals*, which can be interpreted as the set of quadrilaterals with consecutively labelled vertices up to oriented similarity.

Notice that $P(4)$ is biholomorphic to the complex projective plane $\mathbb{CP}^2$ (since $Z, W \in V := \{(0, z_2, z_3, z_4) \in \mathbb{C}^4\}$ belong to the same $\mathcal{A}_{\mathbb{C}}$-orbit if and only if $Z = aW$ with $a \in \mathbb{C}^*$, so $P(4)$ is the complex projectivisation of V).

Let $\mathcal{S}, \mathcal{K} \subset P(4)$ be the subsets corresponding respectively to simple and convex positively oriented quadrilaterals. These subsets were described in [3] and the following properties of $\mathcal{K}$ were proved:

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(1) the interior $\mathcal{K}^\circ \subset P(4)$ is homeomorphic to the Euclidean space $\mathbb{R}^4$;
(2) the boundary $\partial\mathcal{K} \subset P(4)$ is homeomorphic to the 3-sphere $\mathbb{S}^3$;
(3) consequently the closure $\overline{\mathcal{K}} \subset P(4)$ is homeomorphic to the closed disc $\mathbb{D}^4$ (by a result of Freedman from the topology of 4-manifolds).

This paper is devoted to understanding the subsets of $P(4)$ corresponding to parallelograms, rectangles, rhombuses, cyclic quadrilaterals and trapezoids. The result of Freedman used in [3] is now replaced with arguments covered in Appendix A. In particular, we use classical results to prove that the closed n-disc $\mathbb{D}^n$ is the unique topological manifold (with boundary) M satisfying $M^\circ = \mathbb{R}^n$ and $\partial M = \mathbb{S}^{n-1}$. We exploit this fact to show that the closures in $P(4)$ of the sets corresponding to positively oriented cyclic quadrilaterals and trapezoids are both homeomorphic to $\mathbb{D}^3$.

2. Preliminaries

We recall basic concepts from [3] to provide a self-contained exposition.

Think of $Z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$ as a quadrilateral whose consecutive vertices are $z_1, z_2, z_3, z_4 \in \mathbb{C}$ and whose edges are the segments $\overline{z_1z_2}, \overline{z_2z_3}, \overline{z_3z_4}, \overline{z_4z_1}$. Let $\mathfrak{c}(Z)$ denote the closed curve $\overline{z_1z_2} \cup \overline{z_2z_3} \cup \overline{z_3z_4} \cup \overline{z_4z_1}$.

Definition 2.1. Let $Z = (z_1, z_2, z_3, z_4)$ be a quadrilateral.

(1) We say that Z is *simple* if its vertices are distinct and $\mathfrak{c}(Z)$ is a Jordan curve. Let $\widetilde{\mathcal{S}} \subset \mathbb{C}^4$ denote the subset of simple quadrilaterals.
(2) A simple quadrilateral Z is *convex* if its diagonals $\overline{z_1z_3}, \overline{z_2z_4}$ are contained in $\mathfrak{c}(Z) \cup (\mathrm{int}Z)$, where $\mathrm{int}Z$ is the bounded component of $\mathbb{C} \setminus \mathfrak{c}(Z)$. Let $\widetilde{\mathcal{K}} \subset \mathbb{C}^4$ denote the subset of convex quadrilaterals.
(3) A simple quadrilateral Z is *positively* or *negatively oriented* if the increasing order on its vertices determines the counter-clockwise or clockwise (respectively) direction along $\mathfrak{c}(Z)$.
(4) The quadrilaterals whose vertices are collinear will be called *4-segments*.

It is easily seen that $\widetilde{\mathcal{S}}$ is the union of two open connected components (see [3]). These components correspond in fact to positively and negatively oriented quadrilaterals, and we can use $(z_1, z_2, z_3, z_4) \mapsto (z_1, z_4, z_3, z_2)$ to define a homeomorphism between them. Clearly, $\widetilde{\mathcal{K}}$ is contained in $\widetilde{\mathcal{S}}$ and also has two homeomorphic components.

Let $\eta\colon \mathbb{C}^4 \setminus \{(z, z, z, z) \mid z \in \mathbb{C}\} \to P(4)$ be the quotient map arising from (1.1). The notions of simple, convex and positively or negatively oriented are preserved by the action of $\mathcal{A}_{\mathbb{C}}$ in $\mathbb{C}^4$. Thus, $\eta(\widetilde{\mathcal{K}}) \subset P(4)$ is also the union of two homeomorphic connected components. Let $\mathcal{K} \subset \eta(\widetilde{\mathcal{K}})$ be the component corresponding to positively oriented convex quadrilaterals.

Let $[z_1, z_2, z_3, z_4] = \eta\,(z_1, z_2, z_3, z_4)$. The elements of $P(4)$ will be called *shapes*.

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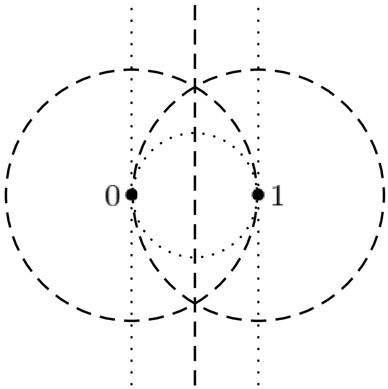

FIGURE 1. Shapes $[0, 1, z]$ with z in the dashed (respectively dotted) lines or circles represent isosceles (respectively right) triangles.

REMARK 2.2 (A choice of local coordinates). Let $\phi\colon \mathcal{U} \to \mathbb{C}^2$ be the local coordinates of $P(4)$ given by

$$[z_1, z_2, z_3, z_4] \mapsto \left(\frac{z_3 - z_1}{z_2 - z_1}, \frac{z_4 - z_1}{z_2 - z_1}\right), \quad \text{where } \mathcal{U} = \{[z_1, z_2, z_3, z_4] \mid z_1 \neq z_2\}.$$

Clearly, $\mathcal{S} \subset \mathcal{U}$ and ϕ restricts to an embedding of $\mathcal{S}$ into $\mathbb{C}^2$. The expression of ϕ is simplest for representatives $[0, 1, z_3, z_4] \in \mathcal{U}$ because $\phi[0, 1, z_3, z_4] = (z_3, z_4)$. Henceforth, we mostly use these local coordinates and these representatives in $\mathcal{U}$. However, we must be careful about limit points of $A \subset \mathcal{U}$ which satisfy $z_1 = z_2$, that is, $\overline{A} \not\subset \mathcal{U}$. For example, $[0, 0, 1, 1] = [1, 1, 0, 0] \notin \mathcal{U}$ is the limit of a sequence of rectangles $\{[0, 1, 1 + ik, ik]\}_{k\in\mathbb{N}} \subset \mathcal{U}$.

Let $\Re(p)$ and $\Im(p)$ denote respectively the real and imaginary parts of $p \in \mathbb{C}$.

EXAMPLE 2.3 (Shapes of triangles). The same arguments allow us to explore the space of shapes of triangles $P(3) := (\mathbb{C}^3 \setminus \{(z, z, z) \mid z \in \mathbb{C}\})/\mathcal{A}_{\mathbb{C}}$. In fact, $P(3)$ is biholomorphic to $\mathbb{CP}^1$, which is homeomorphic to the sphere $\mathbb{S}^2$. For example, $P(3)$ can be seen as $\{[0, 1, z] \mid z \in \mathbb{C} \cup \{\infty\}\} = \{[0, 1, z] \mid z \in \mathbb{C}\} \cup \{[0, 0, 1]\}$. There are three possibilities:

- $\Im(z) > 0$ corresponds to positively oriented triangles;
- $\Im(z) < 0$ corresponds to negatively oriented triangles;
- $\Im(z) = 0$ corresponds to degenerate triangles with collinear vertices.

Thus, the space of shapes of triangles $\mathbb{S}^2$ is divided by a circle of degenerated shapes (the equator of $\mathbb{S}^2$) into two open discs (the hemispheres of $\mathbb{S}^2$) corresponding to the two orientations of triangles. Two sets of special triangles are shown in Figure 1.

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3. Parallelograms, rectangles and rhombuses

Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. The shapes of quadrilaterals whose vertices are collinear will be called *4-segments*.

The space of shapes of positively oriented *parallelograms* is given by

$$\mathcal{P} = \{[0, 1, z, z-1] \in \mathcal{K} \mid z \in \mathbb{H}^2\}.$$

Thus, $\partial\mathcal{P} = \{[0, 1, t, t-1] \mid t \in \mathbb{R} \cup \{\infty\}\}$ is a circle consisting of 4-segments, where $t = \infty$ determines the shape $[0, 0, 1, 1]$. It follows that $\overline{\mathcal{P}} \subset P(4)$ is homeomorphic to $\mathbb{D}^2$.

The spaces of shapes of positively oriented *rectangles* and *rhombuses* are respectively

$$\mathcal{R}_e = \{[0, 1, 1+it, it] \mid t > 0\} \quad \text{and} \quad \mathcal{R}_h = \{[0, 1, 1+e^{i\theta}, e^{i\theta}] \mid \theta \in (0, \pi)\}.$$

Then we have $\partial\mathcal{R}_e = \{[0, 1, 1, 0], [0, 0, 1, 1]\}$, $\partial\mathcal{R}_h = \{[0, 1, 2, 1], [0, 1, 0, -1]\}$ and also $\mathcal{R}_e \cap \mathcal{R}_h = \{[0, 1, 1+i, i]\}$.

4. Cyclic quadrilaterals

Let $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ and $\overline{\mathbb{H}^n} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ for $n \geq 1$.

A simple quadrilateral is *cyclic* if its vertices lie on a circle. Let $C \subset P(4)$ denote the set of shapes of positively oriented cyclic quadrilaterals.

THEOREM 4.1. *$\overline{C} \subset P(4)$ is homeomorphic to $\mathbb{D}^3$.*

PROOF. Every cyclic quadrilateral has a shape of the form $[1, e^{i\alpha}, e^{i\beta}, e^{i\gamma}]$, where $0 < \alpha < \beta < \gamma < 2\pi$, so we view C as $\{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid 0 < \alpha < \beta < \gamma < 2\pi\}$, which clearly is homeomorphic to $\mathbb{R}^3$.

There are two types of shapes in ∂C:

(i) positively oriented quadrilaterals with two coinciding consecutive vertices;
(ii) 4-segments which are limits of cyclic quadrilaterals.

There are four possibilities for shapes of Type (i). One possibility is that the third and fourth vertices coincide, and these are represented by the set $\{[0, 1, z, z] \mid z \in \mathbb{H}^2\}$, whose closure is $\{[0, 1, z, z] \mid z \in \overline{\mathbb{H}^2} \cup \{\infty\}\}$ (where $z = \infty$ determines $[0, 0, 1, 1]$), which is homeomorphic to $\mathbb{D}^2$. This disc can be thought of as the 2-simplex with vertices $[0, 1, 0, 0]$, $[0, 1, 1, 1]$ and $[0, 0, 1, 1]$, and therefore edges $\{[0, 1, t, t] \mid t < 0\}$, $\{[0, 1, t, t] \mid 0 < t < 1\}$ and $\{[0, 1, t, t] \mid t > 0\}$. The other three possibilities for shapes of Type (i) are similar. In this way we obtain four 2-simplices in ∂C such that the intersection of each pair is exactly one vertex (see Figure 2).

The shapes of Type (ii) can be obtained from cyclic quadrilaterals by making the radius of the circumscribed circle tend to infinity. For example, the 4-segment $[0, 1, \frac{3}{4}, \frac{1}{4}]$ is the limit of the sequence of cyclic quadrilaterals

$$\Big\{Z_n = \Big[0, 1, \tfrac{3}{4} + i\Big(\sqrt{n^2 + \tfrac{3}{16}} - n\Big), \tfrac{1}{4} + i\Big(\sqrt{n^2 + \tfrac{3}{16}} - n\Big)\Big]\Big\}_{n=1}^{\infty}$$

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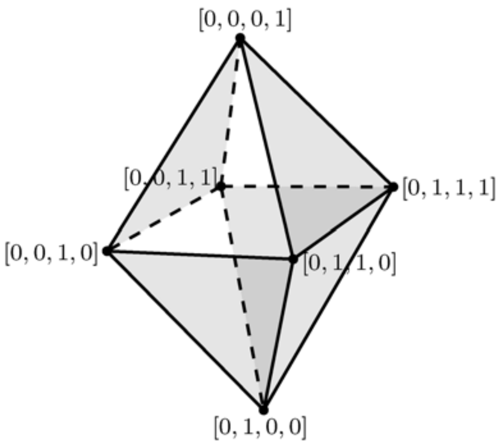

FIGURE 2. ∂C is a 2-complex with octahedral structure. Shapes of Types (i) and (ii) belong to shaded and white triangles, respectively.

(Z_n is inscribed in the circle with centre at $a_n = \frac{1}{2} - in$ and radius $|a_n|$). Also, it is easily seen that

$$\{[0,1,t,s] \mid 0 \le s \le t \le 1\} \subset \partial C, \quad \{[1,t,s,0] \mid 0 \le s \le t \le 1\} \subset \partial C,$$
$$\{[t,s,0,1] \mid 0 \le s \le t \le 1\} \subset \partial C, \quad \{[s,0,1,t] \mid 0 \le s \le t \le 1\} \subset \partial C.$$

Again, the shapes of Type (ii) determine four 2-simplices in ∂C such that the intersection of each pair of triangles is exactly one vertex (see Figure 2).

Identifying the edges of these eight 2-simplices by equivalent shapes, we obtain the octahedron shown in Figure 2. We conclude that ∂C is homeomorphic to $\mathbb{S}^2$.

Now we proceed to prove that shapes in ∂C have neighbourhoods in $\overline{C}$ which are homeomorphic to $\overline{\mathbb{H}^3}$. We will give the argument, without loss of generality, for a single shaded triangle, a single white triangle, a single edge and a single vertex of the octahedron. Let $\mathcal{D}_z \subset \overline{\mathbb{H}^2}$ denote:

- the interval $[0, z]$ if $0 < z < 1$; or
- the interval $[z, \infty)$ if $1 < z$; or
- the arc between 0 and z along the circumscribed circle of the triangle $(0, 1, z)$ if $z \in \mathbb{H}^2$.

Case 1: $[0, 1, p, p]$ *with* $\Im(p) > 0$. Suppose that $B = \{z \in \mathbb{C} \mid |z - p| < \Im(p)/2\}$ and $U = \{[0, 1, z, w] \mid z, w \in B\}$. Then $U \cap \overline{C}$ consists of quadrilaterals such that $z \in B$ and $w \in \mathcal{D}_z \cap B$ (Figure 3(a)). It follows that $U \cap \overline{C}$ is of the form $B \times \overline{\mathbb{H}^1}$, which is homeomorphic to $\overline{\mathbb{H}^3}$. This case corresponds to a shaded triangle.

Case 2: $[0, 1, s, s]$ *with* $0 < s < 1$. Let $r = \min\{s, 1 - s\}$. If $B = \{z \in \mathbb{C} \mid |z - s| < r/2\}$ and $U = \{[0, 1, z, w] \mid z, w \in B\}$, then $U \cap \overline{C}$ consists of quadrilaterals such that $z \in B \cap \overline{\mathbb{H}^2}$

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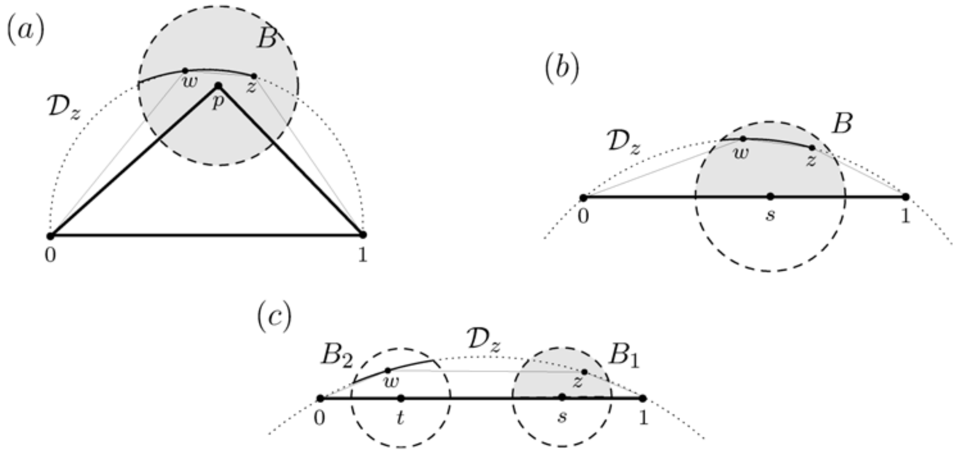

FIGURE 3. Neighbourhoods U of shapes in ∂C using the coordinates from Remark 2.2. The intersections $U \cap \overline{C}$ are described by z in shaded regions and w in the part of $\mathcal{D}_z$ represented by continuous lines.

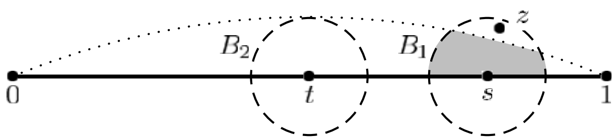

FIGURE 4. The dotted arc is tangent to B_2 and $\mathcal{D}_z \cap B_2 = \emptyset$ when $z \in B_1$ does not lie in the shaded region.

and $w \in \mathcal{D}_z \cap B$ (Figure 3(b)). It follows that $U \cap \overline{C}$ is of the form $\overline{\mathbb{H}^2} \times \overline{\mathbb{H}^1}$, which is homeomorphic to $\overline{\mathbb{H}^3}$. This case corresponds to an edge.

Case 3: $[0, 1, 1, 1]$. If $B = \{z \in \mathbb{C} \mid |z - 1| < 1/2\}$ and $U = \{[0, 1, z, w] \mid z, w \in B\}$, then $U \cap \overline{C}$ consists of quadrilaterals such that $z \in B \cap \overline{\mathbb{H}^2}$ and $w \in \mathcal{D}_z \cap B$. It follows that $U \cap \overline{C}$ is of the form $\overline{\mathbb{H}^2} \times \overline{\mathbb{H}^1}$, which is homeomorphic to $\overline{\mathbb{H}^3}$. This case corresponds to a vertex.

Case 4: $[0, 1, s, t]$ *with* $0 < t < s < 1$. Let $r = \min\{t, s - t, 1 - s\}$. We introduce the discs $B_1 = \{z \in \mathbb{C} \mid |z - s| < r/2\}$ and $B_2 = \{w \in \mathbb{C} \mid |w - t| < r/2\}$ and, in addition, we set $U = \{[0, 1, z, w] \mid z \in B_1, w \in B_2\}$. Then $U \cap \overline{C}$ consists of quadrilaterals such that $z \in B_1 \cap \overline{\mathbb{H}^2}$ and $w \in \mathcal{D}_z \cap B_2$ (Figure 3(c)). It follows that $U \cap \overline{C}$ is of the form $\overline{\mathbb{H}^2} \times \mathbb{H}^1$, which is homeomorphic to $\overline{\mathbb{H}^3}$. A peculiar circumstance arises when $z \in B_1 \cap \overline{\mathbb{H}^2}$ and $\mathcal{D}_z \cap B_2 = \emptyset$ (Figure 4), but the shaded region shown in Figure 4 will serve instead of $B_1 \cap \overline{\mathbb{H}^2}$. This case corresponds to a white triangle.

Since $\overline{C}$ satisfies the hypothesis of Theorem A.1, $\overline{C}$ is homeomorphic to $\mathbb{D}^3$. □

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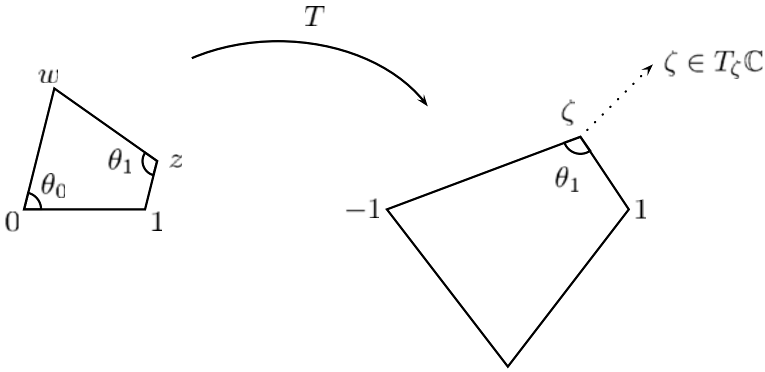

FIGURE 5. A similarity T is helpful to prove that the differential of θ is not the zero function. We find that the derivative of θ is negative when perturbing ζ in the direction of $\zeta \in T_\zeta\mathbb{C}$.

Recall that $\mathcal{K}^\circ \subset P(4)$ is the interior of the set of shapes of positively oriented convex quadrilaterals.

THEOREM 4.2. *$\mathcal{K}^\circ \setminus C$ is the union of two connected components which are homeomorphic to $\mathbb{R}^4$.*

PROOF. We will work in the local coordinates $\phi(\mathcal{U}) = \mathbb{C}^2$ of Remark 2.2. There exists a smooth embedding $i : \mathbb{R}^4 \hookrightarrow \phi(\mathcal{U})$ whose image is $\phi(\mathcal{K}^\circ)$ (see [3, Theorem 3.1]). We also have a smooth embedding $h : C \hookrightarrow i(\mathbb{R}^4)$ of the form

$$[1, e^{i\alpha}, e^{i\beta}, e^{i\gamma}] \mapsto \Big(\frac{e^{i\beta}-1}{e^{i\alpha}-1}, \frac{e^{i\gamma}-1}{e^{i\alpha}-1}\Big).$$

Let θ_0 and θ_1 be the angles of the quadrilateral at the vertices 0 and z (see Figure 5). Then $h(C)$ is the level set $\theta_0 + \theta_1 = \pi$ of the smooth function $\theta_0 + \theta_1 : i(\mathbb{R}^4) \to \mathbb{R}$. We would like to prove that the gradient vector field of $\theta_0 + \theta_1$ has no singularities. To do this, we show that a certain deformation of the vertex z (chosen so as to leave θ_0 unchanged) does not make the derivative of θ_1 vanish. Consider an element of $T \in \mathcal{A}_\mathbb{C}$ such that $T(w) = -1$ and $T(1) = 1$ (see Figure 5). A calculation shows that the angle $\arccos \mathfrak{R}(\mu\bar{\nu})/|\mu|\,|\nu|$ between $\mu(\tau), \nu(\tau) : (-1, 1) \to \mathbb{C}$ satisfies

$$\frac{d}{d\tau}\arccos\frac{\mathfrak{R}(\mu\bar{\nu})}{|\mu|\,|\nu|} = \frac{1}{|\mathfrak{I}(\mu\bar{\nu})|}\Big(\frac{\mathfrak{R}(\mu\bar{\nu})\mathfrak{R}(\bar{\nu}'\nu)}{|\nu|^2} + \frac{\mathfrak{R}(\mu\bar{\nu})\mathfrak{R}(\mu'\bar{\mu})}{|\mu|^2} - \mathfrak{R}(\mu\bar{\nu}') - \mathfrak{R}(\bar{\nu}\mu')\Big).$$

Taking $\mu = -1 - \zeta - \tau\zeta$ and $\nu = 1 - \zeta - \tau\zeta$,

$$\frac{d}{d\tau}\Big|_{\tau=0}\theta(\tau) = \frac{-|\zeta|^4 - 2\mathfrak{I}(\zeta)^2}{\mathfrak{I}(\zeta)\,|1-\zeta|^2\,|1+\zeta|^2} < 0.$$

The conclusion is that the gradient vector field of $\theta_0 + \theta_1$ has no singularities.

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Claim. Let U be an open subset of $\mathbb{R}^n$ diffeomorphic to $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a smooth function without critical points such that the level set $f^{-1}(c)$ is homeomorphic to $\mathbb{R}^{n-1}$ for some $c \in \mathbb{R}$. Then $f^{-1}(-\infty, c)$ and $f^{-1}(c, \infty)$ are both diffeomorphic to $\mathbb{R}^n$. This is an easy exercise in the use of the smooth flow of a smooth vector field. The hypothesis that there are no critical points implies that the flow of the gradient vector field ∇f is injective and carries $f^{-1}(c)$ diffeomorphically onto the other level sets. Both $f^{-1}(-\infty, c)$ and $f^{-1}(c, \infty)$ are smooth products of level sets and flow lines.

Applying the claim concludes the proof of the theorem. □

Remark 4.3. The above proof shows that we can replace 'homeomorphic' by 'diffeomorphic' in Theorem 4.2.

5. Trapezoids

A *trapezoid* is a simple quadrilateral with a pair of parallel edges. Let $\mathcal{T} \subset P(4)$ denote the set of shapes of positively oriented trapezoids.

Remark 5.1. Notice that $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, where $\mathcal{T}_1$ is the set of trapezoids with $\overline{z_1 z_2}$ parallel to $\overline{z_3 z_4}$, and $\mathcal{T}_2$ is the set of trapezoids with $\overline{z_2 z_3}$ parallel to $\overline{z_4 z_1}$. Clearly, $[z_1, z_2, z_3, z_4] \mapsto [z_2, z_3, z_4, z_1]$ is a homeomorphism from $\mathcal{T}_1$ to $\mathcal{T}_2$ and $\mathcal{P} = \mathcal{T}_1 \cap \mathcal{T}_2$.

Theorem 5.2. *$\overline{\mathcal{T}_1}, \overline{\mathcal{T}_2} \subset P(4)$ are both homeomorphic to $\mathbb{D}^3$.*

Proof. By Remark 5.1, it is enough to prove the result for $\mathcal{T}_1$. We will proceed in the same way as for Theorem 4.1.

Every shape in $\mathcal{T}_1$ has a representative of the form $[0, 1, z, z - s]$ with $z \in \mathbb{H}^2$ and $s > 0$. Thus, $\mathcal{T}_1 = \mathbb{H}^2 \times \mathbb{H}^1$, which is homeomorphic to $\mathbb{R}^3$.

There are two types of shapes in $\partial\mathcal{T}_1$.

(i) Positively oriented quadrilaterals $[z_1, z_2, z_3, z_4]$ with $z_1 = z_2$ or $z_3 = z_4$. Shapes of this type determine two 2-simplices (see the proof of Theorem 4.1).

(ii) 4-segments which are the limit of trapezoids. There are a number of possibilities for the 4-segments $[0, 1, t, t - s]$ of this type, each of which determines a 2-simplex:

- $\{1 \le t, 0 \le s \le t - 1\}$;
- $\{1 \le t, t - 1 \le s \le t\}$;
- $\{1 \le t, t \le s\}$;
- $\{0 \le t \le 1, 0 \le s \le t\}$;
- $\{0 \le t \le 1, t \le s\}$;
- $\{t \le 0 \le s\}$.

Identifying the edges of these eight 2-simplices by equivalent shapes, we conclude that $\partial\mathcal{T}_1$ is homeomorphic to $\mathbb{S}^2$ (see Figure 6).

Now we proceed to prove that shapes in $\partial\mathcal{T}_1$ have neighbourhoods in $\overline{\mathcal{T}_1}$ which are homeomorphic to $\overline{\mathbb{H}^3}$. We will give the argument, without loss of generality, for

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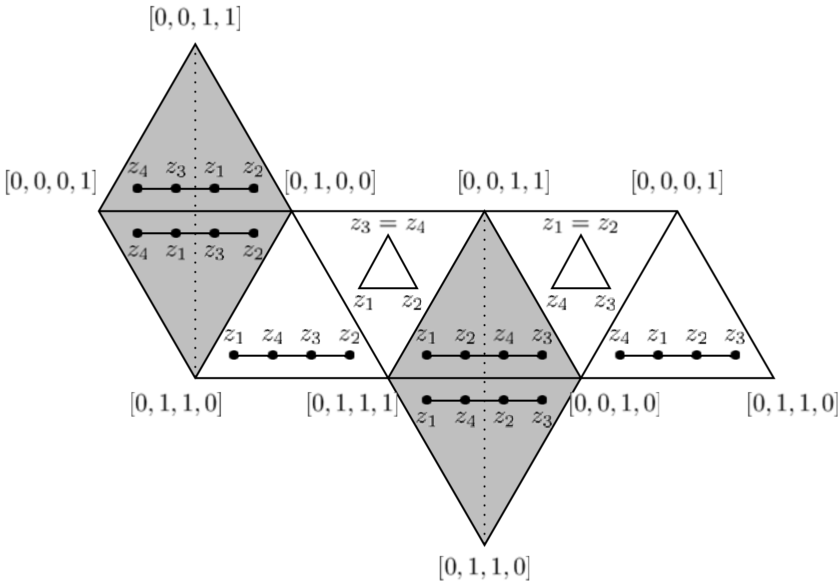

FIGURE 6. The eight 2-simplices in $\partial\mathcal{T}_1$. Inside each 2-simplex we illustrate an example of the shapes occurring in it. The diagram shows some identifications between the 2-simplices, but it remains to glue edges with equivalent vertices to obtain $\mathbb{S}^2$. The shading indicates 2-simplices belonging also to $\partial\mathcal{T}_2$. The dotted segments form the circle $\partial\mathcal{P}$.

a single triangle of Type (i), a single triangle of Type (ii), a single edge and a single vertex in $\partial\mathcal{T}_1$. Let $I_p = \{z \in \mathbb{C} \colon \Re(z) \le \Re(p), \Im(z) = \Im(p)\}$ for $p \in \mathbb{C}$.

Case 1: $[0, 1, p, p]$ *with* $p \in \mathbb{H}^2$. Suppose that $B = \{z \in \mathbb{C} \mid |z - p| < \Im(p)/2\}$ and $U = \{[0, 1, z, w] \mid z, w \in B\}$. Then $U \cap \overline{\mathcal{T}}_1$ consists of quadrilaterals such that $z \in B$ and $w \in I_z \cap B$. It follows that $U \cap \overline{\mathcal{T}}_1$ is of the form $B \times \overline{\mathbb{H}^1}$, which is homeomorphic to $\overline{\mathbb{H}^3}$. This case corresponds to a triangle of Type (i).

Case 2: $[0, 1, t, t]$ *with* $0 < t < 1$. Let $r = \min\{t, 1 - t\}$. If $B = \{z \in \mathbb{C} \mid |z - t| < r/2\}$ and $U = \{[0, 1, z, w] \mid z, w \in B\}$, then $U \cap \overline{\mathcal{T}}_1$ consists of quadrilaterals such that $z \in B \cap \overline{\mathbb{H}^2}$ and $w \in I_z \cap B$. It follows that $U \cap \overline{\mathcal{T}}_1$ is of the form $\overline{\mathbb{H}^2} \times \overline{\mathbb{H}^1}$, which is homeomorphic to $\overline{\mathbb{H}^3}$. This case corresponds to an edge.

Case 3: $[0, 1, 1, 1]$. If $B = \{z \in \mathbb{C} \mid |z - 1| < 1/2\}$ and $U = \{[0, 1, z, w] \mid z, w \in B\}$, then $U \cap \overline{\mathcal{T}}_1$ consists of quadrilaterals such that $z \in B \cap \overline{\mathbb{H}^2}$ and $w \in \mathcal{I}_z \cap B$. It follows that $U \cap \overline{\mathcal{T}}_1$ is of the form $\overline{\mathbb{H}^2} \times \overline{\mathbb{H}^1}$, which is homeomorphic to $\overline{\mathbb{H}^3}$. This case corresponds to a vertex.

Case 4: $[0, 1, t, t - s]$ *with* $0 < t - s < t < 1$. Let $r = \min\{t - s, s, 1 - t\}$. Suppose that $B_t = \{z \in \mathbb{C} \mid |z - t| < r/2\}$ and $B_s = \{w \in \mathbb{C} \mid |w + s - t| < r/2\}$ and, in addition, set $U = \{[0, 1, z, w] \mid z \in B_t, w \in B_s\}$. Then $U \cap \overline{\mathcal{T}}_1$ consists of quadrilaterals such that

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$z \in B_t \cap \overline{\mathbb{H}^2}$ and $w \in I_z \cap B_s$. It follows that $U \cap \overline{\mathcal{T}}_1$ is of the form $\overline{\mathbb{H}^2} \times \mathbb{H}^1$, which is homeomorphic to $\overline{\mathbb{H}^3}$. This case corresponds to a triangle of Type (ii).

Since $\overline{\mathcal{T}_1}$ satisfies the hypothesis of Theorem A.1, $\overline{\mathcal{T}_1}$ is homeomorphic to $\mathbb{D}^3$. □

COROLLARY 5.3. *$\mathcal{T}$ is homeomorphic to $\{(x_1, x_2, x_3, 0) \in \mathbb{R}^4\} \cup \{(x_1, x_2, 0, x_4) \in \mathbb{R}^4\}$ and $\mathcal{K}^\circ \setminus \mathcal{T}$ is the union of four connected components which are homeomorphic to $\mathbb{R}^4$.*

PROOF. The first assertion is immediate because $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, each $\mathcal{T}_j$ is homeomorphic to $\mathbb{R}^3$ by the proof of Theorem 5.2 and $\mathcal{T}_1 \cap \mathcal{T}_2 = \mathcal{P}$, which is homeomorphic to $\mathbb{R}^2$ by Section 3.

The second assertion can be proved as in Theorem 4.2. Let θ_0, θ_1 and θ_2 be the angles of the quadrilateral at vertices 0, z and w (see Figure 5). Notice that $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{K}^\circ$ are defined respectively by $\theta_0 + \theta_2 = \pi$ and $\theta_0 + \theta_1 = \pi$. First apply the Claim in the proof of Theorem 4.2 to the function $\theta_0 + \theta_2$ to see that $\mathcal{K}^\circ \setminus \mathcal{T}_1$ is the union of two connected components U_1 and U_2 which are copies of $\mathbb{R}^4$. Next apply the Claim again to remove the level set $\theta_1 + \theta_2 = \pi$ from each U_j and conclude the proof. □

REMARK 5.4. The proof shows that we can replace 'homeomorphic' by 'diffeomorphic' in the second assertion of Corollary 5.3.

REMARK 5.5. Observe that $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ is mysteriously embedded into $\mathcal{K}^\circ$. We have proved that $\overline{\mathcal{T}_1}$ and $\overline{\mathcal{T}_2}$ are two copies of the unit disc $\mathbb{D}^3$ embedded into $\overline{\mathcal{K}}$, which is the unit disc $\mathbb{D}^4$. However, the obvious embeddings $\mathbb{D}^3_1 = \{(x_1, x_2, x_3, 0)\} \subset \mathbb{D}^4$ and $\mathbb{D}^3_2 = \{(x_1, x_2, 0, x_4)\} \subset \mathbb{D}^4$ suggested by Corollary 5.3 are wrong. The restrictions $(\mathbb{D}^3_1)^\circ \hookrightarrow (\mathbb{D}^4)^\circ$ and $(\mathbb{D}^3_2)^\circ \hookrightarrow (\mathbb{D}^4)^\circ$ of these embeddings satisfy the conclusions of the corollary and give $\partial\mathbb{D}^3_1 \cap \partial\mathbb{D}^3_2 = \mathbb{S}^1$ (which would make sense because $\partial(\mathcal{T}_1 \cap \mathcal{T}_2) = \partial\mathcal{P} = \mathbb{S}^1$), but $\partial\mathcal{T}_1 \cap \partial\mathcal{T}_2$ is the shaded set shown in Figure 6.

PROPOSITION 5.6. *$\overline{C \cap \mathcal{T}} \subset P(4)$ is homeomorphic to*

$$\{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\} \cup \{(x, 0, z) \in \mathbb{R}^3 \mid x^2 + z^2 \leq 1\}.$$

REMARK 5.7. $C \cap \mathcal{T}$ is known to be the set of *isosceles trapezoids*.

PROOF. The proof is a matter of checking properties of $\overline{C \cap \mathcal{T}_1}$ and $\overline{C \cap \mathcal{T}_2}$ because $\overline{C \cap \mathcal{T}} = \overline{C \cap \mathcal{T}_1} \cup \overline{C \cap \mathcal{T}_2}$. Observe that

$$\overline{C \cap \mathcal{T}_1} = \{[0, 1, z, z + 1 - 2\mathfrak{R}(z)] \mid z \in \overline{\mathbb{H}^2}, \mathfrak{R}(z) \geq 1/2\} \cup \{[0, 0, 1, 1]\},$$
$$\overline{C \cap \mathcal{T}_2} = \{[z + 1 - 2\mathfrak{R}(z), 0, 1, z] \mid z \in \overline{\mathbb{H}^2}, \mathfrak{R}(z) \geq 1/2\} \cup \{[1, 0, 0, 1]\},$$

which are clearly homeomorphic to $\mathbb{D}^2$, and $\overline{C \cap \mathcal{T}_1} \cap \overline{C \cap \mathcal{T}_2} = \overline{C \cap \mathcal{P}} = \overline{\mathcal{R}_e}$. □

6. The trivialisation of the bundle η

Now a word about the principal bundle

$$\begin{array}{ccc} \mathcal{A}_{\mathbb{C}} & \longrightarrow & \mathbb{C}^4 \setminus \{(z,z,z,z)\} \\ & & \Big\downarrow \eta \\ & & P(4) \end{array}$$

It is not trivial since $\pi_1(\mathbb{C}^4 \setminus \{(z,z,z,z)\}) = 1$ and $\pi_1(P(4) \times \mathcal{A}_{\mathbb{C}}) = \pi_1(\mathbb{CP}^2 \times \mathbb{C} \times \mathbb{C}^*) = \mathbb{Z}$ (recall from (1.1) that $\mathcal{A}_{\mathbb{C}}$ is the complex affine group, which is homeomorphic to $\mathbb{C} \times \mathbb{C}^*$).

Corollary 6.1. *$\eta^{-1}(\overline{\mathcal{J}})$ is homeomorphic to $\overline{\mathcal{J}} \times \mathbb{C} \times \mathbb{C}^*$ for $\mathcal{J} = \mathcal{P}$, C and $\mathcal{T}$.*

Proof. Since $\overline{\mathcal{J}}$ is contractible, the bundle is trivial. □

7. Negatively oriented quadrilaterals

Let $\mathcal{P}^-, C^-, \mathcal{T}^- \subset P(4)$ be the spaces of negatively oriented parallelograms, cyclic quadrilaterals and trapezoids, respectively. The function $[z_1, z_2, z_3, z_4] \mapsto [z_1, z_4, z_3, z_2]$ defines homeomorphisms between $\overline{\mathcal{P}}$, $\overline{C}$, $\overline{\mathcal{T}}$ and $\overline{\mathcal{P}^-}$, $\overline{C^-}$, $\overline{\mathcal{T}^-}$, respectively. It is clear that $\overline{\mathcal{P}} \cap \overline{\mathcal{P}^-}$, $\overline{C} \cap \overline{C^-}$ and $\overline{\mathcal{T}} \cap \overline{\mathcal{T}^-}$ are exactly the subsets of 4-segments contained in their respective boundaries. For example, $\overline{\mathcal{P}} \cup \overline{\mathcal{P}^-}$ is obtained by pasting two closed discs through their boundaries (see Section 3) and therefore it is homeomorphic to $\mathbb{S}^2$. Similarly, the spaces of rectangles $\overline{\mathcal{R}_e} \cup \overline{\mathcal{R}_e^-}$ and rhombuses $\overline{\mathcal{R}_h} \cup \overline{\mathcal{R}^-}_h$ are copies of $\mathbb{S}^1$.

Let $v_1 = (1, 0, 0)$, $v_2 = (-1/2, \sqrt{3}/2, 0)$, $v_3 = (-1/2, -\sqrt{3}/2, 0)$, $v_4 = (0, 0, \sqrt{2})$ and B_i be the open ball $B_{\sqrt{3}/2}(v_i)$.

Proposition 7.1. *$\overline{C} \cup \overline{C^-}$ is homeomorphic to $\mathbb{R}^3 \cup \{\infty\} \setminus (B_1 \cup B_2 \cup B_3 \cup B_4)$.*

Proof. By Theorem 4.1, $\overline{C} \cup \overline{C^-}$ is an identification of the boundaries of two copies of $\mathbb{D}^3$ along the white triangles in Figure 2. The boundary $\partial(\overline{C} \cup \overline{C^-})$ consists of four copies of $\mathbb{S}^2$ obtained as an identification of the boundaries of the shaded triangles in Figure 2. These spheres are tangent by pairs. Then $\overline{C} \cup \overline{C^-}$ is $\mathbb{S}^3$ (formed by gluing boundaries of two copies of $\mathbb{D}^3$) drilled along four open balls whose boundaries are tangent by pairs. We conclude the proof by using $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ and removing the balls B_1, B_2, B_3 and B_4 from $\mathbb{R}^3$. □

Proposition 7.2. *$\overline{\mathcal{T}_1} \cup \overline{\mathcal{T}_1^-}$ and $\overline{\mathcal{T}_2} \cup \overline{\mathcal{T}_2^-}$ are homeomorphic to $\mathbb{R}^3 \cup \{\infty\} \setminus (B_1 \cup B_2)$.*

Proof. The argument is similar to Proposition 7.1 using Theorem 5.2 instead of Theorem 4.1. □

We are not able to describe $\overline{\mathcal{T}} \cup \overline{\mathcal{T}^-}$ as a known space (see Remark 5.5).

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Appendix A. How to recognise a closed disc

THEOREM A.1. *Let M be a topological manifold with boundary. If $M \smallsetminus \partial M$ and ∂M are respectively homeomorphic to $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$, then M is homeomorphic to $\mathbb{D}^n$.*

PROOF. First we argue that M is compact. Let $\Omega \subset M$ be a collar neighbourhood of ∂M, that is, Ω is closed and there is a homeomorphism $\psi\colon \partial M \times [0, 1] \to \Omega$ with $\psi(x, 1) = x$ (see [2, Theorem 2]). Then $\psi(\partial M \times \{1/2\})$ is a sphere $\mathbb{S}^{n-1}$ topologically embedded into $M \smallsetminus \partial M$, which separates $M \smallsetminus \partial M$ into two components by the Jordan separation theorem (see [4, Section 2.B]). Let $K \subset M \smallsetminus \partial M$ be a compact ball containing $\psi(\partial M \times \{1/2\})$. Since the curve $\psi(\{p\} \times (1/2, 1))$ cannot lie in the bounded component of $(M \smallsetminus \partial M) \smallsetminus \psi(\partial M \times \{1/2\})$ for all $p \in \partial M$, we conclude that M is a union of two compact sets $K \cup \psi(\partial M \times [1/2, 1])$.

Now recall a well-known fact: the uniqueness of the one-point compactification. Given a second-countable Hausdorff space X, there is a space X^* such that:

(1) X is a subspace of X^*;
(2) $X^* \smallsetminus X$ consists of a single point;
(3) X^* is a compact Hausdorff space.

Moreover, if Y is another space satisfying these conditions for X^*, then there is a homeomorphism $X^* \to Y$ that equals the identity map on X (see [5, Theorem 29.1]).

It follows directly from the uniqueness of the one-point compactification that the quotient space $M/\partial M$ is homeomorphic to the one-point compactification of $M \smallsetminus \partial M$. Notice that $\rho \circ \psi\colon \partial M \times [0, 1) \to M/\partial M$ is an embedding, where $\rho\colon M \to M/\partial M$ is the quotient projection. By applying Brown's generalised Schöenflies theorem [1, Theorem 5], we get a homeomorphism between $\mathbb{D}^n$ and each connected component in $(M/\partial M) \smallsetminus \rho \circ \psi(\partial M \times \{1/2\})$. In particular, $M \smallsetminus \psi(\partial M \times (1/2, 1])$ is homeomorphic to $\mathbb{D}^n$ and the map $M \to M \smallsetminus \psi(\partial M \times (1/2, 1])$ given by

$$x \mapsto \begin{cases} x & \text{if } x \in M \smallsetminus \Omega, \\ \psi^{-1}(p, t/2) & \text{if } x \in \Omega \text{ and } \psi(x) = (p, t) \end{cases}$$

is a homeomorphism. □

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