

SOLUTION OF IRVING'S RAMSEY PROBLEM

by HEIKO HARBORTH and HEINZ-MICHAEL NITZSCHKE

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In [1] the following question was posed by R. W. Irving (see also Conjecture 4.10 in [4]): Is there an edge 2-colouring of the complete bipartite graph $K_{13,17}$ with no monochromatic $K_{3,3}$? We give a negative answer in this note (Theorem 2). Furthermore we prove Conjecture 4.11 (i) of [4] (Theorem 1), that is, any edge 2-coloured $K_{2n+1,4n-3}$ contains a monochromatic $K_{2,n}$ with the 2 and n vertices a subset of the $2n+1$ and $4n-3$ vertices, respectively. Theorem 1 is a consequence of Satz 4 in [3], however, we give a direct proof here.

Instead of edge coloured complete bipartite graphs $K_{x,y}$ we use 0-1-matrices $M = (m_{i,j})$, where $m_{i,j} = 0$ or 1 ($1 \leq i \leq x, 1 \leq j \leq y$), if the edge (i, j) of $K_{x,y}$ is of the first or second colour, respectively.

LEMMA 1. *If p_{ij} , p_{ik} , and p_{jk} denote the numbers of equal columns (both entries 0 or both entries 1) in the three pairs of rows of any triple of rows (i, j, k) in any 0-1-matrix with c columns, then*

$$p_{ij} + p_{ik} + p_{jk} \equiv c \pmod{2}. \tag{1}$$

Proof. Each column contributes 1 or 3 to the sum on the left-hand side.

In the following we denote by $[x]$ and $\{x\}$ the greatest integer $\leq x$ and the smallest integer $\geq x$, respectively.

THEOREM 1. *Any $(2n+1, 4n-3)$ -0-1-matrix contains a $(2, n)$ -submatrix with entries 0 only, or 1 only.*

Proof. Any column of a $(2n+1, 4n-3)$ -0-1-matrix M contains at least $\binom{n}{2} + \binom{n+1}{2}$ pairs of equal entries. Thus for the total number A of equal pairs in all columns of M we obtain

$$A \geq (4n-3) \left(\binom{n}{2} + \binom{n+1}{2} \right) = 4n^3 - 3n^2. \tag{2}$$

Using the pigeonhole principle there is at least one pair of rows in M with

$$p = \left\lceil \frac{4n^3 - 3n^2}{\binom{2n+1}{2}} \right\rceil = \left\lceil 2n - 2 - \frac{n-2}{2n+1} \right\rceil = 2n - 2 \tag{3}$$

equal columns for $n \geq 2$ (if $n = 1$, then Theorem 1 is trivial).

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We assume that no pair of rows in M has more than p equal columns. In any triple of rows at most two pairs have p equal columns (Lemma 1). Then the famous theorem of Turán ([2], p. 17) implies that at most $\lfloor (2n+1)^2/4 \rfloor = n^2+n$ pairs of rows in M have p equal columns. It follows that

$$A \leq (n^2+n)(2n-2) + \left(\binom{2n+1}{2} - (n^2+n) \right) (2n-3) = 4n^3 - 3n^2 - 2n, \tag{4}$$

which contradicts (2). Thus at least one pair of rows in M has $p+1=2n-1$ equal columns, that means, $\left\lfloor \frac{2n-1}{2} \right\rfloor = n$ columns have entries 0 only, or 1 only.

THEOREM 2. Any $(13, 17)$ -0-1-matrix contains a $(3, 3)$ -submatrix with entries 0 only, or 1 only.

Proof. We denote by $B = (b_{i,j})$ a $(13, 17)$ -0-1-matrix, by N the $(3, 3)$ -matrix with entries 0 only, and by \bar{M} the matrix M with 0 and 1 interchanged. If M contains a submatrix S , we will write $S \subset M$. The proof is divided into the following Lemmas. Those parts of their proofs which follow by changing 0 and 1 are omitted.

We consider the matrices S_i ($1 \leq i \leq 10$) shown opposite as submatrices of B up to exchanges of rows or columns.

LEMMA 2. If S_1 or $\bar{S}_1 \subset B$, then N or $\bar{N} \subset B$,

Proof. If $S_1 \subset B$, then either $N \subset B$, or every row of that $(11, 6)$ -submatrix M of B determined by the columns of S_1 contains at least 4 entries 1, that is, $\binom{4}{3} = 4$ triples of entries 1. Then any distribution of these 44 triples among the 6 columns of M guarantees $\bar{N} \subset M$, since $2 \binom{6}{3} < 44$.

LEMMA 3. If S_2, S_2^T, \bar{S}_2 or $\bar{S}_2^T \subset B$, then N or $\bar{N} \subset B$.

Proof. Let $S_2 \subset B$, or $S_2 \subset B^T$, which corresponds to $S_2^T \subset B$. The first 5 columns and the last 7 rows of S_2 determine $M \subset S_2$. Either $N \subset S_2$, or every row of M contains at least 3 entries 1, that is, $\binom{3}{2} = 3$ pairs of entries 1. In any distribution of these 21 pairs among the 5 columns of M there are 2 columns with 3 pairs of entries 1 in a row, since $2 \binom{5}{2} < 21$. Together with column 6 of S_2 , it follows that $\bar{N} \subset S_2$.

LEMMA 4. If S_3 or $\bar{S}_3 \subset B$, then N or $\bar{N} \subset B$.

Proof. If S_3 is in the first rows and columns of B , then rows 6 to 13, and columns 3 to 17 determine a $(8, 15)$ -submatrix M of B . If there is one row of M with more than 6 entries 1, then Lemma 3 can be used. Otherwise M has at least $8 \times 9 = 72$ entries 0. Let s_i

$$S_5: \begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot \end{array}$$

$$S_9: \begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot \\ \cdot & \cdot \end{array}$$

$$S_8: \begin{array}{cccc} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}$$

$$S_{10}: \begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot \end{array}$$

$$S_3: \begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}$$

$$S_4: \begin{array}{cccc} 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & \cdot \\ \cdot & 1 & 1 & 1 & 1 & 1 \\ \cdot & 1 & 1 & 1 & 1 & 1 \end{array}$$

$$S_7: \begin{array}{cccc} 0 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}$$

$$S_1: \begin{array}{cccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$S_2: \begin{array}{cccc} 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \end{array}$$

$$S_6: \begin{array}{cccc} \cdot & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdot \\ 1 & 1 & 0 & 0 & \cdot \\ 1 & 1 & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & 1 \end{array}$$

($0 \leq i \leq 8$) denote the number of columns of M with exactly i entries 0. Then

$$\sum_{i=0}^8 s_i = 15, \quad \text{and} \quad \sum_{i=1}^8 i s_i \geq 72$$

yield

$$s_5 + 2s_6 + 3s_7 + 4s_8 \geq 12 + 4s_0 + 3s_1 + 2s_2 + s_3 \geq 12. \quad (5)$$

Together with (5) there are

$$\sum_{i=3}^8 \binom{i}{3} s_i \geq 10(s_5 + 2s_6 + 3s_7 + 4s_8) \geq 120 > 2 \binom{8}{3}$$

triples of entries 0 in the columns of M , so that $N \subset M$.

LEMMA 5. *If a column of B has 9 entries 0 or 9 entries 1, then N or $\bar{N} \subset B$.*

Proof. Let $b_{i,1} = 0$ for all i with $1 \leq i \leq 9$. Rows 1 to 9 and columns 2 to 17 determine $M \subset B$. If there is a column of M with at least 5 entries 0, we use Lemma 4. Otherwise at least $16 \binom{5}{2} = 160$ pairs with both entries 1 in the columns of M distributed among all pairs of rows of M guarantee 2 rows of M having 5 columns with both entries 1, since $4 \binom{9}{2} < 160$. Then $\bar{S}_2 \subset B$, and we use Lemma 3.

LEMMA 6. *If S_4 or $\bar{S}_4 \subset B$, then N or $\bar{N} \subset B$.*

Proof. Let S_4 be in the first rows and columns of B . Rows 5 to 13 and columns 1 to 2, 3 to 5, 6 to 7 determine M_1, M_2, M_3 , respectively. Either N or $\bar{N} \subset B$, or every row of M_2 has at most 2 entries 0, and at most 2 entries 1. In at least 5 rows of M_2 there are 2 entries 0 (1). Then either $N \subset B$ ($\bar{N} \subset B$), or $\bar{S}_3 \subset M_1$ ($S_3 \subset M_3$), and Lemma 4 can be used.

LEMMA 7. *If S_5 or $\bar{S}_5 \subset B$, then N or $\bar{N} \subset B$.*

Proof. Let S_5 occur in the first rows and columns of B . Rows 4 to 13 and columns 1 to 5 of B determine M_1 , and rows 4 to 13 together with columns 11 to 14 of B determine M_2 . At first $b_{3,j} = 1$ for all j with $1 \leq j \leq 5$, or Lemma 2 can be used. Next either $N \subset B$, or every row in M_1 has at least 3 entries 1. If one row of M_2 exists with more than one entry 1, then together with row 3 of B we have $S_4 \subset B$, and use Lemma 6. Otherwise every row of M_2 has 3 entries 0, and any distribution of 10 triples 000 among the 4 columns of M_2 guarantees $N \subset M_2$, since $2 \binom{4}{3} < 10$.

LEMMA 8. *N or \bar{N} exist in any (5, 5)-matrix obtained by changing rows or columns of S_6, S_6^T, \bar{S}_6 or \bar{S}_6^T .*

Proof. If the second or third element of column 5 of S_6 is 1, then $\bar{N} \subset S_6$, and otherwise $N \subset S_6$.

LEMMA 9. If $S_7 \subset B$ (respectively $\bar{S}_7 \subset B$), then either N or $\bar{N} \subset B$, or S_8 (respectively \bar{S}_8) are in rows 3 to 8 and columns 1 to 5 of S_7 (respectively \bar{S}_7), up to exchanges of rows or columns.

Proof. Let $S_7 \subset B$, and M denotes the elements of S_7 in rows 3 to 8 and columns 1 to 5. At least 3 entries 1 exist in every row of M , or $N \subset S_7$. If one row has more than 3 entries 1, then $\binom{4}{2} + 5\binom{3}{2} > 2\binom{5}{2}$ implies that 2 columns exist in M with 3 pairs 11, and together with column 6 of S_7 it follows $\bar{N} \subset S_7$. It remains that M has exactly 18 entries 1. No column of M has more than 4 entries 1, since otherwise $\bar{S}_3 \subset B$ (Lemma 4). No column of M has less than 3 entries 1, since otherwise 4 columns have 4 entries 1, and this forces 2 rows of these 4 columns to have 2 entries 1 and 4 rows to have 3 entries 1, and then $2\binom{2}{2} + 4\binom{3}{2} > 2\binom{4}{2}$ together with column 6 of S_7 guarantees $\bar{N} \subset S_7$. So M can have only 3 columns with 4, and 2 columns with 3 entries 1. Then $N \subset S_7$ (with column 6), or by changing of rows or columns we obtain $M = S_8$.

LEMMA 10. If S_9 or $\bar{S}_9 \subset B$, then N or $\bar{N} \subset B$.

Proof. If we find S_9 in the first rows and columns of B , then by Lemma 2 we can assume $b_{3,j} = b_{2,j+5} = 1$ for $j = 1, 2, 3, 4$. Let M_1 be determined by rows 4 to 13 and columns 1 to 9. Either $N \subset B$, or in columns 1 to 5, and in columns 5 to 9 of M_1 , respectively, there are at least 30 entries 1. Every column $\neq 5$ of M_1 has at most 6 entries 1, or we can use Lemma 3. Thus column 5 must have at least 6 entries 1. Since Lemma 5 yields N or $\bar{N} \subset B$, column 5 has at most 8 entries 1.

(i) $b_{i,5} = 1$ for all i with $8 \leq i \leq 13$, $b_{i,5} = 0$ otherwise: The elements in rows 4 to 7 of B and in columns 1 to 4, 6 to 9, 10 to 14 are denoted by M_2, M_3, M_4 , respectively. Every row of M_2 and of M_3 has at least 3 entries 1, otherwise $N \subset B$. Then every pair of rows of M_2 and of M_3 has at least 2 columns with both entries 1. Together with rows 2 or 3, and M_4 , we find $\bar{N} \subset B$, or every column of M_4 has at least 3 entries 0. Then 2 columns of M_4 together with column 5 of B yield $N \subset B$.

(ii) $b_{i,5} = 1$ for all i with $7 \leq i \leq 13$, $b_{i,5} = 0$ otherwise: Let M_5 and M_6 denote the elements of B in rows 4 to 6, and in columns 1 to 4, and 6 to 9, respectively. Rows 7 to 13 and columns 1 to 4 determine M_7 . Every row of M_5 and of M_6 has at least 3 entries 1 (otherwise $N \subset B$), and then at most 3 entries 1 (otherwise $\bar{N} \subset B$). Every column of M_5 and of M_6 has at most one entry 0, otherwise $\bar{N} \subset B$. Thus we can assume $b_{4,1} = b_{5,2} = b_{6,3} = b_{4,6} = b_{5,7} = b_{6,8} = 0$, and 1 for all other elements of M_5 and of M_6 . As in every pair of columns of M_7 at most 2 pairs 11 occur (otherwise together with column 5 of B we obtain $\bar{N} \subset B$), there is a row in M_7 with 11 in columns 1 and 4, 2 and 4, or 3 and 4, and we can assume $b_{7,3} = b_{7,4} = 1$. At least 2 elements of $b_{7,6}$ to $b_{7,9}$ are 1, otherwise $N \subset B$. These have to be $b_{7,6}$ and $b_{7,7}$, since rows 4, 5, 7 and columns 3, 4, 8, 9 yield $\bar{N} \subset B$ or $b_{7,8} = b_{7,9} = 0$. Then rows 5, 6, 7 and columns 1, 4, 6 imply $\bar{N} \subset B$ or $b_{7,1} = 0$, and rows 4, 6, 7 and columns 2, 4, 7 imply $\bar{N} \subset B$ or $b_{7,2} = 0$.

At least 14 entries 1 occur in M_7 . Column 4 of M_7 contains at most 3 entries 1, since otherwise we can use Lemma 3. No column of M_7 has more than 4 entries 1, and thus two of columns 1 to 3 of M_7 contain exactly 4 entries 1. After possibly changing columns 1 and 2, and rows 4 and 5 we can assume $b_{8,1} = b_{9,1} = b_{10,1} = b_{11,1} = 1$, and $b_{12,1} = b_{13,1} = 0$. Now Lemma 9 guarantees N or $\bar{N} \subset B$, or we can assume S_8 in rows 5, 6, 8, 9, 10, 11 and columns 6, 9, 5, 7, 8, in these sequences.

Then rows 5, 6 and 8, 9, 10 or 11 and columns 1, 4 and 6, 6, 9 or 9, respectively, yield $\bar{N} \subset B$ or $b_{8,4} = b_{9,4} = b_{10,4} = b_{11,4} = 0$. Rows 4, 5, 10 and columns 3, 8, 9 yield $\bar{N} \subset B$ or $b_{10,3} = 0$. Rows 4, 6, 11 and columns 2, 7, 9 yield $\bar{N} \subset B$ or $b_{11,2} = 0$. Then rows 1, 8, 10 and columns 3, 4, 7, rows 1, 7, 9 and columns 2, 8, 9, rows 1, 2, 10 and columns 2, 3, 4, rows 1, 2, 11 and columns 2, 3, 4 yield $N \subset B$ or $b_{8,3} = b_{9,2} = b_{10,2} = b_{11,3} = 1$, respectively. Furthermore $\bar{N} \subset B$ or $b_{8,2} = b_{9,3} = 0$ follow from rows 8, 9, 10 and columns 1, 2, 5, and from rows 8, 9, 11 and columns 1, 3, 5.

If $b_{12,3} = b_{13,3} = 1$, then Lemma 4 can be used. Therefore we can assume $b_{12,3} = 0$ after possibly changing rows 12 and 13. Then rows 1, 2, 12 and columns 1 to 4 imply $N \subset B$ or $b_{12,2} = b_{12,4} = 1$. Rows 4, 6, 12 and columns 2, 4, 7, and columns 2, 4, 9 yield $\bar{N} \subset B$ or $b_{12,7} = b_{12,9} = 0$. Then rows 1, 3, 12 and columns 6 to 9 yield $N \subset B$ or $b_{12,6} = b_{12,8} = 1$. It follows $b_{13,6} = 0$, or Lemma 4 can be used. If $b_{13,2} = 0$, then rows 1, 2, 13 and columns 1 to 4 yield $N \subset B$ or $b_{13,3} = b_{13,4} = 1$, rows 1, 7, 13 and columns 1, 2, 8 yield $N \subset B$ or $b_{13,8} = 1$, and then we find \bar{N} in rows 4, 5, 13 and columns 3, 4, 8. If, however, $b_{13,2} = 1$, then rows 10, 12, 13 and columns 2, 5, 8 yield $\bar{N} \subset B$ or $b_{13,8} = 0$, rows 1, 3, 13 and columns 6 to 9 yield $N \subset B$ or $b_{13,7} = b_{13,9} = 1$, and then we find \bar{N} in rows 4, 6, 13 and columns 2, 7, 9.

(iii) $b_{i,5} = 1$ for all i with $6 \leq i \leq 13$, $b_{i,5} = 0$ otherwise: At least 2 of the elements $b_{3,10}$ to $b_{3,13}$ of B (say $b_{3,10}$ and $b_{3,11}$) are 0, otherwise $\bar{N} \subset B$ (with rows 1 and 2). At least one of the elements $b_{4,10}$, $b_{4,11}$, $b_{5,10}$, $b_{5,11}$ (say $b_{4,10}$) is 1, otherwise $N \subset B$ (with column 5). Columns 6 to 9, rows 4 to 5 and rows 6 to 13 of B determine M_8 and M_9 , respectively. Both rows of M_8 have at least 3 entries 1 (otherwise $N \subset B$), and then at most 3 entries 1 (otherwise $\bar{N} \subset B$). If one column of M_8 has both entries 0, then $\bar{N} \subset B$. Thus we can assume $b_{4,9} = b_{5,8} = 0$, and all other elements of M_8 are 1.

In every column of M_9 there are at most 4 entries 1, or we can use Lemma 4 (with column 5). At least 30 entries 1 are in columns 5 to 9 of M_1 , or $N \subset B$. Thus exactly 4 entries 1 occur in every column of M_9 , and exactly 2 entries 1 in every row of M_9 . We can assume $b_{6,9} = b_{7,9} = b_{8,9} = b_{9,9} = 0$. If in 2 of rows 1 to 4 of M_9 , and in columns 1 and 2 of M_9 there are 4 entries 1, then we have found \bar{S}_3 , and Lemma 4 can be used. Otherwise we can assume $b_{6,6} = b_{6,8} = b_{7,6} = b_{7,8} = 1$, and $b_{6,7} = b_{7,7} = 0$ (after possibly changing columns 6 and 7). Then in these sequences the elements of rows 3, 6, 7, 2, 4 and of columns 6, 8, 7, 9, 10 of B represent S_6 , and Lemma 8 completes the proof.

LEMMA 11. If S_{10} or $\bar{S}_{10} \subset B$, then N or $\bar{N} \subset B$.

Proof. Let S_{10} be in the first rows and columns of B . Then $b_{2,6} = b_{2,7} = b_{2,8} = b_{3,1} = b_{3,2} = b_{3,3} = 1$, or $N \subset B$. The elements of B in rows 4 to 13 and columns 1 to 3, 4 to 5,

and 6 to 8 are denoted by M_1 , M_2 and M_3 , respectively. If we observe Lemma 4, it suffices to discuss 3 cases: M_2 has (i) one row 00, (ii) 4 rows 11, or (iii) 4 rows 01.

(i) $b_{4,4} = b_{4,5} = 0$. Then $N \subset B$, or $b_{4,1} = b_{4,2} = b_{4,3} = b_{4,6} = b_{4,7} = b_{4,8} = 1$. By Lemma 4 we can assume one entry 0 in each of rows 2 to 5 of M_2 . The corresponding rows of M_1 and M_3 have exactly 2 entries 1, since otherwise N or $\bar{N} \subset B$. Then we can assume 2 equal rows in M_1 (say $b_{5,1} = b_{6,1} = b_{5,2} = b_{6,2} = 1$ and $b_{5,3} = b_{6,3} = 0$). As M_3 has at least one column with both entries 1 in rows 2 and 3 (say $b_{5,6} = b_{6,6} = 1$), we find \bar{N} in rows 4, 5, 6 and columns 1, 2, 6.

(ii) $b_{i,4} = b_{i,5} = 1$ for all i with $4 \leq i \leq 7$. In M_1 and M_3 rows 4 to 7 have at least one, and rows 8 to 13 at least 2 entries 1 (otherwise $N \subset B$). Together there are at least 16 entries 1, which guarantee at least one column with 6 entries 1 in M_1 , and in M_3 , since by Lemma 3 we can assume at most 6 entries 1 in every column. At most 2 entries 1 exist in the first 4 rows of every column of M_1 and M_3 (otherwise $\bar{N} \subset B$). If a column with 6 entries 1 in M_1 or M_3 has exactly one entry 1 in the first 4 rows, then N or $\bar{N} \subset B$ by Lemma 9, since the existence of S_8 would force a row 11 in rows 8 to 13 of M_2 , and then Lemma 4 can be used. As M_1 and M_3 cannot both have a column with 4 entries 0 in the first 4 rows (otherwise we use Lemma 4), we can assume that in M_1 a column exists, which has 6 entries 1, and 2 of them in the first 4 rows. Thus we may choose $b_{i,3} = 1$ for $i = 6, 7, \dots, 11$ and $b_{4,3} = b_{5,3} = b_{12,3} = b_{13,3} = 0$. Then by Lemma 9 we can assume S_8 in rows 6 to 11 and columns 4 to 8 of B . After possibly changing rows 12 and 13 we have $b_{12,4} = b_{13,5} = 1$, and $b_{12,5} = b_{13,4} = 0$ (otherwise we have cases (i) or (iii), or \bar{S}_3 , and Lemma 4 can be used). Together with rows 1 and 2 of B we get $N \subset B$, or $b_{12,1} = b_{12,2} = b_{13,1} = b_{13,2} = 1$.

We next prove, that 6 entries 1 in columns 2 or 3 of M_3 yield N or $\bar{N} \subset B$. After possibly changing rows 6 and 7, 8 and 9, 10 and 11, so as columns 7 and 8 of B we can assume $b_{12,8} = b_{13,8} = b_{5,8} = 1$, and $b_{4,8} = 0$ (rows 4 and 5 can be changed). By Lemma 9 we find N or $\bar{N} \subset B$, or S_8 in rows 5, 6, 12, 8, 13, 10 and columns 4, 5, 1, 3, 2 after possibly changing columns 1 and 2 of B . Thus we assume $b_{5,1} = b_{6,1} = b_{6,2} = b_{8,2} = b_{10,2} = 0$, $b_{8,1} = b_{10,1} = b_{5,2} = 1$. Then rows 8, 10, 12, 13 and columns 1, 6, 8 of B yield $N \subset B$, or $b_{12,6} = b_{13,6} = 0$. Rows 12 and 13 together with rows 1 and 3 of B yield $N \subset B$, or $b_{12,7} = b_{13,7} = 1$. Rows 8, 9, 10, 11 and columns 1, 3, 6 of B yield $\bar{N} \subset B$, or $b_{9,1} = b_{11,1} = 0$. Together with rows 1 and 2 it follows $b_{9,2} = b_{11,2} = 1$ (otherwise $N \subset B$). Rows 7, 9, 11 and columns 2, 3, 7 yield $\bar{N} \subset B$, or $b_{7,2} = 0$. Rows 1, 6, 7 and columns 1, 2, 6 force $N \subset B$, or $b_{7,1} = 1$. Rows 5, 12, 13 and columns 2, 7, 8 yield $\bar{N} \subset B$, or $b_{5,7} = 0$. Rows 1, 5, 6 and columns 1, 6, 7 force $N \subset B$, or $b_{5,6} = 1$. If now $b_{4,1} = 1$, then 6 entries 1 are in column 1 of M_1 , and by Lemma 9 it remains $b_{4,6} = 1$, $b_{4,7} = 0$ (S_8 is in rows 4, 7, 8, 12, 10, 13 and columns 4, 5, 8, 7, 6 of B). Then in case $b_{4,2} = 0$ we find S_3 in columns 2, 7 and rows 1, 4, 6, 8, 10 of B , and we use Lemma 4, and in case $b_{4,2} = 1$ rows 4, 5, 9 and columns 2, 4, 6 yield $\bar{N} \subset B$. If otherwise $b_{4,1} = 0$, then rows 1, 2, 4 and columns 1, 2, 3 yield $N \subset B$, or $b_{4,2} = 1$. Then $b_{4,7} = 0$ forces $N \subset B$ (rows 1, 4, 5 and columns 1, 3, 7), and $b_{4,7} = 1$ gives $\bar{S}_3 \subset B$ (rows 4, 9, 11, 12, 13, columns 2, 7), and we use Lemma 4.

In the following we can assume that at most 5 entries 1 exist in columns 2 and 3 of M_3 , that is, 6 entries 1 occur in column 1 of M_3 . By Lemma 9, this is possible only in two

cases, either $b_{4,6} = b_{5,6} = 1$ and $b_{12,6} = b_{13,6} = 0$, or $b_{4,6} = b_{5,6} = 0$ and $b_{12,6} = b_{13,6} = 1$. In the first case we find N in rows 1, 3, 12, 13 and columns 4 to 8, or $b_{12,7} = b_{12,8} = b_{13,7} = b_{13,8} = 1$. Then more than 5 entries 1 occur in column 2 or 3 of M_3 , or $N \subset B$ (rows 1, 4, 5, columns 3, 7, 8). In the second case we use Lemma 9, and we can assume S_8 in columns 3, 2, 1, 4, 5 of B , and (j) in rows 11, 9, 10, 8, 13, 12, or (jj) in rows 11, 8, 10, 9, 13, 12 of B , after possibly changing columns 1 and 2. In other words, we can assume $b_{10,2} = b_{11,1} = 0$, $b_{10,1} = b_{11,2} = 1$ and (j) $b_{8,2} = b_{9,1} = 0$, $b_{8,1} = b_{9,2} = 1$, or (jj) $b_{8,2} = b_{9,1} = 1$, $b_{8,1} = b_{9,2} = 0$.

In case (j) rows 9, 11, 12, 13 and columns 2, 6, 7 yield $\bar{N} \subset B$ or $b_{12,7} = b_{13,7} = 0$, rows 8, 10, 12, 13 and columns 1, 6, 8 yield $\bar{N} \subset B$ or $b_{12,8} = b_{13,8} = 0$, and then we have N in rows 1, 12, 13 and columns 3, 7, 8.

In case (jj) $b_{4,7} = b_{5,7} = 0$ or 1 yield N in rows 1, 4, 5 and columns 3, 6, 7, or \bar{N} in rows 4, 5, 7 and columns 4, 5, 7, respectively, and thus we can assume $b_{4,7} = 0$, $b_{5,7} = 1$, after possibly changing rows 4 and 5. Then rows 1, 3, 4 and columns 6, 7, 8 imply $N \subset B$ or $b_{4,8} = 1$, and rows 4, 5, 6 and columns 4, 5, 8 imply $\bar{N} \subset B$ or $b_{5,8} = 0$. If $b_{4,2} = 0$, then $b_{5,2} = 1$ or N is in rows 1, 4, 5 and columns 2, 3, 6. Rows 5, 11, 13 and columns 2, 5, 7 yield $\bar{N} \subset B$ or $b_{13,7} = 0$. Then rows 1, 3, 13 and columns 4, 7, 8 imply $N \subset B$ or $b_{13,8} = 1$. Rows 4, 10, 13 and columns 1, 5, 8 then yield $\bar{N} \subset B$ or $b_{4,1} = 0$, and we find N in rows 1, 2, 4 and columns 1, 2, 3. If, otherwise, $b_{4,2} = 1$, then $b_{6,2} = 0$ or \bar{N} is in rows 4, 6, 8 and columns 2, 4, 8, and $b_{12,8} = 0$ or \bar{N} is in rows 4, 8, 12 and columns 2, 4, 8. Rows 1, 3, 12 and columns 5, 7, 8 yield $N \subset B$ or $b_{12,7} = 1$. Then rows 5, 9, 12 and columns 1, 4, 7 yield $\bar{N} \subset B$ or $b_{5,1} = 0$. Rows 11, 12, 13 and columns 2, 6, 7 yield $\bar{N} \subset B$ or $b_{13,7} = 0$. Then rows 1, 3, 13 and columns 4, 7, 8 imply $N \subset B$ or $b_{13,8} = 1$. At last rows 4, 10, 13 and columns 1, 5, 8 yield $\bar{N} \subset B$ or $b_{4,1} = 0$, and we find N in rows 1, 4, 5 and columns 1, 3, 6.

(iii) $b_{i,4} = 0$, $b_{i,5} = 1$ for all i with $4 \leq i \leq 7$. There are at most 13 entries 1 in M_2 (otherwise (ii)). At least 3 entries 1 are in every row of rows 4 to 13 in columns 1 to 5, or in 4 to 8 (otherwise $N \subset B$). Thus at least 17 entries 1 exist in M_1 , and in M_3 , and either we use Lemma 3, or at least 2 columns of M_1 , and at least 2 columns of M_3 have exactly 6 entries 1.

Every column in the first 4 rows of M_1 or of M_3 has at least 2 entries 1 (otherwise $S_3 \subset B$, and we use Lemma 4). In the first 4 rows of M_1 and M_3 there exists at most one column with 4 entries 1 (otherwise $\bar{N} \subset B$), which we can assume not to be in M_1 . Each of the first 4 rows of M_1 and of M_3 has at least 2 entries 1 (otherwise $N \subset B$). Altogether the first 4 rows of M_1 contain at least 8 entries 1, so that 2 columns have exactly 3 entries 1, and we can assume $b_{4,3} = b_{5,3} = b_{6,3} = b_{4,2} = b_{5,2} = b_{7,2} = 1$, $b_{6,2} = b_{7,3} = 0$ (otherwise $\bar{N} \subset B$). Then $b_{6,1} = b_{7,1} = 1$, or $N \subset B$.

After possibly changing rows 6 and 7, and columns 2 and 3 of B we can assume column 3 to be one of the 2 columns of M_1 with 6 entries 1, that is, $b_{8,3} = b_{9,3} = b_{10,3} = 1$. Then Lemma 9 can be used, and we find S_8 in rows 4, 8, 5, 9, 6, 10 and columns 8, 7, 6, 4, 5 of B (columns 4 and 5 of S_8 have to be in M_2), that is,

$$\begin{aligned} b_{4,7} = b_{4,8} = b_{5,6} = b_{5,8} = b_{6,6} = b_{6,7} = b_{8,4} = b_{8,7} = b_{8,8} \\ = b_{9,4} = b_{9,6} = b_{9,8} = b_{10,4} = b_{10,6} = b_{10,7} = 1, \end{aligned}$$

and

$$b_{4,6} = b_{5,7} = b_{6,8} = b_{8,5} = b_{8,6} = b_{9,5} = b_{9,7} = b_{10,5} = b_{10,8} = 0.$$

In rows 4, 5, 7 and columns 2, 5, 8 of B we find \bar{N} , or $b_{7,8} = 0$, and then $b_{7,6} = b_{7,7} = 1$ (otherwise $N \subset B$). Rows 4, 5, 8, 9 and columns 2, 3, 8 of B yield $\bar{N} \subset B$, or $b_{8,2} = b_{9,2} = 0$.

If 6 entries 1 are in column 1 of M_1 , then N or $\bar{N} \subset B$ by Lemma 9 (S_8 is impossible, since rows 6 and 7 are identical in columns 4 to 8). Thus 6 entries 1 exist in column 2 of M_1 . Then Lemma 9 can be used, and columns 4 and 5 of S_8 have to be in M_2 . Then at most 11 entries 1 exist in M_2 , and therefore M_1 has more than 18 entries 1, that means, at least one column of M_1 has 7 entries 1, and Lemma 3 can be used.

LEMMA 12. Let Z_i denote the number of pairs of rows of B with i equal columns. Then either N or $\bar{N} \subset B$, or

$$Z_9 + Z_{10} \geq 12 + \sum_{i \leq 6} Z_i. \tag{6}$$

Proof. If $Z_i > 0$ for $i > 10$, then by Lemma 2 we have N or $\bar{N} \subset B$. Otherwise, if s_i denotes the number of columns with i entries 0, then

$$\sum_{i \leq 10} Z_i = \binom{13}{2} = 78, \quad \text{and} \quad \sum_{i \geq 0} s_i = 17, \tag{7}$$

$$\sum_{i \leq 10} iZ_i = \sum_{i \geq 0} \left(\binom{i}{2} + \binom{13-i}{2} \right) s_i \geq 36 \sum_{i \geq 0} s_i = 612. \tag{8}$$

At most 2 pairs of rows in every triple of rows of B have an even number of equal columns (Lemma 1, $c = 17$). Again the Theorem of Turán ([2], p. 17) implies then

$$Z_0 + Z_2 + Z_4 + Z_6 + Z_8 + Z_{10} \leq [13^2/4] = 42. \tag{9}$$

With (7) and $Z_8 + Z_{10}$ from (9) we obtain

$$\begin{aligned} \sum_{i \leq 10} iZ_i &= 2(Z_9 + Z_{10}) + Z_8 + Z_{10} + 7 \sum_{i \geq 10} Z_i - \sum_{i \leq 6} (7-i)Z_i \\ &\leq 2(Z_9 + Z_{10}) + 588 - 2 \sum_{i \leq 6} Z_i. \end{aligned} \tag{10}$$

From (8) and (10) we obtain (6).

LEMMA 13. If there are three rows a, b, c in B , so that the number of equal columns is at least 9 in rows a and b , and in rows a and c , then N or $\bar{N} \subset B$.

Proof. Let a, b, c be rows 1, 2, 3 of B . If more than 5 columns with both entries 0 (or both entries 1) occur in one pair of rows we use Lemma 2. If then the pairs of rows 1 and 2, and of rows 1 and 3 both contain 5 columns with both entries 0, or both contain 5 columns with both entries 1, then N or $\bar{N} \subset B$ follows directly, or by use of Lemmas 7, 10, or 11. Thus in the following we can assume 5 pairs with both entries 0 in the first 5 columns of rows 1 and 2, 5 pairs with both entries 1 in columns 6 to 10 of rows 1 and 3, and exactly 4

columns with both entries 1 in rows 1 and 2, and exactly 4 columns with both entries 0 in rows 1 and 3.

If in rows 2 and 3 there are less than 7 equal columns, then by Lemma 12 we have $Z_9 + Z_{10} \geq 13$ in B . Then the pigeonhole principle guarantees that one row of B exists which has together with each of two other rows at least 9 equal columns, among which in both pairs of rows occur 5 columns with both entries 0, or 5 columns with both entries 1. Thus we can use Lemmas 7, 10 or 11 to get N or $\bar{N} \subset B$.

In columns 11 to 17 occur at least 2 columns with both entries 0 in rows 1 and 3, and at least 2 columns with both entries 1 in rows 1 and 2. We can assume $b_{1,11} = b_{1,12} = b_{3,11} = b_{3,12} = 0$ and $b_{1,13} = b_{1,14} = b_{2,13} = b_{2,14} = 1$. Then S_1 or $\bar{S}_1 \subset B$, or $b_{2,11} = b_{2,12} = 1$ and $b_{3,13} = b_{3,14} = 0$. Now at least 7 equal columns in rows 2 and 3 are possible only if at least 4 of them occur in columns 1 to 10. More than 4 columns, however, yield N or $\bar{N} \subset B$. Thus we can choose $b_{3,1} = b_{3,2} = b_{2,8} = b_{2,9} = b_{2,10} = 0$, $b_{3,3} = b_{3,4} = b_{3,5} = b_{2,6} = b_{2,7} = 1$, and without loss of generality $b_{2,15} = b_{2,16} = b_{3,15} = b_{3,16} = 1$. Then $\bar{N} \subset B$, or $b_{1,15} = b_{1,16} = 0$.

If column 15 in rows 4 to 13 contains more than 6 entries 0, or more than 5 entries 1, then S_2 or $\bar{S}_2 \subset B$, and we can use Lemma 3. It remains to consider that column 15 in rows 4 to 13 contains (i) 5 entries 1, and (ii) 6 entries 0.

(i) $b_{i,15} = 1$ for all i with $4 \leq i \leq 8$. By Lemma 9 we can assume S_8 in rows 3 to 8 and columns 1 to 5. There exist 3 rows in this S_8 which together with row 3 of B have 2 columns with both entries 1. If one of these rows contains more than one entry 1 in columns 6 to 10, then $\bar{S}_{10} \subset B$, and we use Lemma 11. Otherwise in columns 6 to 10 we find \bar{N} , or we have 3 rows with at least 4 entries 0, and 8 rows with at least 3 entries 0.

Then any distribution of $3 \binom{4}{2} + 8 \binom{3}{2} = 42$ pairs 00 among the columns 6 to 10 guarantees

$S_3 \subset B$ (since $4 \binom{5}{2} < 42$), and we use Lemma 4.

(ii) $b_{i,15} = 0$ for all i with $4 \leq i \leq 9$. By Lemma 9 we can assume \bar{S}_8 in rows 4 to 9 and columns 6 to 10. If column 16 has at least 4 entries 0 in rows 4 to 9, then $S_3 \subset B$ (together with row 1 of B), and Lemma 4 can be used. Thus we can assume $b_{4,16} = b_{5,16} = b_{6,16} = 1$. If then in columns 3 to 5 in one of rows 4 to 6 there are 2 entries 1, then $\bar{S}_{10} \subset B$, and we use Lemma 11. Otherwise $N \subset B$, or in rows 4 to 6 and columns 1 and 2 occur entries 1 only. Then, however, we find \bar{N} in rows 4 to 6 and columns 1, 2 and 16, and Lemma 13 is proved.

The proof of Theorem 2 is complete, since Lemma 12 for any B guarantees the existence of three rows which enable us to apply Lemma 13.

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TECHNISCHE UNIVERSITÄT BRAUNSCHWEIG
D-3300 BRAUNSCHWEIG,
WEST GERMANY