BLOCK INTERSECTIONS IN BALANCED INCOMPLETE BLOCK DESIGNS

R. G. Stanton and D. A. Sprott

(received January 27, 1964)

1. Introduction. One of the most interesting of the smaller BIBD's is the system (8, 14, 7, 4, 3), where we write the parameters in the standard order v, b, r, k, λ. One representation of a design with these parameters is 1248, 3567; 2358, 1467; 3468, 1257; 4578, 1236; 5618, 2347; 6728, 1345; 7138, 2456. This particular design has the feature that every block B is paired with a complementary block B' consisting of all varieties not lying in B. Thus $B \cap B' = 0$. If we seek to generalize this type of design, we obtain

THEOREM 1. If a design contains one pair of complementary blocks, then it must have parameters

$$2x+2,\ t(4x+2),\ t(2x+1),\ x+1,\ tx .$$

Proof. Let the number of plots in a block be $k = x+1$. Since all varieties occur in a pair of complementary blocks B and B', it follows that $v = 2(x+1)$. Also, the basic BIBD relations give

$$\lambda(2x+1) = rx, \quad 2r = b .$$

Since x is relatively prime to $2x+1$, x must divide λ, say $\lambda = tx$. The theorem now follows.

It will be convenient to refer to the designs with parameters as specified in Theorem 1 as designs $H_2(t, x)$; a generalization will be given later. It should of course be pointed out that,

Canad. Math. Bull. vol. 7, no. 3, July 1964.

while we have shown that every design which splits into pairs of complementary blocks is automatically a design $H_2(t, x)$, it does not follow that a design $H_2(t, x)$ necessarily possesses the splitting property which we have discussed.

The simplest designs $H_2(t, x)$ are the designs $H_2(1, x) = H_2(x)$; we shall obtain certain results about these designs, and extend one of the results to designs in general. However, designs which have a factor in common among b, r, and λ, need not be ignored as implied by Parker [3]. The useful design made up of all selections of triplets from 7 varieties has parameters (7, 35, 15, 3, 5), yet no blocks are repeated; on the other hand, one can get a different design with these parameters by repeating the Fano design (7, 7, 3, 3, 1) a total of five times. Also, the design $(16, 8\lambda, 3\lambda, 6, \lambda)$ exists for all $\lambda > 1$, but not for $\lambda = 1$.

2. Block Intersection Properties. Parker [3] showed that for x odd it was not possible for two blocks of a design $H_2(x)$ to be identical; Seiden [4] extended this result to all x by using the theory of orthogonal arrays. In Theorem 2, we shall deduce this result by using a technique which is originally due to Fisher [2] and which has also been used by Bose [1].

THEOREM 2. *In a design $H_2(x)$, it is impossible to have two identical blocks.*

Proof. Let B_1 be a specific block and let $x_i = x_{1i}$ be the number of elements in $B_1 \cap B_i$, where i ranges from 2 to b. It immediately follows that, in general,

$$\bar{x} = \Sigma x_i/(b-1) = k(r-1)/(b-1),$$

$$\Sigma (x_i - \bar{x})^2 = k(\lambda k-k-\lambda+r) - k^2(r-1)^2/(b-1)$$

For the designs $H_2(x)$, we find

$$\bar{x} = 2x(x+1)/(4x+1) ,$$

(2.1) $$\Sigma (x_i - \bar{x})^2 = (x+1)^2 x/(4x+1) .$$

If there is another block B_j identical with B_1, then $x_j = k$ and

$$(x_j - \bar{x})^2 = (x+1)^2(2x+1)^2/(4x+1)^2 .$$

Since $\Sigma (x_i - \bar{x})^2 - (x_j - \bar{x})^2 \geq 0$,

we arrive at the contradiction

$$(x+1)^2 (-1 - 3x)/(4x+1)^2 \geq 0 .$$

It follows that there cannot be a block B_j identical with B_1. Since B_1 was arbitrary, the theorem follows.

The same method allows us to discuss the possibility of complementary blocks in $H_2(x)$.

THEOREM 3. If $x_2 = 0$, then x must be odd. Furthermore, all other values of x_i must be equal to $\frac{1}{2}(x+1)$.

Proof. The first part of this theorem was proved by Parker [3], using incidence matrices. We note that if a block, say B_2, is complementary to B_1, then $x_2 = 0$. Consider the b-2 variates $x_3, x_4, \ldots, x_b$. Then

$$\sum_{i=3}^{b} x_i = \sum_{i=2}^{b} x_i = k(r-1) = 2x(x+1), \quad \bar{x} = \frac{1}{2}(x+1) ;$$

$$\sum_{i=3}^{b} x_i^2 = (x+1)^2 x .$$

https://doi.org/10.4153/CMB-1964-050-0 Published online by Cambridge University Press

Clearly $\sum_{i=3}^{b} (x_i - \bar{x})^2 = (x+1)^2 x - 4x[\frac{1}{2}(x+1)]^2 = 0$.

Thus $x_i = \bar{x} = \frac{1}{2}(x+1)$ for $i \geq 3$. Since x_i is an integer, we see that x must be odd. Furthermore, if x is odd and there are two complementary blocks, then these two blocks intersect any other block in the same number of varieties, namely, $\frac{1}{2}(x+1)$.

3. A Generalization of the Fisher Inequality. Fisher's inequality $b \geq v$ was proved in [2]; we use the method of Section 2 to prove

THEOREM 4. If a BIBD contains $\alpha > 0$ blocks other than B_1 which are identical with a specified block B_1, then $b \geq (\alpha+1)v - (\alpha-1)$.

Proof. Define T by the equation

$$T = \Sigma (x_i - \bar{x})^2 = k(k\lambda - k - \lambda + r) - k^2(r-1)^2/(b-1) .$$

Using the basic relation

$$r-\lambda = rk - \lambda v , \tag{3.1}$$

we write

$$\begin{aligned} T &= k(k\lambda - k + rk - \lambda v) - k^2(r-1)^2/(b-1) \\ &= k\lambda(k-v) + k^2(r-1) - k^2(r-1)^2/(b-1) \\ &= k\lambda(k-v) + k^2(r-1)(b-r)/(b-1) . \end{aligned}$$

Now the basic relation $b/r = v/k$ can be written as

$$(b-r)/r = (v-k)/k , \tag{3.2}$$

so we obtain

$$T = k^2(r-1)(b-r)/(b-1) - k^2\lambda(b-r)/r ;$$

https://doi.org/10.4153/CMB-1964-050-0 Published online by Cambridge University Press

but the contribution from the blocks identical with B_1 is

$$\alpha(k - \bar{x})^2 = \alpha k^2 (b-r)^2/(b-1)^2 ,$$

and this cannot exceed T. We thus find

$$\alpha k^2 (b-r)/(b-1)^2 \leq k^2 (r-1)/(b-1) - k^2 \lambda / r .$$

This relation may be written as

$$\alpha(b-r)/(b-1) \leq (r^2 - b\lambda - r + \lambda)/r .$$

Now we may use (3. 1) to write

$$b - r = (bk - rk)/k = (rv - rk)/k = (rv + \lambda - r - \lambda v)/k$$

$$= (r-\lambda)(v-1)/k .$$

Also, by another use of (3. 1),

$$r^2 - b\lambda = (kr^2 - bk\lambda)/k = (kr^2 - rv\lambda)/k = r(r-\lambda)/k .$$

Our inequality may then be written

$$\alpha(r-\lambda)(v-1)/k(b-1) \leq (r-\lambda)/k - (r-\lambda)/r ;$$

since $r-\lambda > 0$, we find

$$\alpha\, bk(v-1)/v \leq (b-1)(r-k) = (b-1)(bk/v - k) ,$$

(3. 3) $$b\, \alpha(v-1) \leq (b-1)(b-v) .$$

If we put $\alpha = 0$ in (3. 3), we immediately obtain Fisher's result $b \geq v$. Assuming $\alpha \neq 0$, we can write (3. 3) as

$$b^2 - bv + v \geq \alpha b(v-1) + b ,$$

and so obtain

$$b^2 + v \geq bv(\alpha+1) - b(\alpha-1) ,$$

https://doi.org/10.4153/CMB-1964-050-0 Published online by Cambridge University Press

$$b \geq v(\alpha+1) - (\alpha-1) - \frac{v}{b} .$$

In this inequality $b = v$ is not possible. Thus $v/b < 1$; but b is an integer, and so

$$b \geq (\alpha+1)v - (\alpha-1) .$$

This establishes the theorem.

We note that $\alpha = 1$ implies that $b \geq 2v$; consequently, the condition that there be no repeated block leads to the restriction $b < 2v$. We then obtain

THEOREM 5. For a given value of v, the design having largest b for which there is no possibility of a repeated block is just the design $H_2(x)$.

Proof. If there is to be no repeated block, the restriction $b < 2v$ forces us to try $b = 2v - 1$. This value is impossible, since the equation

$$(2v-1)k = rv$$

leads to the contradiction that v must divide k. Thus we must try $b = 2v-2$. Then we obtain

$$(2v-2)k = rv, \; \lambda(v-1) = r(k-1) .$$

It follows from the first of these equations that $v = 2k$, $r = v-1$; from the second we then obtain $\lambda = k-1$. Our design is then

$$(2k, 4k-2, 2k-1, k, k-1) = H_2(k-1) .$$

4. The Family $H_n(x)$. If we seek to generalize the results of Section 2, we obtain

THEOREM 6. If a design contains a set S of n disjoint blocks forming a complete replication, then $r \geq k + \lambda$.

Proof. Let the blocks in S be $B_1, \ldots, B_n$. Then

https://doi.org/10.4153/CMB-1964-050-0 Published online by Cambridge University Press

$v = nk$ and $b = nr$, so there are $n(r-1)$ blocks outside S. Also, let x_j be the number of varieties in $B_1 \cap B_j$, where $j = n+1, \ldots, b$. We find, as usual,

$$\Sigma x_j = k(r-1), \quad \Sigma x_j^2 = k(\lambda k - \lambda - k + r),$$

$$\bar{x} = \frac{k(r-1)}{n(r-1)} = \frac{k}{n}.$$

Then

$$\Sigma (x_j - \bar{x})^2 = k(\lambda k - \lambda - k + r) - \frac{k^2(r-1)}{n} \geq 0.$$

So

$$nk(\lambda k - \lambda - k + r) - k^2(r-1) \geq 0,$$

$$v(\lambda k - \lambda - k + r) - k^2(r-1) \geq 0,$$

$$k(\lambda v - rk) + v(r-\lambda) - kv + k^2 \geq 0,$$

$$-k(r-\lambda) + v(r-\lambda) - k(v-k) \geq 0.$$

Divide by $v-k > 0$ to give the result $r-\lambda-k \geq 0$, that is $r \geq k+\lambda$.

It is well known (see for example Stanton [5]) that the condition $r \geq k+\lambda$ is equivalent to the condition $v \geq b+r-1$ given by Bose [1] for a resolvable design; However, we see here that this condition follows from the existence of a single set S (in a resolvable design, there are r sets of blocks, each forming a complete replication).

Bose [1] showed that if one had an affine resolvable design, that is, a resolvable design in which blocks from different replications have the same number of elements in common, then $b = v+r-1$; conversely, if $b = v+r-1$, the design is affine resolvable. This idea generalizes to give

THEOREM 7. If a design contains a set S of n disjoint blocks forming a complete replication, and if $r = k+\lambda$, then each block of S has the same number of elements in common--

with blocks outside S; moreover, v divides k^2.

Proof. If $r = k+\lambda$ in Theorem 6, then $\Sigma (x_j - \bar{x})^2 = 0$, that is,

$$x_j = \bar{x} = \frac{k}{n} = \frac{k^2}{nk} = \frac{k^2}{v} .$$

This result shows that x_j is constant; furthermore, since x_j is an integer, v must divide k^2.

We can now use the results of Theorems 6 and 7 to obtain a series $H_n(x)$ generalizing the results of Section 2.

THEOREM 8. Let a design contain a set S of n disjoint blocks forming a complete replication; also, let $r = k+\lambda$. Then the design, which we shall call $H_n(x)$, has parameters

$$n(nx-x+1),\ n(nx+1),\ nx+1,\ nx-x+1,\ x .$$

Proof. We have

$$v = nk, \quad b = nr ;$$

$$\lambda(v-1) = r(k-1) ,$$

$$r = k+\lambda .$$

Then

$$\lambda(nk-1) = r(k-1) = (k+\lambda)(k-1) ,$$

$$\lambda n = k+\lambda - 1 = r-1 .$$

So n divides $r-1$, and we may thus set $r-1 = nx$; the theorem follows.

COROLLARY 1. n is a factor of $x-1$.

Proof. For v divides k^2, that is, $n(nx-x+1)$ divides $(nx-x+1)^2$.

https://doi.org/10.4153/CMB-1964-050-0 Published online by Cambridge University Press

COROLLARY 2. Each block of S in the design $H_n(x)$ intersects all blocks outside S in $x - (x-1)/n$ varieties.

COROLLARY 3. If we drop the assumption $r = k+\lambda$ in Theorem 8, we obtain parameters

$$v = n[1 + \frac{x(n-1)}{y}], \quad b = n(y+nx) ,$$

$$r = y+nx, \quad k = 1 + \frac{x(n-1)}{y}, \quad \lambda = x ,$$

where $r-\lambda = ky$.

Proof. The relations $v = nk$, $b = nr$, $\lambda(v-1) = r(k-1)$, at once give $r-\lambda = k(r-\lambda n)$. So we may set $r-\lambda = ky$. We then obtain $y = r-\lambda n$, whence $k = \frac{nx - x + y}{y}$. The corollary follows. Evidently it is necessary that y divide $x(n-1)$; the theorem corresponds to the case $y = 1$. We can also use Theorem 4 to prove

THEOREM 9. The general family $H_n(x)$ cannot have repeated blocks.

Proof. Let α $(\alpha \geq 1)$ be the number of blocks, other than B_1 itself, identical with B_1. Then

$$b \geq (\alpha+1)v - (\alpha-1) .$$

For $H_n(x)$, we find

$$n(nx+1) \geq (\alpha+1)\, n(nx-x+1) - \alpha + 1 ,$$

$$n^2 x - n(n-1)(\alpha+1)x \geq n(\alpha+1) - \alpha+1 - n ,$$

$$nx(-n\alpha + \alpha + 1) \geq \alpha(n-1) + 1 .$$

Now n and x are fixed and positive; α must be chosen so that $\alpha+1 - n\alpha > 0$, that is, $\alpha(1-n) + 1 > 0$. This cannot occur since $n \geq 2$, $\alpha \geq 1$, We have thus established the theorem.

https://doi.org/10.4153/CMB-1964-050-0 Published online by Cambridge University Press

5. Conclusion. Interesting questions arise concerning the designs $H_2(t, x)$ with $t > 1$, non-isomorphic designs $H_2(x)$, the existence of designs $H_2(x)$ with prescribed block intersection numbers satisfying the relation (2.1), discussion of other series of designs. Studies along these lines are under way.

REFERENCES

1. R. C. Bose, A note on the resolvability of balanced incomplete block designs, Sankhya 6 (1942), 105-110.

2. R. A. Fisher, An examination of different possible solutions in incomplete blocks, Ann. Eugenics 10 (1940), 52-75.

3. E. T. Parker, Remarks on balanced incomplete block designs, Proc. Amer. Math. Soc. 14 (1963), 729-730.

4. E. Seiden, A supplement to Parker's "Remarks on balanced incomplete block designs", Proc. Amer. Math. Soc. 14 (1963), 731-732.

5. R. G. Stanton, A Note on BIBDs, Ann. Math. Stat. 28 (1957), 1054-1055.

University of Waterloo

https://doi.org/10.4153/CMB-1964-050-0 Published online by Cambridge University Press