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On Some Twistor Spaces Over $4\mathbb{CP}^2$

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Abstract. We show that for any positive integer τ there exist on $4\mathbb{CP}^2$, the connected sum of four complex projective planes, twistor spaces whose algebraic dimensions are two. Here, τ appears as the order of the normal bundle of C in S, where S is a real smooth half-anti-canonical divisor on the twistor space and C is a real smooth anti-canonical divisor on S. This completely answers the problem posed by Campana and Kreussler. Our proof is based on the method developed by Honda, which can be regarded as a generalization of the theory of Donaldson and Friedman.

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1. Introduction

Let $n\mathbb{CP}^2$ be the connected sum of n copies of the complex projective plane, where $0\mathbb{CP}^2$ denotes the four-sphere S^4 by convention. Let g be a self-dual metric on $n\mathbb{CP}^2$ and Z the associated twistor space. Throughout this paper we always assume that the type of the scalar curvature of g is positive. From the works of Poon, LeBrun, Kreussler, Kurke, Campana and others [P1, P2, LB, KK, Kr1, Kr2, C] it has turned out that twistor spaces associated with such self-dual metrics have rich structures as compact complex threefolds.

In this paper we focus our attention on the case $n = 4$. This case is interesting because we have $c_1(Z)^3 = 4 - n = 0$ [Hi], where $c_1(Z)$ denotes the first Chern class of Z and $c_1(Z)^3$ is a positive multiple of the coefficient of the leading term of the Riemann–Roch for pluri-anti-canonical system of Z. Another reason is that for $n \leqslant 3$ twistor spaces over $n\mathbb{CP}^2$ have already been described [P1, P2, KK] and the case $n = 4$ is the next one to be studied.

Some important families of twistor spaces over $4\mathbb{CP}^2$ are known. (a) LeBrun twistor spaces [LB]: They are explicitly given as bimeromorphic transforms of conic bundles over $\mathbb{CP}^1 \times \mathbb{CP}^1$. In particular they are Moishezon threefolds. They have a holomorphic $\mathbb{C}^*$-action. They are naturally parameterized by distinct four points on H^3, the upper half three-space, and form a six-dimensional family. (b) Twistor spaces with a $(\mathbb{C}^*)^2$-action [PP2, Hon1]: They have a pencil whose general elements

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are nonsingular toric surfaces, and the base locus of the pencil is the anti-canonical curve of the surfaces. In particular, they are Moishezon. They are naturally parameterized by distinct four points on the circle and form a one-dimensional family. (c) Another Moishezon twistor spaces are also known [Kr2, Hon2]: They have a net of rational surfaces and the associated meromorphic maps give on the twistor spaces (meromorphic) conic bundle structures over $\mathbb{CP}^2$. (d) Recently the author and M. Itoh [HI] have proved that there exist twistor spaces over $4\mathbb{CP}^2$ with a $\mathbb{C}^*$-action whose corresponding self-dual metrics are not LeBrun's or Joyce's.

On the other hand Campana and Kreussler [CK] showed that there exist twistor spaces over $4\mathbb{CP}^2$ whose algebraic dimensions are two. More precisely they showed the following: Let $|-\frac{1}{2}K_Z|^\sigma$ be the real sub-system of the half-anti-canonical system of Z, where σ denotes the real structure of Z. Let $S \in |-\frac{1}{2}K_Z|^\sigma$ be an irreducible element. (It is relatively easy to see that such an S always exists on any twistor space over $4\mathbb{CP}^2$.) Then by a result of Pedersen and Poon [PP1] S is an eight points blown-up of $\mathbb{CP}^1 \times \mathbb{CP}^1$. Hence we always have $\dim|-K_S| \geqslant 0$, and if C is an irreducible nonsingular anti-canonical curve of S the degree of the normal bundle, which we will denote by $N_{C/S}$, is zero. Then Campana and Kreussler showed the following: (i) Let $a(Z)$ denote the algebraic dimension of Z. Then $1 \leqslant a(Z) \leqslant 2$ and the equality $a(Z) = 2$ holds if and only if the order of $N_{C/S}$ in $\mathrm{Pic}^0 C$ is finite. (ii) For *some* $\tau \geqslant 1$, there exists a twistor space Z over $4\mathbb{CP}^2$ with $S \in |-\frac{1}{2}K_Z|^\sigma$ and $C \in |-K_S|^\sigma$ such that the order of $N_{C/S}$ in $\mathrm{Pic}^0 C$ is τ.

Then they asked [CK, Open Problem] which values of τ can be realized as above for some twistor spaces over $4\mathbb{CP}^2$. The purpose of this paper is then to give an answer to this problem in the following form:

THEOREM 1.1. *For* any $\tau \geqslant 1$ *there exist twistor spaces over* $4\mathbb{CP}^2$ *with the following property: There exist smooth and irreducible members* $S \in |-\frac{1}{2}K_Z|^\sigma$ *and* $C \in |-K_S|^\sigma$ *respectively such that the order of* $N_{C/S}$ *in* $\mathrm{Pic}^0 C$ *is* τ.

It is easy to see that for distinct τ the twistor spaces are not biholomorphic. Thus for each $\tau \geqslant 1$ there exist twistor spaces over $4\mathbb{CP}^2$ whose algebraic dimensions are two. We also remark that all of the twistor spaces (a)–(c) cited above contain only reducible $C \in |-K_S|^\sigma$.

Our proof of Theorem 1.1 is based on the method developed in [Hon2], which is a generalization, in a sense, of the theory of Donaldson and Friedman [DF]. That is, for any given integer $\tau \geqslant 1$ we construct a 'triple' (Z', S', A') of normal crossing varieties, where S' (resp. A') is a (real) Cartier divisor on Z' (resp. S'). Then we will show that this triple can be smoothed to give a twistor space over $4\mathbb{CP}^2$ in Theorem 1.1.

Finally we should mention that in the previous paper [HI] we have already shown the existence of twistor spaces in the case that $\tau = 1$. But the twistor spaces considered in that paper are different from the one in this paper even in the case that

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$\tau = 1$. For example, as was mentioned above, twistor spaces in [HI] have a $\mathbb{C}^*$-action, whereas the identity component of the automorphism group of twistor spaces in Theorem 1.1 is trivial.

2. Main Construction

In this section we shall construct a triple (Z', S', A') of normal crossing varieties which depends on an integer $\tau \geqslant 1$. This will be used in Section 3 to prove Theorem 1.1.

Let g be a self-dual metric on $3\mathbb{CP}^2$ whose scalar curvature is of positive type. That is, there exists a C^∞-function φ on $3\mathbb{CP}^2$ such that the scalar curvature of $e^\varphi g$ is a positive constant. Let Z be the twistor space associated to g. Such a twistor space belongs to either of the following (Sections 2 and 3 of [P2]; See also [KK, Kr1]):

(i) (generic type [P2, KK, Kr1]) Assume that the complete linear system $|-\frac{1}{2}K_Z|$ has no base points. Then $|-\frac{1}{2}K_Z|$ is three-dimensional and defines a morphism $f : Z \to \mathbb{CP}^3$. f is a double covering map branched along a (real) quartic surface.

(ii) (LeBrun twistor spaces [LB]) Assume that $|-\frac{1}{2}K_Z|$ has base points. In this case we also have $\dim|-\frac{1}{2}K_Z| = 3$, but the image under the associated meromorphic map is $\mathbb{CP}^1 \times \mathbb{CP}^1$, a (real) quadric surface. Further a bimeromorphic model of Z has a conic bundle structure over the quadric surface. Z has a $\mathbb{C}^*$-action and is one of twistor spaces constructed by LeBrun [LB].

For the proof of Theorem 1.1 we use a twistor space over $3\mathbb{CP}^2$ which is type (i). From now on let Z_1 be such a twistor space, σ_1 the real structure of Z_1 and $f : Z_1 \to \mathbb{CP}^3$ the double covering map induced by $|-\frac{1}{2}K_{Z_1}|$. Moreover, let B denote the branch quartic surface, which is real with respect to σ_1. It was shown in [P2, KK, Kr1] that B has exactly 13 ordinary double points, one of which is the unique real point on B.

Let H_1 be a real plane on $\mathbb{CP}^3$ which intersects B transversally along a nonsingular curve. We further assume that H_1 does not go through any of the singular points of B. Then we put $S_1 := f^{-1}(H_1)$. By construction S_1 is a real nonsingular element of $|-\frac{1}{2}K_{Z_1}|^{\sigma_1}$. Adjunction formula and the vanishing theorem of Hitchin imply that the restriction of f onto S_1 is the morphism induced by $|-K_{S_1}|$, which is two-dimensional without base points. It is easy to see that S_1 is a rational surface with $c_1^2 = 2$. But the reality implies more [PP1]: S_1 is obtained from $\mathbb{CP}^1 \times \mathbb{CP}^1$ by blowing-up six points. Let $p : S_1 \to \mathbb{CP}^1$ be the composition of the blowing-down and the projection to one of the $\mathbb{CP}^1$s. Then twistor lines (on Z_1) which are contained in S_1 are parameterized by $S^1 \subseteq \mathbb{CP}^1$, the real circle. Let $\{L_s := p^{-1}(s) \mid s \in S^1\}$ be the family of twistor lines. By choosing H_1 sufficiently general we may suppose that the blown-up six points are in general position. That is, no two (resp. four) points among the six points are on a curve of bidegree (1,0) or (0,1) (resp. a curve of bidegree (1,1)), and the six points are not on a curve of bidegree (1,2) or (2,1). Then we have

$(*)$ For any twistor line $L_s\,(=p^{-1}(s))$ on S_1 $f|_{L_s}$, the restriction of f onto L_s, is a biholomorphic map onto a real *conic* on H_1.

In fact if the image $f(L_s)$ is a line then there must be an effective curve D on S_1 such that $D+L_s$ is an anti-canonical curve of S_1. But since L_s is an element of the system $|\beta^*\mathcal{O}(0,1)|$, where $\beta: S_1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ is the above blowing-down map, $\beta(D)$ must be a curve of bidegree (2, 1) which goes through all of the (blown-up) six points. This contradicts to the above generality condition.

Next let m_1 be a real line on H_1 which intersects $B \cap H_1$ transversally, and put $C_1 := f^{-1}(m_1)$. Clearly C_1 is a non-singular elliptic curve with a real structure and is a real anti-canonical curve of S_1. Since $C_1 \cdot L_s = -K_{S_1} \cdot L_s = -\frac{1}{2}K_{Z_1} \cdot L_s = 2$ and both C_1 and L_s are real, $C_1 \cap L_s$ consists of two distinct points for every $s \in S^1$. Therefore the set $\{C_1 \cap L_s \mid s \in S^1\}$ defines an unramified double covering over the circle S^1. We denote this by $\mathcal{T}$. $\mathcal{T}$ is obviously a real subset of C_1. By choosing m_1 sufficiently general we may assume that the following holds:

$(**)$ The four ramification points of the double covering map $f|_{C_1}: C_1 \to m_1$ are not on any twistor lines on S_1.

Further let $m\,(\neq m_1)$ be also a real line on H_1 and set $y := m_1 \cap m$ and $f^{-1}(y) = \{w_1, \overline{w}_1\}$, where we put $\overline{w}_1 := \sigma_1(w_1)\,(\neq w_1)$. The situation is illustrated as follows:

$$\begin{array}{ccccccc} Z_1 & \supseteq & S_1 & \supseteq & C_1 & \supseteq & \{w_1, \overline{w}_1\} \\ f\downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{CP}^3 & \supseteq & H_1 & \supseteq & m_1 & \ni & y \end{array}$$

Then we have isomorphisms

$$N_{C_1/S_1} \simeq \mathcal{O}_{C_1}(w_1 + \overline{w}_1) \simeq f^*\mathcal{O}_{m_1}(1).$$

Now we consider a map $\alpha: C_1 \longrightarrow \mathrm{Pic}^0 C_1$ which is defined by

$$z \longmapsto \mathcal{O}_{C_1}(w_1 + \overline{w}_1 - z - \overline{z})$$

$$(\simeq \mathcal{O}_{C_1}(-z - \overline{z}) \otimes f^*\mathcal{O}_{m_1}(1)).$$

Then the structure of α is described as follows: The image of α, which we denote by $\mathcal{S}^1$, is the circle. $\mathcal{S}^1$ is the identity component of $(\mathrm{Pic}^0 C_1)^{\sigma_1} := \{F \in \mathrm{Pic}^0 C_1 \mid \overline{\sigma_1^* F} \simeq F\}$. α gives on C_1 the structure of a fiber bundle over $\mathcal{S}^1$. When $(\mathrm{Pic}^0 C_1)^{\sigma_1}$ is connected, that is $(\mathrm{Pic}^0 C_1)^{\sigma_1} = \mathcal{S}^1$, the typical fiber of α is a circle. When $(\mathrm{Pic}^0 C_1)^{\sigma_1}$ is disconnected, which is two disjoint circles, the typical fiber of α is two disjoint circles. These can be proved, for example, by classifying all of anti-holomorphic involutions on elliptic curves and writing down explicitly the equation of the fibers of α (using a coordinate on the universal cover $\mathbb{C}$).

Now we show that:

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LEMMA 2.1 *For any positive integer τ there exists a point $z \in C_1 \backslash \mathcal{T}$ such that the order of $\alpha(z)$ in* $\mathrm{Pic}^0 C_1$ *is τ.*

Proof. Let $\varphi(\tau)$ denote the Euler function of τ. That is, for a positive integer τ, $\varphi(\tau)$ denotes the number of integers n with $1 \leqslant n \leqslant \tau$ such that $(n, \tau) = 1$. Then there exist $\varphi(\tau)$ points on $\mathcal{S}^1$ whose order (in $\mathrm{Pic}^0 C_1$) is τ. If $\tau \geqslant 7$ or $\tau = 5$ we have $\varphi(\tau) \geqslant 3$ and hence it is obvious that the claim of the lemma holds. When $\varphi(\tau) = 2$, that is $\tau = 3, 4$ or 6, it suffices to show that $\mathcal{T}$ cannot coincide with the fiber over the two-torsion points. Suppose that. Then $\mathcal{T}$ consists of disjoint two circles $\mathcal{T}_1$ and $\mathcal{T}_2$ and each are the fibers over two-torsion points. But this cannot happen, since we have $\mathcal{T}_2 = \sigma_1(\mathcal{T}_1)$, α preserves the real structures, and hence even if $\mathcal{T}$ is contained in some fiber of α, it must be *a* fiber of α.

Therefore to prove the lemma it suffices to show that $\mathcal{T}$ cannot be the fiber (of α) over the trivial bundle or the real line bundle whose order is two. First we show that $\alpha(z) \not\simeq \mathcal{O}_{C_1}$ for any $z \in \mathcal{T}$. Assume that $\alpha(z) \simeq \mathcal{O}_{C_1}$. Then since in such a case $\mathcal{O}_{C_1}(z + \overline{z}) \simeq f^*\mathcal{O}_{m_1}(1)$ and the system $|\mathcal{O}_{C_1}(z + \overline{z})|$ is one-dimensional we have $f(z) = f(\overline{z})$. On the other hand if $z \in \mathcal{T}$ there exists a twistor line $L_s \subseteq S_1$ such that $z \in L_s$. Therefore $f(z) \neq f(\overline{z})$ since $z \neq \overline{z}$ and $f|_{L_s}$ is an isomorphism by $(*)$. This is a contradiction. Hence $\alpha(z) \not\simeq \mathcal{O}_{C_1}$ for any $z \in \mathcal{T}$, and the case $\tau = 1$ is proved.

Next let $z \in C_1$ be a ramification points of $f|_{C_1} : C_1 \to m_1$. Then we have

$$\alpha(z)^{\otimes 2} = \mathcal{O}_{C_1}(-2z - 2\overline{z}) \otimes f^*\mathcal{O}_{m_1}(2) \simeq f^*\mathcal{O}_{m_1}(-2) \otimes f^*\mathcal{O}_{m_1}(2) \simeq \mathcal{O}_{C_1}.$$

Moreover z does not lie on $\mathcal{T}$ by $(**)$. Hence, $\alpha(z)$ is a torsion point whose order is two. Therefore the case $\tau = 2$ is also proved. □

Let $\tau \geqslant 1$ be a given integer, $z_{10} \in C_1 \backslash \mathcal{T}$ a point such that the order of $\alpha(z_{10})$ in $\mathrm{Pic}^0(\mathrm{C}_1)$ is τ, L_1 the twistor line on Z_1 through z_{10} (and $\overline{z}_{10}$), and $\mu_1 : Z_1' \to Z_1$ the blowing-up along L_1. Further we set $S_1' := \mu_1^{-1}(S_1)$, $Q_1 := \mu_1^{-1}(L_1)$, $l_1 := \mu_1^{-1}(z_{10})$ and $\overline{l}_1 := \mu_1^{-1}(\overline{z}_{10})$. Let $C_1' (\subseteq S_1')$ denote the strict transform of C_1. Since $L_1 S_1$ $\mu_1|_{S_1'}$ is the blowing-up at z_{10} and $\overline{z}_{10}$, and l_1 and $\overline{l}_1$ are the exceptional curves. Then we have

$$N_{C_1'/S_1'} \simeq \mathcal{O}_{C_1'}(w_1 + \overline{w}_1 - z_{10} - \overline{z}_{10}) = \alpha(z_{10}),$$

where we regard $w_1, \overline{w}_1, z_{10}$ and $\overline{z}_{10}$ as points on C_1'. Hence, by the choice of z_{10}, the order of $N_{C_1'/S_1'}$ in $\mathrm{Pic}^0\mathrm{C}_1'$ is τ. It is easy to show the following claim:

CLAIM 2.2. *The τth anti-canonical system $|-\tau K_1'|$ of S_1' is one-dimensional without base points, and defines an elliptic fibration $g : S_1' \to \mathbb{CP}^1$.*

We note that $\tau C_1'$ is a real element of $|-\tau K_1'|$ and that the anti-Kodaira dimension (cf. [S]) of S_1' is one.

Next let f_1 be a real nonsingular fiber of g. Since f_1 is linearly equivalent to $\tau C_1'$ and we have $C_1' \cdot l_1 = 1$, we have $f_1 \cdot l_1 = \tau$ and may suppose that f_1 intersects l_1 transversally at τ distinct points. Let $\{z_{11}, \cdots, z_{1\tau}\}$ be the intersections. Then we have $\{\overline{z}_{11}, \cdots, \overline{z}_{1\tau}\} = f_1 \cap \overline{l}_1$ by the reality of f_1.

On the other hand let Z_2 be the flag twistor space of $\mathbb{CP}^2$ with Fubini–Study metric, σ_2 the real structure and $L_2 \subseteq Z_2$ any twistor line. Then there exists a divisor D_2 on Z_2 which satisfies (i) $D_2 \cdot L_2 = 1$; (ii) D_2 and $\overline{D}_2 := \sigma_2(D_2)$ intersect transversally along L_2. Let $\mu_2 : Z_2' \to Z_2$ be the blowing-up along L_2, Q_2 the exceptional divisor, and D_2' and $\overline{D}_2'$ the proper transforms of D_2 and $\overline{D}_2$ respectively. D_2 and $\overline{D}_2$ are isomorphic to Σ_1, the non-minimal Hirzebruch surface. Further we set $l_2 := D_2' \cap Q_2$ and $\overline{l}_2 := \overline{D}_2' \cap Q_2$. These define disjoint sections of $\mu_2|_{Q_2} : Q_2 \to L_2$.

Next we choose a biholomorphic map $\phi : Q_1 \to Q_2$ which preserves the real structures and satisfies $\phi(l_1) = l_2$ and $\phi(\overline{l}_1) = \overline{l}_2$. Then we set ([DF, KP]) $Z' := Z_1' \cup_\phi Z_2'$, and

$$S' := S_1' \cup (D_2' \amalg \overline{D}_2') = D_2' \underset{l}{\cup} S_1' \underset{\overline{l}}{\cup} \overline{D}_2'.$$

Here, we put $l := l_1 \simeq l_2$ and $\overline{l} := \overline{l}_1 \simeq \overline{l}_2$. S' is clearly a Cartier divisor which is invariant by the natural real structure of Z'.

Next for each i with $0 \leqslant i \leqslant \tau$ we set $z_{2i} := \phi(z_{1i}) \in l_2$, $\overline{z}_{2i} := \phi(\overline{z}_{1i}) \in \overline{l}_2$ and let f_{2i} (resp. $\overline{f}_{2i}$) be the fiber of $D_2 \to \mathbb{CP}^1$ (resp. $\overline{D}_2 \to \mathbb{CP}^1$) through z_{2i} (resp. $\overline{z}_{2i}$). Then we put

$$C' := C_1' \underset{\phi}{\cup} (f_{20} \amalg \overline{f}_{20}) = f_{20} \cup C_1' \cup \overline{f}_{20}',$$

and

$$f' := f_1 \underset{\phi}{\cup} \Big(\coprod_{i=1}^{\tau} (f_{2i} \amalg \overline{f}_{2i})\Big) = \Big(\coprod_{i=1}^{\tau} f_{2i}\Big) \cup f_1 \cup \Big(\coprod_{i=1}^{\tau} \overline{f}_{2i}\Big).$$

(See next page for figures.) We note that C' and f' are Cartier divisors on S' which are invariant by the real structure. Furthermore we put $A' := C' + f'$.

3. Proof of Theorem 1.1

In the previous section for each $\tau \geqslant 1$ we have constructed a triple (Z', S', A') of normal crossing varieties, where S' (resp. A') is a real Cartier divisor on Z' (resp. S'). In this section using the results of [Hon2] we study smoothing of this triple and prove Theorem 1.1. For notations we refer to Sections 3 and 5 of [Hon2].

First we consider smoothing of the pair (S', A'). The following lemma can be proved in the same way as Proposition 3.1 and Lemma 3.2 of [Hon2].

LEMMA 3.1. *We have* $\Theta^1_{S',A'} \simeq \mathcal{O}_l \oplus \mathcal{O}_{\overline{l}}$ *and* $\Theta^i_{S',A'} = 0$ *for* $i \geqslant 2$, *and there exists an exact sequence of vector spaces*

$$0 \longrightarrow H^1(\Theta_{S',A'}) \longrightarrow T^1_{S',A'} \xrightarrow{r} H^0(\Theta^1_{S',A'})$$

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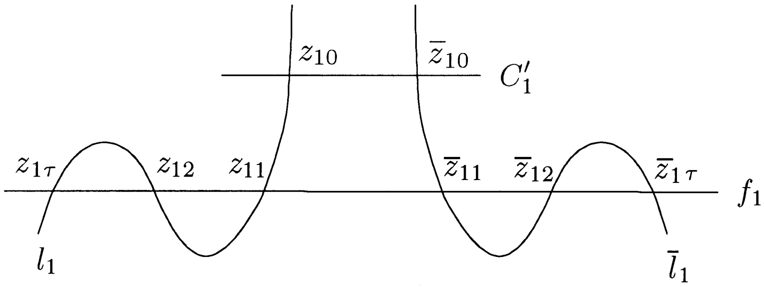

Figure. Curves on S_1'.

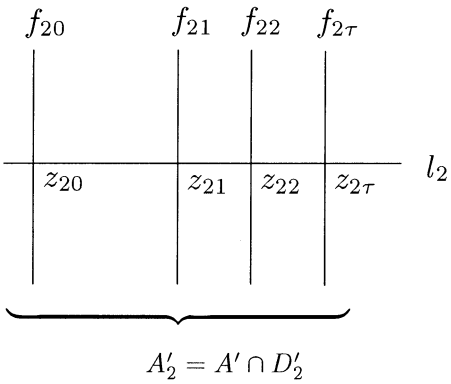

Figure. Curves on D_2'.

$$\longrightarrow H^2(\Theta_{S',A'}) \longrightarrow T^2_{S',A'} \longrightarrow H^1(\Theta^1_{S',A'}) = 0.$$

Next we show (after Proposition 3.3):

LEMMA 3.2. *We have* $H^2(\Theta_{S',A'}) = 0$.

Lemmas 3.1 and 3.2 imply

PROPOSITION 3.3. *We have* $T^2_{S',A'} = 0$. *In particular deformations of the pair* (S', A') *are unobstructed.*

Proof of Lemma 3.2. For simplicity we put $A_1' := C_1' + f_1 \,(\subseteq S_1')$, $A_2' := \Sigma_{i=0}^{\tau} f_{2i} \,(\subseteq D_2')$ and $\overline{A}_2' := \Sigma_{i=0}^{\tau} \overline{f}_{2i} \,(\subseteq \overline{D}_2')$. Then we have $A' = A_1' + A_2' + \overline{A}_2'$. First we consider the exact sequence

$$0 \to \Theta_{S',A'} \to \Theta_{S_1',A_1'+l_1+\bar{l}_1} \oplus (\Theta_{D_2',A_2'+l_2} \oplus \Theta_{\overline{D}_2',\overline{A}_2'+\bar{l}_2}) \\ \to \Theta_l(-1-\tau) \oplus \Theta_{\bar{l}}(-1-\tau) \to 0. \qquad (1)$$

(Strictly speaking we must take the normalization of S' into consideration. But for simplicity of notations we omit it.)

CLAIM 3.4. *We have* $H^2(\Theta_{D'_2,A'_2+l_2}) = H^2(\Theta_{\overline{D}'_2,\overline{A}'_2+\bar{l}_2}) = 0.$

Proof. The cohomology exact sequence of

$$0 \to \Theta_{D'_2,A'_2+l_2} \to \Theta_{D'_2,l_2} \to \mathcal{O}_{A'_2} \to 0$$

shows that $H^2(\Theta_{D'_2,A'_2+l_2}) \simeq H^2(\Theta_{D'_2,l_2})$. But the latter cohomology group is easily seen to vanish by using the exact sequence

$$0 \to \Theta_{D'_2,l_2} \to \Theta_{D'_2} \to \mathcal{O}_{l_2}(1) \to 0.$$

By the reality we also have $H^2(\Theta_{\overline{D}'_2,\overline{A}'_2+\bar{l}_2}) = 0$. (qed for Claim 3.4) □

CLAIM 3.5. *The natural map* $H^1(\Theta_{D'_2,A'_2+l_2}) \to H^1(\Theta_{l_2}(-1-\tau))$ *is surjective.*

Proof. The cohomology exact sequence of

$$0 \to \Theta_{D'_2,A'_2}(-l_2) \to \Theta_{D'_2,A'_2+l_2} \to \Theta_{l_2}(-1-\tau) \to 0$$

shows that we have only to show that $H^2(\Theta_{D'_2,A'_2}(-l_2)) = 0$. But the exact sequence

$$0 \to \Theta_{D'_2,A'_2}(-l_2) \to \Theta_{D'_2}(-l_2) \to \mathcal{O}_{A'_2}(-1) \to 0$$

implies that $H^2(\Theta_{D'_2,A'_2}(-l_2)) \simeq H^2(\Theta_{D'_2}(-l_2))$. Further the exact sequences

$$0 \to \Theta_{D'_2}(-l_2) \to \Theta_{D'_2,l_2} \to \Theta_{l_2} \to 0$$

and

$$0 \to \Theta_{D'_2,l_2} \to \Theta_{D'_2} \to N_{l_2/D'_2} \to 0$$

show that we have

$$H^2(\Theta_{D'_2}(-l_2)) \simeq H^2(\Theta_{D'_2,l_2}) \simeq H^2(\Theta_{D'_2}) = 0,$$

as desired. (qed for Claim 3.5) □

CLAIM 3.6. *We have* $H^2(\Theta_{S'_1,A'_1+l_1+\bar{l}_1}) = 0.$

Proof. The exact sequence

$$0 \to \Theta_{S'_1,A'_1+l_1+\bar{l}_1} \to \Theta_{S'_1,A'_1} \to N_{l_1/S'_1} \oplus N_{\bar{l}_1/S'_1} \to 0$$

implies that $H^2(\Theta_{S'_1,A'_1+l_1+\bar{l}_1}) \simeq H^2(\Theta_{S'_1,A'_1})$.

To prove that $H^2(\Theta_{S'_1,A'_1})$ is zero we first show that $H^0(\Omega_{S'_1}(C'_1)) = 0$, where Ω_X denotes the cotangent sheaf of a complex manifold X. We choose a blowing-down map $\beta : S_1 \to S_0 := \mathbb{CP}^1 \times \mathbb{CP}^1$ and put $\alpha := \beta \cdot (\mu_1|_{S'_1})$. α is eights points blown-up of S_0. We put $C_0 := \alpha(C'_1)(= \beta(C_1))$, which is an anti-canonical curve of S_0. Then the eight points, which we denote by $P := \{p_1, \cdots, p_8\} (\subseteq S_0)$, clearly lie on C_0 and we may assume that $p_i \neq p_j$ for $i \neq j$. Further, we set $E_i := \alpha^{-1}(p_i)$ for $1 \leqslant i \leqslant 8$, the exceptional curves of α, and put $E := \Sigma_{i=1}^8 E_i$. Then we have an exact

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sequence

$$0 \to \alpha^*\Omega_{S_0} \to \Omega_{S_1'} \to \Omega_E \to 0,$$

from which, by taking tensor product with $\mathcal{O}_{S_1'}(C_1')$, we get an exact sequence

$$0 \longrightarrow (\alpha^*\Omega_{S_0}) \otimes \mathcal{O}_{S_1'}(C_1') \longrightarrow \Omega_{S_1'}(C_1') \longrightarrow \Omega_E \otimes \mathcal{O}_{S_1'}(C_1') \longrightarrow 0. \tag{2}$$

Since C_1' is the strict transform of C_0 we have $\mathcal{O}_{S_1'}(C_1') \simeq (\alpha^*\mathcal{O}_{S_0}(C_0)) \otimes \mathcal{O}_{S_1'}(-E)$ and hence the first nontrivial term of (2) becomes $(\alpha^*\Omega_{S_0}(C_0)) \otimes \mathcal{O}_{S_1'}(-E)$. On the other hand being $E_i \cdot C_1' = 1$ for each i $(1 \leqslant i \leqslant 8)$ the last nontrivial term of (2) becomes $\oplus_{i=1}^{8}(\Omega_{E_i} \otimes \mathcal{O}_{E_i}(1))$, which we denote by $\mathcal{O}_E(-1)$ for simplicity. Therefore (2) can be rewritten as

$$0 \longrightarrow (\alpha^*\Omega_{S_0}(C_0)) \otimes \mathcal{O}_{S_1'}(-E) \longrightarrow \Omega_{S_1'}(C_1') \longrightarrow \mathcal{O}_E(-1) \longrightarrow 0.$$

Hence we get an isomorphism

$$H^0(S_1', \Omega_{S_1'}(C_1')) \simeq H^0(S_0, \Omega_{S_0}(C_0) \otimes \mathcal{I}_P), \tag{3}$$

where $\mathcal{I}_P$ denotes the ideal sheaf of P in S_0. On the other hand, the second Chern class of $\Omega_{S_0}(C_0)$ is

$$\begin{aligned} c_2(\Omega_{S_0}(C_0)) &= c_2(\Omega_{S_0}) + c_1(\Omega_{S_0}) \cdot c_1(\mathcal{O}_{S_0}(C_0)) + C_0^2 \\ &= e(S_0) + K_{S_0} \cdot (-K_{S_0}) + (-K_{S_0})^2 \\ &= e(S_0) = 4, \end{aligned}$$

where $e(S_0)$ denotes the Euler number of S_0. Therefore if a section of $\Omega_{S_0}(C_0)$ has only isolated zeros it vanishes at four points. Hence a nonzero element s of $H^0(S_0, \Omega_{S_0}(C_0) \otimes \mathcal{I}_P)$ must vanish along a curve containing P. But since P is on an anti-canonical curve C_0 and the six points among P are in general position (see $(*)$ in Section 2) this implies that s determines a nonzero section of $\Omega_{S_0}(C_0) \otimes \mathcal{O}_{S_0}(-C_0) \simeq \Omega_{S_0}$. But this cannot happen because S_0 is rational. Hence, by using (3) and Serre duality we get

$$H^0(\Omega_{S_1'}(C_1')) = H^2(\Theta_{S_1'} \otimes 2K_{S_1'}) = 0. \tag{4}$$

Now assume that $\tau = 1$. Then we have $\Theta_{S_1'}(-A_1') \simeq \Theta_{S_1'} \otimes 2K_{S_1'}$. Hence, (4) and the cohomology exact sequence of

$$0 \to \Theta_{S_1'}(-A_1') \to \Theta_{S_1', A_1'} \to \Theta_{A_1'} \to 0$$

imply that $H^2(\Theta_{S_1', A_1'}) = 0$, which is the claim for the case $\tau = 1$.

Next we show that $H^2(\Theta_{S_1', A_1'}) = 0$ for $\tau \geqslant 2$. Since in this case $N_{C_1'/S_1'}$ is a non-trivial line bundle of degree zero we have $H^1(N_{C_1'/S_1'}) = 0$. Hence the

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cohomology exact sequence of

$$0 \to \Theta_{S'_1, A'_1} \to \Theta_{S'_1} \to N_{C'_1/S'_1} \oplus N_{f_1/S'_1} \to 0$$

shows that it suffices to show that the map $H^1(\Theta_{S'_1}) \to H^1(N_{f_1/S'_1})$ is surjective. Considering further the cohomology exact sequences of

$$0 \to \Theta_{S'_1, f_1} \to \Theta_{S'_1} \to N_{f_1/S'_1} \to 0$$

and

$$0 \to \Theta_{S'_1}(-f_1) \to \Theta_{S'_1, f_1} \to \Theta_{f_1} \to 0,$$

we have only to show that $H^2(\Theta_{S'_1}(-f_1)) = 0$. Moreover, by Serre duality, this is equivalent to $H^0(\Omega_{S'_1}((\tau - 1)C'_1)) = 0$.

Fix $\tau \geqslant 2$. We show by induction on k that

$$H^0(\Omega_{S'_1}(kC'_1)) = 0 \tag{5}$$

for any $1 \leqslant k \leqslant \tau - 1$. The case $k = 1$ is nothing but (4). Assume that (5) holds for some k, $1 \leqslant k \leqslant \tau - 2$. The exact sequence $0 \to N^*_{C'_1/S'_1} \to \Omega_{S'_1}|_{C'_1} \to \Omega_{C'_1} \to 0$ splits because $N^*_{C'_1/S'_1}$ is non-trivial. That is, we have

$$\Omega_{S'_1}|_{C'_1} \simeq N^*_{C'_1/S'_1} \oplus \Omega_{C'_1}. \tag{6}$$

By taking the tensor product of $\Omega_{S'_1}$ with the exact sequence $0 \to \mathcal{O}_{S'_1}(kC'_1) \to \mathcal{O}_{S'_1}((k+1)C'_1) \to (k+1)N_{C'_1/S'_1} \to 0$, we get an exact sequence

$$0 \longrightarrow \Omega_{S'_1}(kC'_1) \longrightarrow \Omega_{S'_1}((k+1)C'_1) \longrightarrow \Omega_{S'_1}|_{C'_1} \otimes (k+1)N_{C'_1/S'_1} \longrightarrow 0. \tag{7}$$

But by (6) the last nontrivial term of this sequence is isomorphic to $kN_{C'_1/S'_1} \oplus (k+1)N_{C'_1/S'_1}$, whose cohomology groups vanish since we have assumed that $1 \leqslant k \leqslant \tau - 2$. Thus by using (7) we have $H^0(\Omega_{S'_1}(kC'_1)) \simeq H^0(\Omega_{S'_1}((k+1)C'_1))$. Hence by assumption we get $H^0(\Omega_{S'_1}((k+1)C'_1)) = 0$. In particular we have $H^0(\Omega_{S'_1}((\tau - 1)C'_1)) = 0$. This is the required result. (qed for Claim 3.6)

Completion of the Proof of Lemma 3.2. Then the cohomology exact sequence of (1) and Claims 3.4–3.6 (and the reality) imply $H^2(\Theta_{S',A'}) = 0$. □

Let $\{\mathcal{S} \overset{p}{\to} B, \mathcal{A} \overset{q}{\to} B \text{ with } \mathcal{A} \hookrightarrow \mathcal{S}\}$ be the Kuranishi family of deformations of the pair (S', A'). By Proposition 3.3 B can be regarded as a small open ball in $T^1_{S',A'}$ containing the origin and we have isomorphisms $p^{-1}(0) \simeq S'$ and $q^{-1}(0) \simeq C'$. Again by Proposition 3.3 and the exact sequence of Lemma 3.1 we have an exact sequence

$$0 \longrightarrow H^1(\Theta_{S',A'}) \longrightarrow T^1_{S',A'} \overset{r}{\longrightarrow} H^0(\mathcal{O}_l \oplus \mathcal{O}_{\bar{l}}) \longrightarrow 0. \tag{8}$$

Then the following proposition can be proved along the same line as in the proof of Proposition 2.3 in [Hon2].

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PROPOSITION 3.7. *Let* $t(\neq 0) \in B(\subseteq T^1_{S',A'})$ *be an element such that both of the factors of* $r(t)$ *in* (8) *are non-zero. Then* $S_t := p^{-1}(t)$ *satisfies the following:* (i) S_t *is nonsingular and is an eight points blown-up of* $\mathbb{CP}^1 \times \mathbb{CP}^1$, (ii) *the* τ*th anti--canonical system of* S_t *is one-dimensional without base points and defines an elliptic fibration* $S_t \to \mathbb{CP}^1$, (iii) *there exists real and nonsingular anti-canonical curve* C_t *of* S_t, *such that* (iv) *the order of* N_{C_t/S_t} *in* $\mathrm{Pic}^0 C_t$ *is* τ.

Proof. By the choice of t it is obvious that $S_t = p^{-1}(t)$ is nonsingular and $A_t := q^{-1}(t)$ consists of two smooth curves C_t and f_t which are invariant by the natural real structure of S_t. It is also obvious that both C_t and f_t are elliptic curves, because C_t (resp. f_t) is obtained as a smoothing of C' (resp. f') and the curves f_{20} and $\overline{f}_{20}$ (resp. f_{2i} and $\overline{f}_{2i}$ $(1 \leqslant i \leqslant \tau)$) are smooth rational curves.

Now following the idea of [KP] we proceed as follows. Let $\Delta \subseteq \mathbb{C}$ be a small open disk around the origin and $\varpi : S'_1 \times \Delta \to \Delta$ the projection. Let $\gamma : \mathcal{S}_\Delta \to S'_1 \times \Delta$ be the blowing-up with center $(l_1 \amalg \overline{l}_1) \times \{0\}$, and put $\varpi' := \varpi \cdot \gamma : \mathcal{S}_\Delta \to \Delta$. Then it is easily shown that $\varpi'^{-1}(0)$ is biholomorphic to S'. That is, the pair $(S', A' = C' + f')$ can be smoothed to obtain the pair $(S'_1, A'_1 = C'_1 + f_1)$. Hence the versality of the Kuranishi family of deformations of the pair (S', A') implies that $(S_t = p^{-1}(t), A_t = q^{-1}(t))$ can be obtained as smooth deformation of (S'_1, A'_1). In particular we have $c_1^2(S_t) = c_1^2(S'_1) = 0$.

We choose a blowing-down map $\beta: S_1 \to S_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ as in Section 2 and set $\beta' := \mu_1|_{S'_1} \cdot \beta$, where $\mu_1: Z'_1 \to Z_1$ is the blowing-up with center L_1 as before. β' is eight points blowing-up of S_0. Let n and n' be curves on S_0 whose bidegrees are $(1, 0)$ and $(0, 1)$ respectively. We suppose that they do not go through the blown-up eight points on S_0. Set $n_1 := \beta'^{-1}(n)$ and $n'_1 := \beta'^{-1}(n')$. We regard n_1 and n'_1 as curves on S' which do not go through the singular locus of S'. Then since both of $N_{n_1/S'}$ and $N_{n'_1/S'}$ are trivial n_1 and n'_1 are stable by any small deformations of S'. Let n_t and n'_t be preserved curves on S_t, and let β_t be the rational map associated to the linear system $|n_t + n'_t|$. Then β_t gives a blowing-down map $S_t \to \mathbb{CP}^1 \times \mathbb{CP}^1$ since both of N_{n_t/S_t} and $N_{n'_t/S_t}$ are trivial and we have $H^1(\mathcal{O}_{S_t}) = 0$ by upper-semi-continuity. Combining with $c_1^2(S_t) = 0$ we have (i).

We now know that (S_t, A_t) is obtained as a smooth deformation of (S'_1, A'_1) as rational surfaces. Then recalling that C'_1 (resp. f_1) is an anti-canonical curve (resp. a τ-th anti-canonical curve) of S'_1, we may conclude that C_t (resp. f_t) is also an anti-canonical curve (resp. a τ-th anti-canonical curve).

Thus we get two distinct τth anti-canonical curves τC_t and f_t and, hence, the τth anti-canonical system of S_t is at least one-dimensional. But since $f_t^2 = 0$ (because f_t is pluri-anti-canonical curve of S_t with $c_1^2(S_t) = 0$), $|f_t|$ is at most one-dimensional without base points. Hence we have completed the proof of (ii) and (iii). For a proof of (iv) see [BPV, III (8.3)], for example. □

Next we investigate deformations of the triple (Z', S', A') which was constructed in Section 2.

LEMMA 3.8. *We have* $H^2(\Theta_{Z'}(-S'))=0$.

Proof. By Proposition 4.1 in [Hon2] we have only to show that $H^2(\Theta_{Z_1}(-S_1))=0$. Since Z_1 is a Moishezon twistor space a result of Campana [C, Lemma 1.9] shows that it suffices to show that the restriction map $H^2(Z_1,\mathbb{C})\to H^2(S_1,\mathbb{C})$ is injective. But the latter is shown by Kreussler [Kr1, p. 258]. □

The following Proposition can be proved in the same way as Propositions 4.5 and 4.6 in [Hon2], using Lemmas 3.2, 3.8 and Proposition 3.3. So we omit the proof.

PROPOSITION 3.9. *We have* $T^2_{Z',S',A'}=H^2(\Theta_{Z',S',A'})=0$. *In particular deformations of the triple* (Z',S',A') *are unobstructed. Further we have a commutative diagram*

$$\begin{array}{ccc} \mathbf{T}^1_{Z',S',A'} & \longrightarrow & \mathbf{T}^1_{S',A'} \\ \downarrow & & \downarrow \\ H^0(\mathcal{O}_Q) & \xrightarrow{h} & H^0(\mathcal{O}_l)\oplus H^0(\mathcal{O}_{\bar{l}}), \end{array}$$

where the vertical arrows are surjective and h is given by $t\mapsto(t,t)$.

Let $\{\mathcal{Z}'\xrightarrow{\rho}B',\mathcal{S}'\xrightarrow{p'}B',\mathcal{A}'\xrightarrow{q'}B',\ \text{with}\ \mathcal{A}'\hookrightarrow\mathcal{S}'\hookrightarrow\mathcal{Z}'\}$ be the Kuranishi family of deformations of the triple (Z',S',A'), where B' can be identified with a small open ball in $T^1_{Z',S',A'}$ containing the origin by Proposition 3.9.

Let $\xi\in T^1_{Z',S',A'}$ be any real vector whose image in $H^0(\mathcal{O}_Q)$ (see the above diagram) is non-zero. Let $B''\subseteq B'$ be any real holomorphic curve in B' through the origin whose tangent vector at the origin is ξ. Let $\{\mathcal{Z}''\to B'',\mathcal{S}''\to B'',\mathcal{A}''\to B''$ with $\mathcal{A}''\hookrightarrow\mathcal{S}''\hookrightarrow\mathcal{Z}''\}$ be the restriction of the Kuranishi family onto B'' and $t\in B''$ be a non-zero real element. Then by Donaldson and Friedman [DF] $Z_t:=\rho^{-1}(t)$ is a twistor space of $4\mathbb{CP}^2$. Further as in the proof of Proposition 2.5 in [Hon2] $S_t:=p'^{-1}(t)$ is a real nonsingular element of $|-\frac{1}{2}K_{Z_t}|$. Moreover by Proposition 3.7 there exists a real nonsingular anti-canonical curve C_t of S_t such that the order of N_{C_t/S_t} in $\mathrm{Pic}^0 C_t$ is τ. (The reality of C_t easily follows from that of t.)

That is, we have proved

THEOREM 3.10 (= Theorem 1.1). *Z_t is a twistor space over $4\mathbb{CP}^2$ with the following property: There exist real, smooth and irreducible members $S_t\in|-\frac{1}{2}K_{Z_t}|$ and $C_t\in|-K_{S_t}|$ respectively such that the order of N_{C_t/S_t} in* $\mathrm{Pic}^0 C_t$ *is* τ.

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