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# AN INEQUALITY FOR GAMMA FUNCTIONS

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ABSTRACT. By using Bellman–Wishart distribution, Bellman [1], an inequality for gamma functions is derived. This inequality generalizes a recent inequality given by Selliah [4].

**1. Introduction.** Selliah [4] proves that

(1) $$[\Gamma_p^2(\delta+\alpha)/\Gamma_p(\delta)\Gamma_p(\delta+2\alpha)]\le\delta/(\delta+p\alpha^2),$$

where $\delta+\alpha>2t(p-1)+1$, $\delta>0$, $p>0$, $t=\frac{1}{4},\frac{1}{2}$, and

(2) $$\Gamma_p(\delta)=\pi^{tp(p-1)}\prod_{i=1}^{p}\Gamma(\delta-2t(i-1)).$$

When $t=\frac{1}{4}$ the result (1) holds for the real Wishart distribution and when $t=\frac{1}{2}$ it holds for the complex Wishart distribution.

The Bellman–Wishart density of a $p\times p$ positive definite Hermitian ($p\,dH$) matrix $S$, with $N$ degrees of freedom, is defined as follows. Let $S$ be a $p\times p$ $p\,dH$ matrix and

(3) $$S_k=(S_{ij}),\qquad 1\le i,\qquad j\le k,$$

and for convenience $\Lambda=\operatorname{diag}(\lambda_1,\ldots,\lambda_p)$. Then the Bellman–Wishart density $\phi(S,\Lambda)=\phi$ of $S$ is

(4) $$(\Gamma_p^*(a))^{-1}\prod_{i=1}^{p}\lambda_i^{-a_{p-i+1}}|S|^{a_p-2t(p-1)-1}$$
$$\exp\{-\operatorname{tr}\Lambda^{-1}S\}\prod_{j=2}^{p}|S_j|^{-k_{j-1}},$$

where $a_p=2NT$, $a_j=k_{p-j+1}+\cdots+k_p$, $k=(k_1,\ldots,k_p)$, $a=(a_1,\ldots,a_p)$ and $a_j>2t(p-1)+1$, and

(5) $$\Gamma_p^*(a)=\pi^{tp(p-1)}\prod_{i=1}^{p}\Gamma(a_1-2t(i-1)).$$

When $t=\frac{1}{4}$ we assume $S$ in (4) to be a real $p\times p$ positive definite symmetric matrix, and when $t=\frac{1}{2}$ it is a $p\,dH$ matrix.

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Now to derive an inequality for (4) similar to (1), we need a modification of Cramer–Rao multiparameter lower bound for the variance of an unbiased estimate. We state the result as follows.

Let $E(T)=\psi(\lambda_1,\ldots,\lambda_p)=\psi$, then the multiparameter Cramer–Rao lower bound is

$$V(T)\geq\sum_{i=1}^{p}\sum_{j=1}^{p} j^{ij}\frac{d\psi}{d\lambda_1}\frac{d\psi}{d\lambda_j}, \tag{6}$$

where

$$(J_{ij})=(J^{ij})^{-1}=\operatorname{cov}\left(\frac{\partial\log\phi}{\partial\lambda_i},\frac{\partial\log\phi}{\partial\lambda_j}\right) \tag{7}$$

Now consider the matrix

$$\Delta=\begin{bmatrix} V(T) & \sigma' \\ \sigma & (J_{ij})\end{bmatrix},\qquad \sigma'=\left(\frac{\partial\psi}{\partial\lambda_1},\ldots,\frac{\partial\psi}{\partial\lambda_p}\right), \tag{8}$$

then the squared multiple correlation coefficient between $T$ and

$$\frac{\partial\log\phi}{\partial\lambda_1},\frac{\partial\log\phi}{\partial\lambda_2},\ldots,\frac{\partial\log\phi}{\partial\lambda_p}$$

is

$$\theta^2=(\sigma'(J_{ij})^{-1}\sigma)/V(T). \tag{9}$$

If $\theta_1$ and $\theta_2$ are the smallest and largest roots of $\Delta$, then Eaton [2] shows that

$$\sigma^2=(\sigma'(J_{ij})^{-1}\sigma)/V(T)\leq\frac{(\theta_2-\theta_1)^2}{(\theta_2+\theta_1)^2}. \tag{10}$$

Now it follows from (10) that

$$V(T)\geq[(\theta_1+\theta_2)^2/(\theta_1-\theta_2)^2](\sigma'(J_{ij})^{-1}\sigma). \tag{11}$$

We use (11) in the next section to generalize the gamma function inequality given by Selliah [4].

2. **Gamma function inequality.** From (4) an unbiased estimate $T$ of $|\Lambda|^\alpha$, $\alpha>0$, is

$$T=C_\alpha|S|^\alpha,\qquad C_\alpha=(\Gamma_p^*(a+\alpha))^{-1}\Gamma_p^*(a), \tag{12}$$

and

$$V(T)=|\Lambda|^{2\alpha}([C_\alpha^2/C_{2\alpha}]-1). \tag{13}$$

Further, we have that

$$J_{ij}=\operatorname{cov}\left(\frac{\partial \log \phi}{\partial \lambda_i},\frac{\partial \log \phi}{\partial \lambda_j}\right)=0, \qquad i\neq j, \tag{14}$$

$$J_{ii}=E\left(-\frac{\partial^2 \log \phi}{\partial \lambda_i^2}\right)=(a_{p-i+1})\lambda_i^{-2}, \tag{15}$$

$$\partial\,|\Lambda|^{\alpha}/\partial\lambda_i=\alpha\,|\Lambda|^{\alpha}/\lambda_i, \qquad i=1,\ldots,p, \tag{16}$$

and hence from (11) it follows that

$$[\Gamma_p^{*2}(a+\alpha)/\Gamma_p^*(a)\Gamma_p^*(a+2\alpha)] \leq \left(1+\frac{(\theta_1+\theta_2)^2}{(\theta_1-\theta_2)^2}\sum_{j=1}^{p}(a_{p-j+1})^{-1}\alpha^2\right). \tag{17}$$

Note that (17) is a sharper upper bound than one given by Selliah [4] by assuming $a_1=\cdots=a_p=\delta$.

It is possible to derive similar inequalities for beta functions by using Olkin's [3] modified multivariate beta distributions.

Let

$$\begin{bmatrix} 1 & p_1' \\ p_1 & p_{22} \end{bmatrix} \tag{18}$$

be the correlation matrix corresponding to the covariance matrix $\Delta$ of (8), then $\theta^2$ of (9) equals

$$\theta^2=p_1'p_{22}^{-1}p_1. \tag{19}$$

Since $p_{22}^{-1}-I$ is known to be at least positive semidefinite, as $I-p_{22}$ is at least positive semidefinite, we have that

$$\theta^2=p_1'p_{22}^{-1}p_1\geq p_1'p_1\leq 1. \tag{20}$$

Thus we find that

$$\text{Min } p_1p_{22}^{-1}p_1, \quad \text{subject to} \quad p_1'p_1\leq 1, \tag{21}$$

is $\lambda$, where $\lambda$ is the smallest root of $p_{22}^{-1}$. It follows

$$\lambda\leq\theta^2=(\sigma'(J_{i_j}^{-1})\sigma)/V(T)\leq(\theta_2-\theta_1)^2/(\theta_2+\theta_1)^2. \tag{22}$$

Now by using (22) we may set a lower brand for the left hand side of (17).

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