# ON COMMUTATIVE $V^*$-ALGEBRAS

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## 1. Introduction

We shall use results of Palmer (**10**, **11**) and of Edwards and Ionescu Tulcea (**6**) to show that a commutative $V^*$-algebra (with identity) of operators on a weakly complete Banach space is isomorphic to such an algebra on a Hilbert space, the isomorphism extending to the weak closures of the algebras. This result leads to an extension of Stone's theorem on unitary groups (a similar extension is proved by different methods in (**2**, p. 350) and of Nagy's theorems on semigroups of normal operators. The same technique yields an easy proof of Dunford's theorem on the existence of a $\sigma$-complete extension of a bounded Boolean algebra of projections on a weakly complete Banach space. We are indebted to H. R. Dowson for suggesting this topic and for help and guidance in pursuing it.

## 2. Preliminaries

$\mathbf{R}$ will denote the reals, $\mathbf{R}^+$ the non-negative reals, $\mathbf{Q}$ the rationals, $\mathbf{Q}^+$ the non-negative rationals, $\mathbf{C}$ the complex numbers, $\mathbf{Z}$ the positive integers. For each $n \in \mathbf{Z}$, let $e_n = [-n, n]$.

Let $\Lambda$ be a compact Hausdorff space, $C(\Lambda)$ the space of continuous (complex) functions on $\Lambda$ under the supremum norm. Let $\chi_\sigma$ be the characteristic function of $\sigma$ for each $\sigma \subset \Lambda$. Let $S(\Lambda)$ ($S_0(\Lambda)$) be the family of Borel (Baire) sets of $\Lambda$. Let $B(\Lambda)$ ($B_0(\Lambda)$) be the family of bounded Borel (Baire) measurable functions on $\Lambda$. Let $S_0(\mathbf{R})$ and $S_0(\mathbf{C})$ be the family of Borel (= Baire) sets of $\mathbf{R}$ and $\mathbf{C}$ respectively.

Completeness and $\sigma$-completeness of Boolean algebras of projections are defined in (**1**, pp. 345-346). Spectral measures and spectral operators are defined in (**3**, pp. 291-292).

Let $X$ be a complex Banach space with dual space $X'$ and $\mathscr{A}$ be a subalgebra of $L(X)$, the algebra of bounded operators on $X$. Let $s(\mathscr{A})$ and $w(\mathscr{A})$ be the strong and weak closure of $\mathscr{A}$. It is well known that $w(\mathscr{A}) = s(\mathscr{A})$.

For any subset $\mathscr{B}$ of $L(X)$ let $\mathscr{B}^c = \{S \in L(X)\colon \forall T \in \mathscr{B}, ST = TS\}$. If $H$ is a Hilbert space and $\mathscr{A}$ a $*$-subalgebra (with identity) of $L(H)$, then $\mathscr{A}^{cc} = \mathscr{A}$ if and only if $\mathscr{A} = s(\mathscr{A})$.

Let $T \in L(X)$. $T$ is said to be hermitian in Vidav's sense if, for real $\lambda$, $\| I + i\lambda T \| = 1 + o(\lambda)$ as $\lambda \to 0$.

Let $\mathscr{A}$ be a subalgebra (with identity) of $L(X)$ and $\mathscr{H}$ the set of hermitian elements in $\mathscr{A}$. $\mathscr{A}$ is called a $V^*$-algebra if $\mathscr{A} = \mathscr{H}+i\mathscr{H}$, in the sense that $\forall S \in \mathscr{A}\ \exists\ R, J \in \mathscr{H}$ such that $S = R+iJ$. It follows from the theorem of (**10**, p. 539) that in these circumstances $R$ and $J$ are determined uniquely, and that $*: R+iJ \mapsto R-iJ$ is the (Vidav) involution on $\mathscr{A}$. By the same theorem, a subalgebra with identity of $L(X)$ is $V^*$ if and only if it is $C^*$.

Let $S$ be an unbounded operator on $X$. Following (**11**, p. 386) we call $S$ self-conjugate if it generates a strongly continuous group of isometries: i.e. if there exists a family $\{U(t, S):\ t \in \mathbf{R}\}$ of operators such that

(1) $U(0, S) = I$,

(2) $\forall s, \forall t,\ U(s, S)U(t, S) = U(s+t, S)$,

(3) $\forall x \in X, \forall s \in \mathbf{R}, \lim_{t\to s} \| U(t, S)x - U(s, S)x \| = 0$,

(4) $\forall t \in \mathbf{R}, \| U(t, S)\| = 1$,

(5) $\mathscr{D}(S) = \{x: \lim_{t\to 0} [(it)^{-1}(U(t, S)x - x)]$ exists$\}$, and

$$\forall x \in \mathscr{D}(S),\ Sx = \lim_{t\to 0} [(it)^{-1}(U(t, S)x - x)].$$

We call $S$ strongly self-conjugate if it is self-conjugate and if the group $\{U(t, S)\}$ is contained in a commutative $V^*$-algebra with identity. We call $S$ normal if $S = R+iJ$, where $R$ and $J$ are both strongly self-conjugate and the groups $\{U(t, R)\}$ and $\{U(t, J)\}$ are contained in the same $V^*$-algebra. We note that a bounded operator is normal if and only if it belongs to a commutative $V^*$-algebra with identity (**11**, p. 402).

We assume throughout that $X$ is weakly (sequentially) complete.

### 3. $V^*$-algebras

Throughout this paper we shall abbreviate " commutative $V^*$-algebra with identity " to " $V^*$-algebra ".

Let $\mathscr{A}$ be a $V^*$-algebra on $X$ with structure space $\Lambda$. There exists an isometric $*$-isomorphism $\psi$: $C(\Lambda)\to\mathscr{A}$. By Theorem 2.5 of (**11**, p. 392) there exists a unique regular strongly countably additive self-conjugate spectral measure $E(\ )$ in $L(X)$, defined on $S(\Lambda)$, such that $\forall f \in C(\Lambda)$

$$\psi(f) = \int_\Lambda f(\lambda)E(d\lambda).$$

Extend $\psi$ in the obvious way to give $\psi$: $B(\Lambda)\to L(X)$.

**Lemma 1.** $\psi B_0(\Lambda) \subset s(\mathscr{A})$.

**Proof.** $B_0(\Lambda)$ is the smallest family of bounded functions which contains $C(\Lambda)$ and is closed under pointwise sequential limits (**8**, p. 164). Hence it is enough to prove that if $(f_n)$ is a bounded sequence in $B_0(\Lambda)$ with pointwise limit $f$ and with $\psi(f_n) \in s(\mathscr{A})$, then $\psi(f) \in s(\mathscr{A})$. But, by (**5**, IV. 10.10), $\forall x \in X$

$$\psi(f)x = \lim_n \psi(f_n)x.$$

Let $\mathscr{R}(E) = \{E(\sigma): \sigma \in S_0(\Lambda)\}$ and let $\mathscr{A}(E)$ be the weakly closed algebra generated by $\mathscr{R}(E)$.

**Lemma 2.** $w(\mathscr{A}) = \mathscr{A}(E)$.

**Proof.** Every $f \in C(\Lambda)$ can be approximated uniformly by simple Baire measurable functions; hence $\mathscr{A} \subset \mathscr{A}(E)$, and so $w(\mathscr{A}) \subset \mathscr{A}(E)$. For each $\sigma \in S_0(\Lambda)$ the function $\chi_\sigma \in B_0(\Lambda)$. Hence, by Lemma 1, $E(\sigma) = \psi(\chi_\sigma) \in s(\mathscr{A})$. Therefore $\mathscr{A}(E) \subset w(\mathscr{A})$.

By Corollary 2 of (**6**, p. 549) there exist:

(1) a Hilbert space $H$,

(2) a strongly countably additive self-adjoint spectral measure $E^h(\ )$ in $L(H)$, defined on $S_0(\Lambda)$, and

(3) a map $\phi$: $\mathscr{A}(E) \to \mathscr{A}(E^h)$ which is a *-isomorphism (on extending the involution to $\mathscr{A}(E)$), onto, norm bicontinuous, and both strongly and weakly bicontinuous on bounded sets. Let $\mathscr{A}^h = \phi\mathscr{A}$.

**Lemma 3.** *$w(\mathscr{A})$ is a $V^*$-algebra.*

**Proof.** Let $\mathscr{S}$ be the set of *-hermitian operators in $w(\mathscr{A})$, and let $S \in \mathscr{S}$. $S$ is the weak limit of a net of *-hermitian operators in $\mathscr{A}$, and these are Vidav-hermitian. Hence $S$ is Vidav-hermitian. Clearly, $w(\mathscr{A}) = \mathscr{S} + i\mathscr{S}$ and $w(\mathscr{A})$ is therefore a $V^*$-algebra. Also, $\phi$ is a $C^*$-algebra isomorphism, and hence an isometry.

**Theorem 1.** *Let $T$ be a bounded normal operator on $X$, and $\mathscr{A}$ the $V^*$-algebra generated by $T$. Let $\Lambda$ be the structure space of $\mathscr{A}$, $E(\ )$ the spectral measure as above. Then $\Lambda$ and $\sigma(T)$ are homeomorphic and $\lambda \in \sigma(T)$ is an eigenvalue of $T$ if and only if $E(\{\lambda\}) \neq 0$.*

**Proof.** Take $H$, $\phi$, $\mathscr{A}^h$ as above. Since $\sigma(T)$ is determined by the set of maximal ideals of $\mathscr{A}$, it follows that $\sigma(T) = \sigma(\phi T)$. By (**5**, IX. 3.15) there exists an isometric *-isomorphism $\theta$: $C(\phi T)) \to \mathscr{A}^h$. Thus $C(\Lambda)$ and $C(\sigma(T))$ are *-isomorphic as algebras and so $\Lambda$ and $\sigma(T)$ are homeomorphic. We identify $\Lambda$ and $\sigma(T)$, and write $\psi$ above as

$$\psi\colon\ C(\sigma(T)) \to \mathscr{A} : f \mapsto \int_{\sigma(T)} f(\lambda)E(d\lambda),$$

and extend $\psi$ to give $\psi$: $B_0(\sigma(T)) \to w(\mathscr{A})$.

That $\lambda$ is an eigenvalue of $T$ if and only if $E(\{\lambda\}) \neq 0$ is shown by direct imitation of the proofs of (**5**, X. 3.3(ii)) and (**5**, X. 2.8 (iii)). (This has been proved for spectral operators of finite type in Theorem 1 of § 4 of (**7**, p. 56).)

## 4. An extension of Stone's theorem

**Theorem 2.** *Let $S$ be strongly self-conjugate with generated group of isometries $\{U(t, S): t \in \mathbf{R}\}$. Then there exists a spectral measure $E_s(\ )$ defined on $S_0(\mathbf{R})$ such that*

$$U(t, S) = \lim_n \int_{e_n} e^{it\lambda} E_s(d\lambda) \quad \forall t \in \mathbf{R}$$

$$Sx = \lim_n \int_{e_n} \lambda E_s(d\lambda)x \quad \forall x \in \mathscr{D}(S).$$

**Proof.** Let $\{U(t, S)\}$ be contained in a $V^*$-algebra $\mathscr{A}$, which, without loss of generality, can be taken to be weakly closed. Introduce $H, \phi, \mathscr{A}^h, E(\ ), E^h(\ )$ as in § 3.

Let $V(t, S) = \phi U(t, S)$. $\{V(t, S): t \in \mathbf{R}\}$ is a group of invertible isometries on $H$, therefore a unitary group. By Stone's theorem there exists a self-conjugate spectral measure $E_s^h(\ )$ on $S_0(\mathbf{R})$ such that $V(t, S) = \lim_n \int_{e_n} e^{it\lambda} E_s^h(d\lambda)$ and such that $E_s^h(\sigma) \in \{V(t, S)\}^{cc}$ for each $\sigma \in S_0(\mathbf{R})$. Since $\{V(t, S)\} \subset \mathscr{A}^h$, we have $\{V(t, S)\}^{cc} \subset \mathscr{A}^{hcc}$: and $\mathscr{A}^{hcc} = \mathscr{A}^h$ since $\mathscr{A}^h$ is weakly closed. Hence each $E_s^h(\sigma) \in \mathscr{A}^h$.

Now define a spectral measure $E_s(\ )$ in $L(X)$ by $E_s(\ ) = \phi^{-1} E_s^h(\ )$. Since $\phi$ is an isometry,

$$U(t, S) = \phi^{-1} V(t, S) = \lim_n \int_{e_n} e^{it\lambda} E_s(d\lambda).$$

Define $S'$ on $X$ by $S' = \operatorname{st} \lim_n \int_{e_n} \lambda E_s(d\lambda)$, wherever this limit exists. We show that $S = S'$.

(1) $S \subset S'$:

$$\begin{aligned} \forall x \in \mathscr{D}(S), \forall n \in \mathbf{Z}, \ E_s(e_n)Sx &= \lim_{t \to 0} E_s(e_n)(it)^{-1}[U(t, S)x - x] \\ &= \lim_{t \to 0} \int_{e_n} (it)^{-1}(e^{it\lambda} - 1) E_s(d\lambda)x \\ &= \int_{e_n} \lambda E_s(d\lambda)x \qquad \text{by (5, IV. 10.10).} \end{aligned}$$

Since $\lim_n E_s(e_n) = I$, we have

$$Sx = \lim_n E_s(e_n)Sx = \lim_n \int_{e_n} \lambda E_s(d\lambda)x.$$

Hence $x \in \mathscr{D}(S')$ and $Sx = S'x$. Therefore $S \subset S'$.

(2) $S' \subset S$:

$$\forall x \in \mathscr{D}(S'),\ \forall n \in \mathbf{Z},\ E_s(e_n)S'x = \int_{e_n} \lambda E_s(d\lambda)x$$
$$= \lim_{t\to 0} \int_{e_n} (it)^{-1}(e^{it\lambda}-1)E_s(d\lambda)x,$$
$$= \lim_{t\to 0} \int_{e_n} (it)^{-1}(e^{it\lambda}-1)E_s(d\lambda)E_s(e_n)x$$
$$= \lim_{t\to 0} \{(it)^{-1}[U(t, S)-I]\}E_s(e_n)x.$$

by (**5**, IV. 10.10).

Hence $E_s(e_n)x \in \mathscr{D}(S)$ and $SE_s(e_n)x = E_s(e_n)S'x$. Now $S$ is closed (**11**, p. 397),

$$x = \lim_n E_s(e_n)x \text{ and } S'x = \lim_n SE_s(e_n)x.$$

Hence $x \in \mathscr{D}(S)$ and $S'x = Sx$. Therefore $S' = S$.

## 5. Semigroups of bounded normal operators

**Lemma 4.** *Let $\mathscr{T} = \{T(t)\colon\ t>0\}$ be a weakly continuous semigroup of bounded normal operators on $X$. Then there exists a $V^*$-algebra containing $\mathscr{T}$.*

**Proof.** Let $\mathscr{A}(t)$ be the weakly closed $V^*$-algebra generated by $T(t)$:

$$\mathscr{A}(t) = \text{wk cl}\,\{p(T(t), T(t)^*)\colon\ p(\ ,\ )\text{ a polynomial}\}.$$

For $t \in \mathbf{Q}^+\backslash\{0\}$, $n \in \mathbf{Z}$ we have $T(t) = T(t/n)^n$; hence $\mathscr{A}(t) \subset \mathscr{A}(t/n)$. Thus if $m/n,\ m'/n' \in \mathbf{Q}^+\backslash\{0\}$ we have $\mathscr{A}(m/n) \cup \mathscr{A}(m'/n') \subset \mathscr{A}(1/nn')$: the system

$$\{\mathscr{A}(t)\colon\ t \in \mathbf{Q}^+\backslash\{0\}\}$$

is a directed system of $V^*$-algebras. Hence $\mathscr{A} = \text{wk cl} \cup \{\mathscr{A}(t)\colon\ t \in \mathbf{Q}^+\backslash\{0\}\}$ is a weakly closed $V^*$-algebra. If $t \in \mathbf{R}^+\backslash\mathbf{Q}^+$ then

$$T(t) = \text{wk}\lim_{s\to t} \{T(s)\colon\ s \in \mathbf{Q}^+\}.$$

Thus $\mathscr{A}$ is the required $V^*$-algebra.

**Theorem 3.** *Let $\mathscr{T} = \{T(t)\colon\ t>0\}$ be a weakly continuous semigroup of bounded strongly self-conjugate operators on $X$. Then $\mathscr{T}$ is uniformly continuous and there exists a unique spectral measure $F(\ )$ with compact support contained in $\mathbf{R}^+$ such that*

$$T(t) = \int_{\mathbf{R}^+} \lambda^t F(d\lambda).$$

*Furthermore, if no $T(t)$ has 0 as an eigenvalue there exists a unique spectral measure $G(\ )$ on $S_0(\mathbf{R})$ such that*

$$T(t) = \int_{\mathbf{R}} e^{\lambda t} G(d\lambda).$$

**Proof.** Let $\mathscr{A}$ be the $V^*$-algebra of Lemma 4. Introduce $H$, $\phi$, $\mathscr{A}^h$ as in §3. The semigroup $\mathscr{T}^h = \{\phi T(t): t>0\}$ satisfies the hypotheses of the theorem of (**9**, p. 73). Therefore $\mathscr{T}^h$ is uniformly continuous and there exists a unique spectral measure $F^h(\ )$ with compact support contained in $\mathbf{R}^+$ such that

$$\phi T(t) = \int_{\mathbf{R}^+} \lambda^t F^h(d\lambda).$$

Since for any $\sigma \in S_0(\mathbf{R}^+)$ we have $F^h(\sigma) \in \mathscr{T}^{hcc} \subset \mathscr{A}^{hcc} = \mathscr{A}^h$ we can define the required spectral measure $F(\ )$ by $F(\ ) = \phi^{-1}F^h(\ )$. If 0 is not an eigenvalue of any $T(t)$ then by Theorem 1 we know that 0 is not an eigenvalue of any $\phi T(t)$. Hence we can find the required $G(\ )$ in the same way from the $G^h(\ )$ given by Nagy's theorem.

**Theorem 4.** *Let $\mathscr{T} = \{T(t): t>0\}$ be a weakly continuous semigroup of bounded normal operators on $X$, and let 0 be eigenvalue of no $T(t)$. Then $\mathscr{T}$ is strongly continuous and there exists a unique spectral measure $F(\ )$ defined on $S_0(\mathbf{C})$ such that*

$$T(t) = \int_{\mathbf{C}} e^{t\lambda} F(d\lambda).$$

**Proof.** This follows by the argument of the preceding theorem, but using instead the theorem of (**9**, p. 74).

The following two lemmas show that Theorems 3 and 4 have the obvious extensions to the case when $\mathscr{T}$ is a group.

**Lemma 5.** *Let $T$ be an invertible element of the $V^*$-algebra $\mathscr{A}$. Then $T^{-1} \in \mathscr{A}$.*

**Proof.** Since $T^{-1}$ exists, $0 \notin \sigma(T)$. Hence $z \mapsto z^{-1}$ is continuous on $\sigma(T)$. By Theorem 1, $T^{-1} = \psi(z \mapsto z^{-1}) \in \mathscr{A}$.

**Lemma 6.** *Let $\mathscr{T} = \{T(t): t \in \mathbf{R}\}$ be a weakly continuous group of bounded normal operators on $X$. Then there exists a $V^*$-algebra containing $\mathscr{T}$.*

**Proof.** By Lemma 5 the algebra $\mathscr{A}$ constructed in Lemma 4 for the semigroup $\{T(t): t>0\}$ contains each $T(-t) = T(t)^{-1}$.

## 6. The extension of a Boolean algebra of projections

Let $\mathscr{B}'$ be a bounded Boolean algebra of projections, $\Lambda$ its Stone space, $M$ an upper bound for $\{\| B \|: B \in \mathscr{B}\}$. Let $K'(\Lambda)$ be the set of characteristic functions of the clopen sets of $\Lambda$. Let $\psi'$ be the representation isomorphism

$$\psi': K'(\Lambda) \rightarrow \mathscr{B}': \chi_\sigma \mapsto B(\sigma).$$

Let $K(\Lambda)$ be the complex commutative algebra of all finite sums $\Sigma c_j \chi_{\sigma_j}$, where $\chi_{\sigma_j} \in K'(\Lambda)$. Let $\mathscr{B}$ be the corresponding algebra of sums $\Sigma c_j B(\sigma_j)$. Extend $\psi'$ to an algebra isomorphism

$$\psi: K(\Lambda) \rightarrow \mathscr{B}: \Sigma c_j \chi_{\sigma_j} \mapsto \Sigma c_j B(\sigma_j).$$

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**Lemma 7.** *$\psi$ is bicontinuous.*

**Proof.** Let $k \in K(\Lambda)$ have the form $\Sigma c_j \chi_{\sigma_j}$: without loss of generality we can take the $\sigma_j$ pairwise disjoint. Then $\| k \| = \sup_j | c_j |$. Let $x \in X$, $y \in X'$. Then $yB(\ )x$ is a finite measure on the Boolean algebra of clopen subsets of $\Lambda$, and for $\sigma$ clopen we have $| yB(\sigma)x | \leqq M \| x \| \| y \|$. Hence, by (**5**, III. 1.5), $yB(\ )x$ is bounded, with bounded total variation satisfying

$$\text{var}\,(yB(\ )x, \Lambda) \leqq 4M \| x \| \| y \|.$$

By (**5**, III. 1.6) the total variation is an additive function on the clopen subsets of $\Lambda$. Hence

$$\begin{aligned} | y\psi(k)x | &\leqq \sum_j | c_j | \text{var}\,(yB(\ )x, \sigma_j) \\ &\leqq \sup_j | c_j | \sum_j \text{var}\,(yB(\ )x, \sigma_j) \\ &= \| k \| \text{var}\,(yB(\ )x, \bigcup_j \sigma_j) \leqq 4M \| x \| . \| y \| . \| k \|. \end{aligned}$$

Thus $\| \psi \| \leqq 4M$ and $\psi$ is continuous.

For $k = \Sigma c_j \chi_{\sigma_j}$, $x \in B(\sigma_j)X$ we have $\psi(k)x = c_j x$. Therefore $\| \psi(k) \| \geqq | c_j |$; hence $\| \psi(k) \| \geqq \| k \|$ and $\psi^{-1}$ is continuous.

Since $\Lambda$ is totally disconnected, $K(\Lambda)$ separates the points of $\Lambda$; hence is norm dense in $C(\Lambda)$. Let $\bar{\mathscr{B}}$ be the norm closure of $\mathscr{B}$ in $L(X)$. Extend $\psi$ to a bicontinuous onto isomorphism $\psi$: $C(\Lambda) \to \bar{\mathscr{B}}$. The argument of the proof of Theorem 2.5 of (**11**, p. 392) shows that there exists a regular strongly countably additive spectral measure $E(\ )$ in $L(X)$ such that

$$\psi(f) = \int_\Lambda f(\lambda) E(d\lambda) \quad \forall f \in C(\Lambda).$$

Extend $\psi$ to a map $\psi$: $B(\Lambda) \to L(X)$. Let $\tilde{\mathscr{B}} = \psi\{\chi_\sigma : \sigma \in S_0(\Lambda)\}$. As in Lemma 1, $\tilde{\mathscr{B}} \subset s(\mathscr{B})$.

**Theorem 5** (Dunford, (**4**), p. 578; Bade, (**1**), p. 351). *Let $\mathscr{B}'$ be a bounded Boolean algebra of projections on a weakly complete Banach space X. Then $\mathscr{B}'$ has a $\sigma$-complete extension contained in the strong closure of $\mathscr{B}'$.*

**Proof.** We show that $\tilde{\mathscr{B}}$ is $\sigma$-complete as a Boolean algebra of projections.

Let $(B_j)$ be a sequence in $\tilde{\mathscr{B}}$: $B_j = \psi\chi_{\sigma_j}$ with $\sigma_j \in S_0(\Lambda)$. Let

$$\sigma = \bigcup_1^\infty \sigma_j, \ B = \psi\chi_\sigma.$$

The sequence $\left(\bigvee_1^n \chi_{\sigma_i}\right)$ converges pointwise to $\chi_\sigma$, therefore is weakly Cauchy in $B(\Lambda)$. Therefore $B = \text{st}\lim_n \left(\bigvee_1^n B_j\right)$ (**5**, VI. 7.4 and remarks on p. 497); i.e. $B = \bigvee_1^\infty B_j$ in the strong topology.

Clearly, $B_jX \subset BX$: hence clm $\{B_jX\} \subset BX$. For any $x \in BX$ we have

$$x = Bx = \lim_n \bigvee_1^n B_j x \in \text{clm}\,\{B_jX\}.$$

Hence $BX = \text{clm}\,\{B_jX\}$.

The proof that $(\wedge B_j)X = \bigcap \{B_jX\}$ is similar. Since $\Lambda$ is totally disconnected, $S_0(\Lambda)$ is contained in the $\sigma$-algebra generated by the clopen sets. Hence $\tilde{\mathscr{B}} \subset s(\mathscr{B}')$.

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