ON THE COHOMOLOGY OF LOOP SPACES OF COMPACT LIE GROUPS

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Introduction. Let G be a compact, simply-connected Lie group. The cohomology of the loop space ΩG has been described by Bott, both in terms of a cell decomposition [**1**] and certain homogeneous spaces called generating varieties [**2**]. It is possible to view ΩG as an infinite dimensional "Grassmannian" associated to an appropriate infinite dimensional group, cf. [**3**], [**7**]. From this point of view the above cell-decomposition of Bott arises from a Bruhat decomposition of the associated group. We choose a generator $H \in H^2(\Omega G, \mathbb{Z})$ and call it the hyperplane class. For a finite-dimensional Grassmannian the highest power of H carries geometric information about the variety, namely, its degree. An analogous question for ΩG is: What is the largest integer $N_k = N_k(G)$ which divides $H^k \in H^{2k}(\Omega G, \mathbb{Z})$?

Of course, if $G = \mathrm{SU}(2) = S^3$, one knows $N_k = h!$. In general, the deviation of N_k from $k!$ measures the failure of H to generate a divided polynomial algebra in $H^*(\Omega G, \mathbb{Z})$.

One approach to the above question is to find a general formula for multiplying H by an arbitrary Bott class in terms of the Bott basis arising from the cell decomposition. This is an analogue of the classical Pieri formula in a Grassmannian and will be described elsewhere.

In fact, the numbers N_k can be computed more efficiently using the generating variety approach. If we interpret the problem mod p, we are led to finding the smallest integer r such that $0 = H^{p^r}$ in $H^{2p^r}(\Omega G, \mathbb{Z}/p)$. A result of Hubbuck [**6**] allows one to glue together this p-primary information so as to answer the original problem. We compute the numbers $N_k(G)$ explicitly for all classical G and G_2. Further computation with the exceptional groups provides an easy alternative proof of the Serre-Kumpel theorem on the regular primes of groups $G \neq E_8$.

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1. Classical groups. Suppose that G is a compact, simply-connected Lie group of rank n with exponents $m_1 \leq \ldots \leq m_n$. In particular the dimensions of the exterior algebra generators of $H^*(G, \mathbb{Q})$ are $2m_i + 1$, $1 \leq i \leq n$. Recall that p is a torsion prime for G if $H^*(G, \mathbb{Z})$ contains p-torsion.

We begin with the following lemma which expresses the basic relation between powers of the hyperplane class H in G and the Steenrod algebra action in G.

LEMMA 1.1. *If p is not a torsion prime and p^r is not an exponent for G, then $H^{p^r} = 0$ in $H^*(\Omega G, \mathbb{Z}/p)$.*

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Proof. There is a commutative diagram in $\mathbb{Z}/p$-cohomology:

$$\begin{array}{ccc} H^2(\Omega G) & \xrightarrow{(\ )^{p^r}} & H^{2p^r}(\Omega G) \\ \uparrow{\scriptstyle \sigma'} & & \uparrow{\scriptstyle \sigma'} \\ H^3(G) & \xrightarrow[\mathcal{P}]{} & H^{2p^r+1}(G) \end{array}$$

where $\mathcal{P} = \mathcal{P}^{p^{r-1}} \cdot \ldots \cdot \mathcal{P}^p \cdot \mathcal{P}^1$ and σ' is cohomology suspension. Since p is not a torsion prime, the indecomposables x_{2m_i+1} of $H^*(G, \mathbb{Z}/p)$ lie in dimensions $2m_i+1$, $1 \le i \le n$. Hence if $p^r \neq m_i$, $\mathcal{P}(x_3)$ is indecomposable. But σ' kills indecomposables and $\sigma'(x_3) = H$, so $H^{p^r} = 0$.

REMARK 1.2. If p is a non-torsion prime for G, then apart from the cases $(\mathrm{Sp}(n), 2)$ and $(G_2, 3)$, p^r is not an exponent for G if and only if $p^r > m_n$. (See Table at the end of this section.)

REMARK 1.3. Bott [1] showed the following are equivalent:
(i) $\pi_4(G) = \mathbb{Z}/2$,
(ii) 2 is neither a torsion prime nor an exponent prime for G,
(iii) $H^2 = 0$ in $H^*(\Omega G, \mathbb{Z}/2)$,
(iv) $G = \mathrm{Sp}(n)$ for some $n \ge 1$.

REMARK 1.4. For the torsionless groups $\mathrm{SU}(n+1)$, $\mathrm{Sp}(n)$ there is another way to see that $H^p = 0$, $p > m_n$, which is of interest in its own right. We observe:

$$H^*(\Omega\,\mathrm{SU}(n+1), \mathbb{Z}) = \mathbb{Z}[\sigma_1, \sigma_2, \ldots]/(\psi_{n+1}, \psi_{n+2}, \ldots),$$
$$H(\Omega\,\mathrm{Sp}(n), \mathbb{Z}) = \mathbb{Z}[\sigma_1, \sigma_2, \ldots]/(\psi_2, \psi_4, \ldots, \psi_{2n-2}, \psi_{2n}, \psi_{2n+1}, \ldots)$$

where the σ_i's can be viewed as elementary symmetric power series and the ψ_i's as the corresponding Newton polynomials. One recalls that $\psi^p \equiv \sigma_1^p \pmod p$ and $H = \sigma_1$, so the result follows.

We now examine the classical groups, family by family, to show that H^{p^r} is non-zero essentially when it is allowed by (1.1). We exploit Bott's construction of generating varieties $G/P \to \Omega G$ [**2**].

PROPOSITION 1.5. *In $H^*(\Omega\,\mathrm{SU}(n+1), \mathbb{Z}/p)$, $H^{p^r} = 0$ if and only if $p^r > m_n = n$.*

Proof. The generating variety is $g: \mathbb{C}P^n \to \Omega\,\mathrm{SU}(n+1)$. (This is the adjoint of the well-known map $\Sigma\mathbb{C}P^n \to \mathrm{SU}(n+1)$.) Since $g^*(H)$ generates $H^2(\mathbb{C}P^n, \mathbb{Z}/p)$, $H^{p^r} \neq 0$ if $p^r \le n$. The other direction is from (1.1).

PROPOSITION 1.6. *In H^* $(\Omega\,\mathrm{Spin}(2n+1), \mathbb{Z}/p)$, $p > 3$, $H^{p^r} = 0$ if and only if $p^r > m_n = 2n-1$. If $p = 2$, $H^{2^r} = 0$ if and only if $2^r \ge n$.*

Proof. The generating variety is $g: Q_{2n-1} \to \Omega\,\mathrm{Spin}(2n+1)$, where Q_{2n-1} is the quadric hypersurface in $\mathbb{C}P^{2n}$. Recall that

$$H^*(Q_{2n-1}, \mathbb{Z}) = \mathbb{Z}[x, y]/(x^{n+1} - 2y, y^2)$$

where $x \in H^2(Q_{2n-1}, \mathbb{Z})$ and $y \in H^n(Q_{2n-1}, \mathbb{Z})$ and $g^*(H) = x$. In particular, if p is odd: $H^*(Q_{2n-1}, \mathbb{Z}/p) = H^*(\mathbb{C}P^{2n-1}, \mathbb{Z}/p)$, so that $H^{p^r} \neq 0$ if $p^r \leqslant 2n-1$. This proves the first assertion. If $p = 2$, $x^i \neq 0$, $i < n$. Hence $H^{2^r} \neq 0$, if $2^r < n$. It remains to show $H^{2^r} = 0$ if $2^r \geqslant n$. We consider the following commutative diagram (with $\mathbb{Z}/2$ coefficients suppressed):

$$\begin{array}{ccc}
H^2(\Omega\,\mathrm{Spin}(2n+1)) & \xrightarrow{(\ )^{2^r}} & H^{2^{r+1}}(\Omega\,\mathrm{Spin}(2n+1)) \\
\uparrow{\scriptstyle \sigma'} & & \uparrow{\scriptstyle \sigma'} \\
H^3(\mathrm{Spin}(2n+1)) & \longrightarrow & H^{2^{r+1}+1}(\mathrm{Spin}(2n+1)) \\
\uparrow{\scriptstyle \sigma'} & & \uparrow{\scriptstyle \sigma'} \\
H^4(B\,\mathrm{Spin}(2n+1)) & \xrightarrow[\mathrm{Sq}]{} & H^{2^{r+1}+2}(B\,\mathrm{Spin}(2n+1))
\end{array}$$

where $\mathrm{Sq} = \mathrm{Sq}^{2^r} \cdot \ldots \cdot \mathrm{Sq}^4 \cdot \mathrm{Sq}^2$. Using the Wu formula one can compute that $\mathrm{Sq}(w_4) = w_{2^{r+1}+2}$ modulo decomposables. Since $H^*(B\,\mathrm{Spin}(2n+1), \mathbb{Z}/2) = \mathbb{Z}/2[w_3, w_4, \ldots, w_{2n+1}]$, $\mathrm{Sq}(w_4)$ is decomposable if $2^{r+1} + 2 > 2n+1$, i.e. $2^r \geqslant n$. Hence the result follows.

Proposition 1.7. *In $H^*(\Omega\,\mathrm{Sp}(n), \mathbb{Z}/p)$, $p > 2$, $H^{p^r} = 0$ if and only if $p^r > m_n = 2n-1$. If $p = 2$, $H^2 = 0$.*

Proof. The last line follows from either (1.1) or (1.4). If p is an odd prime, we can simply quote the p-equivalence $\Omega\,\mathrm{Sp}(n) \simeq_p \Omega\,\mathrm{SO}(2n+1)$ of Harris [**4**] and (1.6) above.

Proposition 1.8. *In $H^*(\Omega\,\mathrm{Spin}(2n), \mathbb{Z}/p)$, $p > 2$, $H^{p^r} = 0$ if and only if $p^r > m_n = 2n-3$. If $p = 2$, $H^{2^r} = 0$ if and only if $2^r \geqslant n$.*

Proof. As in (1.6), there is a generating variety $Q_{2n-2} \to \Omega\,\mathrm{Spin}(2n)$. The integral cohomology of an even-dimensional quadric possesses, at least, a 2-dimensional generator e satisfying e^k is a generator of $H^{2k}(Q_{2n-2})$, $k < n$. In particular, $H^{p^r} \neq 0$ if $p^r \leqslant 2n-2$. This proves the first assertion and the second follows as in (1.6).

We now recall the following lemma from Hubbuck [**6**] which essentially allows us to globalize the above p-primary information.

Lemma 1.9 (Hubbuck). *Suppose X has no p-torsion. If $x \in H^i(X, \mathbb{Z})$ and $x^p = py$, then there is a $z \in H^{p^2 i}(X, \mathbb{Z})$ such that $y^p = pz$. Hence*

$$x^{p^j} = p^{(p^j-1)/(p-1)} v$$

for some $v \in H^{p^j i}(X, \mathbb{Z})$.

We also have the following easy arithmetic fact.

Lemma 1.10. (i) *If $\sigma_p(k)$ denotes the sum of the coefficients in the p-adic expansion of k and v_p denotes p-adic valuation, then*

$$v_p(k!) = \frac{k - \sigma_p(k)}{p-1}.$$

(ii)

$$v_p(p^j!) = \frac{p^j - 1}{p-1}.$$

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The numbers $N_k(G)$, defined in the introduction, will be described using the following function:

$$F_m(p,k)=v_p\left[\frac{k}{p^{[\log_p(m)]}}!\right]$$

where $[x]$ is the greatest integer $\leqslant x$. Observe that if $p>m$ then $F_m(p,k)=v_p(k!)$. It is now possible to combine (1.5) to (1.10), using elementary arguments with cup products and the comultiplication in cohomology induced from the loop multiplication to obtain:

THEOREM 1.11. *If $N_k(G)$ denotes the largest integer which divides $H^k\in H^{2k}(\Omega G,\mathbb{Z})$, then the p-adic valuations of these numbers for the classical groups are given by:*

(i) $v_pN_k(\mathrm{SU}(n+1))=F_n(p,k)$.

(ii) $v_pN_k(\mathrm{Spin}(m))=\begin{cases}F_{m-2}(p,k) & \text{if} \quad p>2,\\ F_{m/2}(p,k) & \text{if} \quad p=2.\end{cases}$

(iii) $v_pN_k(\mathrm{Sp}(n))=\begin{cases}F_{2n-1}(p,k) & \text{if} \quad p>2,\\ F_1(p,k) & \text{if} \quad p=2.\end{cases}$

We can use the following arithmetic result to simplify the statement of this result in low rank cases.

LEMMA 1.12. $v_P(k!)-v_p([k/p]!)=[k/p]$.

Proof. Let $k=ap+j$, $0\leqslant j<p$. Then by (1.10)

$$\begin{aligned}v_p(k!)-v_p([k/p]!)&=\frac{1}{p-1}[k-\sigma_p(k)-(a-\sigma_p(a))]\\&=\frac{1}{p-1}[k-a-\sigma_p(ap+j)-\sigma_p(a))]\\&=\frac{1}{p-1}[ap+j-a-j]=a=[k/p].\end{aligned}$$

Since $H^*(\Omega\,\mathrm{SU}(2),\mathbb{Z})$ is a divided power algebra, $N_k(\mathrm{SU}(2))=k!$, for all k. For the rank 2 groups we have

COROLLARY 1.13.

(i) $N_k\ \mathrm{SU}(3)=k!\,2^{-[k/2]}$.

(ii) $N_k\ \mathrm{Sp}(2)=k!\,3^{-[k/3]}$.

(iii) $N_kG_2=k!\,2^{-[k/2]}5^{-[k/5]}$.

Proof. (i) and (ii) follow from (1.11), (1.12) and (iii) is done in the next section.

TABLE 1

Group	Exponents	Torsion	Generating variety V	$\dim_{\mathbb{C}} V$
$SU(n+1)$	$1, 2, \ldots, n$	$\varnothing$	$\mathbb{C}P^n$	n
$\mathrm{Spin}(2n+1)$	$1, 3, \ldots, 2n-1$	2	Q_{2n-1}	$2n-1$
$\mathrm{Sp}(n)$	$1, 3, \ldots, 2n-1$	$\varnothing$	$\mathrm{Sp}(n)/U(n)$	$\binom{n+1}{2}$
$\mathrm{Spin}(2n)$	$1, 3, \ldots, 2n-3, n-1$	2	Q_{2n-2}	$2n-2$
E_6	$1, 4, 5, 7, 8, 11$	2, 3	$E_6/SO(10)$	16
E_7	$1, 5, 7, 9, 11, 13, 17$	2, 3	E_7/E_6	27
F_4	$1, 5, 7, 11$	2, 3	$F_4/\mathrm{Sp}(3)$	15
G_2	$1, 5$	2	Q_5	5

2. Exceptional groups. The generating variety for the exceptional group G_2 is the 5-dimensional quadric $Q_5 \to \Omega G_2$. The cohomology of Q_5 (as in (1.6) above) contains a 2-dimensional generator x with x^2 a generator and x^3 divisible by 2. Hence we obtain:

PROPOSITION 2.1. *In $H^*(\Omega G_2, \mathbb{Z}/p)$, $p>2$, $H^{p^r}=0$ if and only if p^r is not an exponent, i.e. $p^r \neq 5$. If $p=2$, $H^2 \neq 0$ and $H^4=0$.*

Proof. For the first assertion, it suffices to show $H^5 \neq 0$. This follows from the fact that $x^5 \neq 0$, for odd primes. Similarly, $H^2 \neq 0$. The last assertion follows from the existence of a fibration

$$G_2 \xrightarrow{i} \mathrm{Spin}(7) \to S^7$$

which induces the following diagram in $\mathbb{Z}/2$-cohomology:

$$\begin{array}{ccc} H^3(\mathrm{Spin}(7)) & \xrightarrow[i^*]{} & H^3(G_2) \\ \downarrow{\scriptstyle \mathrm{Sq}} & & \downarrow{\scriptstyle \mathrm{Sq}} \\ H^9(\mathrm{Spin}(7)) & \xrightarrow[i^*]{} & H^9(G_2) \end{array}$$

where $\mathrm{Sq} = \mathrm{Sq}^4 \mathrm{Sq}^2$. Hence by the argument in (1.6), $\mathrm{Sq}(x_3)=0$ on the left-hand side, therefore also on the right. Hence $H^4=0$ in $H^*(\Omega G_2)$ by the usual argument with σ'. This also completes the proof of (1.13).

PROPOSITION 2.2. *In $H^*(\Omega E_6, \mathbb{Z}/p)$, $p>3$, $H^{p^r}=0$ if and only if $p^r > m_6 = 11$. If $p=2$, $H^8 \neq 0$. If $p=3$, $H^3 \neq 0$.*

Proof. For the first assertion it suffices to show $H^p \neq 0$ in $H^*(\Omega E_6, \mathbb{Z}/p)$ for $p=5$, 7, 11. There is a generating map $E_6/SO(10) \to \Omega E_6$. So we need only check our assertions in this homogeneous space. This follows from the following picture of the Schubert cell-decomposition of $E_6/SO(10)$. The numbers attached to the cells give the coefficients of the pullback of powers of H in the Schubert basis (see [**8**], [**5**]).

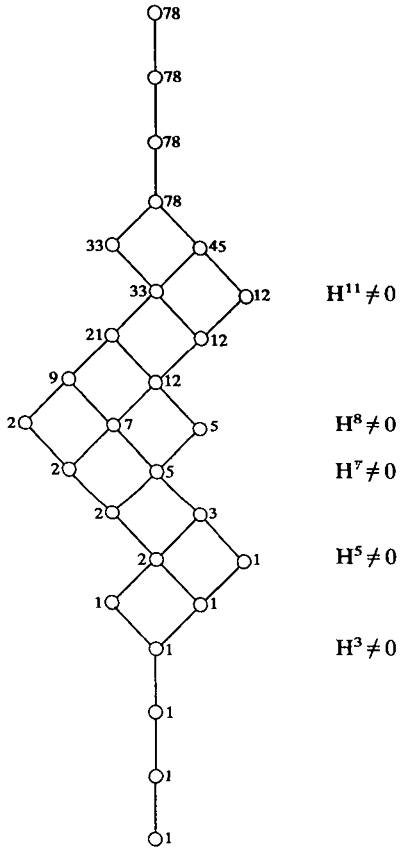

Figure 1
$E_6/SO(10)$

PROPOSITION 2.3. *In* $H^*(\Omega E_7, \mathbb{Z}/p)$ $p>3$, $H^{p^r}=0$ *if and only if* $p^r > m_7 = 17$. *If* $p=2$, $H^8 \neq 0$. *If* $p=3$, $H^9 \neq 0$.

Proof. The generating map is $E_7/E_6 \to \Omega E_7$. As in (2.2) the result follows from the picture.

REMARK 2.4. If $p=2$ in (2.3) and (2.2), in fact, one knows $H^{16} \neq 0$ [**9**].

PROPOSITION 2.5. *In* $H^*(\Omega F_4, \mathbb{Z}/p)$, $p>3$, $H^{p^r}=0$ *if and only if* $p^r \geqslant m_4 = 11$. *If* $p=2$, $H^2 \neq 0$. *If* $p=3$, $H^3 \neq 0$.

Proof. The minimal generating variety for ΩF_4 is $F_4/\mathrm{Sp}(3) \to \Omega F_4$. The corresponding

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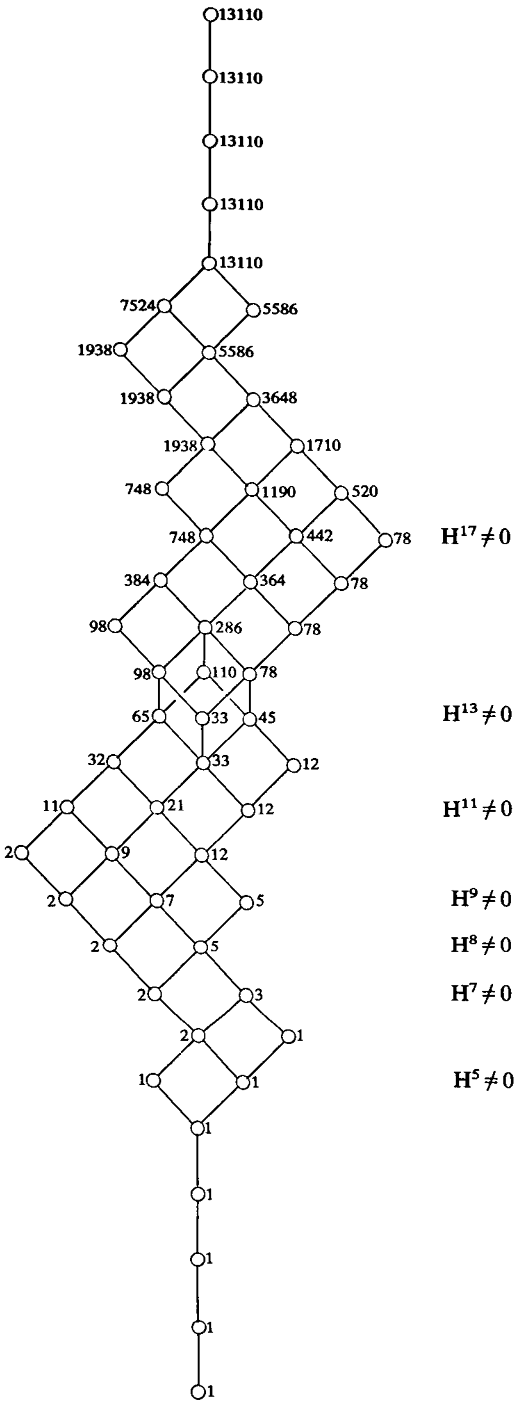

Figure 2
E_7/E_6

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cell-decomposition of the homogeneous space is described by:

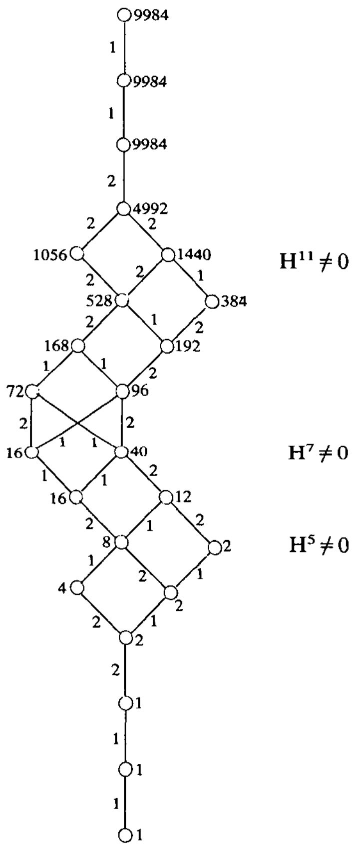

Figure 3
$F_4/\mathrm{Sp}(3)$

The coefficients in this example are more difficult to compute as multiplication by H produces multiplicities. These numbers are computed using the generalized Pieri formula [**5**, p. 151] and appear as edge labels. The assertions are now easy to check.

We record here a geometric consequence of the three propositions above. (The second one corrects a mis-count in [**5**, p. 179].) Recall that the degree of a projective variety can be computed from the top power of a hyperplane class.

PROPOSITION 2.6. *The degrees of the generating varieties are given by:*

$$\deg E_6/\mathrm{SO}(10) = 78,$$
$$\deg E_7/E_6 = 13110,$$
$$\deg F_4/\mathrm{Sp}(3) = 9984.$$

Finally we obtain

COROLLARY 2.7. *If G is a compact, simply-connected Lie group $\neq E_8$ and $p \leq m_n$, then p is not a regular prime for G, i.e. the p-localization of G is not a product of mod p spheres.*

Proof. If p is a torsion prime the result is clear. For all other primes $\leq m_n$, we have shown $H^{p^r} \neq 0$ in $H^*(\Omega G, \mathbb{Z}/p)$. It now follows from the diagram in (1.1) that G has a non-trivial Steenrod reduced power in its mod p cohomology. Hence the claim follows.

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