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ON THE UPPER MAJORANT PROPERTY FOR LOCALLY COMPACT ABELIAN GROUPS

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1. Introduction and preliminaries. Let G be a compact abelian group and form the spaces $L_p(G)$ with respect to the normalized Haar measure on G.

If $f, g \in L_1(G)$ and $|\hat{f}| \leqq \hat{g}$ we say that *g majorizes f* (or *g is a majorant of f*). Let $1 \leqq p \leqq \infty$. We say that $L_p(G)$ has the *upper majorant property* if there is a positive constant D such that whenever $f, g \in L_p(G)$ and g majorizes f we have $||f||_p \leqq D||g||_p$. We say that $L_p(G)$ has the *lower majorant property* if there is a positive constant C such that every $f \in L_p(G)$ has a majorant $g \in L_p(G)$ for which $||g||_p \leqq C||f||_p$.

These properties will be abbreviated to UMP and LMP respectively.

The majorant problem, initiated by Hardy and Littlewood [**5**], is to determine for which p the space $L_p(G)$ has the UMP or the LMP. The known results can be summarized as follows:

(a) $L_p(G)$ has the UMP if and only if p is an even integer or ∞; and when $L_p(G)$ has the UMP the constant is 1.

(b) For $1 < p < \infty$ and $(1/p) + (1/q) = 1$, the space $L_p(G)$ has the UMP if and only if $L_q(G)$ has the LMP, with the same constant.

(c) $L_1(G)$ has the LMP with constant 1.

(d) $L_\infty(G)$ does not have the LMP.

For the proofs of (a), (b), and (c) we refer to Hardy and Littlewood [**5**], Boas [**2**], Bachelis [**1**], and Fournier [**3**]. The papers of Hardy and Littlewood and of Boas are concerned with the circle group, while the latter two contain the results for the general (infinite) compact abelian group. The paper of Bachelis also completes the results for the circle group.

To prove (d) we need only note that a continuous function on G which does not belong to $A(G)$ (see [**14**, p. 9]) has no majorant in $L_\infty(G)$. We now condense statements (b), (c), (d) to:

(e) If $1 \leqq p \leqq \infty$ and $(1/p) + (1/q) = 1$, then $L_p(G)$ has the UMP if and only if $L_q(G)$ has the LMP, with the same constant.

The statements of (e) are called "duality theorems". That $L_p(G)$ has the UMP implies $L_q(G)$ has the LMP was proved, for the circle group, by Hardy and Littlewood [**5**]. The outher statement in (e) is due to Boas [**2**]. We shall be concerned with the generalization of these results to noncompact locally com-

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pact abelian groups. We obtain an analogue of (a) and conclude with a remark on (e). The main theorem is stated in Section 2 and is proved in the succeeding sections. The crux of this proof is the case of the integers (see Section 5) in which we use Rudin's description [**13**] of the functions which operate on real-valued positive definite sequences. Once the theorem for the integer group is known we can easily derive the analogous result for the real line (see Section 6).

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Notation. Throughout this paper G will denote a locally compact abelian group (abbreviated LCAG) with dual group $\hat{G}$. The Lebesgue spaces $L_p(G)$ ($1 \leqq p \leqq \infty$) are constructed with respect to Haar measure on G. This measure is generally normalized. When G is discrete, Haar measure is usually counting measure and for G compact Haar measure is normalized to have total mass 1. One exception to the latter convention occurs when H is an open subgroup of G. The total mass of H is then $[G : H]^{-1}$, where $[G : H]$ is the index of H in G.

For $f \in L_1(G)$ define its Fourier transform by

$$\hat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}dx, \quad \gamma \in \hat{G},$$

The range of the Fourier transform, $L_1(G)^{\wedge}$, is denoted by $A(\hat{G})$.

We denote the group of real numbers by $\mathbf{R}$, the circle group by $\mathbf{T}$, and the integer group by $\mathbf{Z}$. The positive integers are denoted by $\mathbf{N}$ and $\mathbf{Z}(r)$, $r \geqq 1$, denotes the cyclic group of order r.

2. Statement of the main theorem. The definition of the UMP given in the introduction was used by Boas [**2**] and Bachelis [**1**], while in [**3**] Fournier uses an analogous definition involving only trigonometric polynomials. Standard approximate identity arguments show that the definitions of Bachelis and Fournier are equivalent. When G is a noncompact LCAG and $p > 2$ the Fourier transform of an $L_p(G)$-function must be defined in the sense of quasimeasures (see [**4**, Chapter 6]). Then we must somehow interpret the inequality $|\hat{f}| \leqq \hat{g}$. This problem vanishes if we use a definition of the UMP for a noncompact LCAG which involves only functions in a suitable dense subspace of $L_p(G)$. The space we shall use is $S(G) = L_1(G) \cap A(G)$, which is contained in every $L_p(G)$, $1 \leqq p \leqq \infty$. We will remark briefly on the use of other suitable spaces at the end.

2.1. *Definition.* If $f, g \in S(G)$ and $|\hat{f}| \leqq \hat{g}$ we say g *majorizes* f or g *is a majorant of* f. We say $L_p(G)$ has the *upper majorant property* if there is a positive constant D such that whenever $f, g \in S(G)$ and g majorizes f then $\|f\|_p \leqq D\|g\|_p$.

Note that we must have $D \geqq 1$ since $S(G)$ contains nontrivial functions with nonnegative transforms. As before we abbreviate upper majorant property to UMP.

If we proceed as in Hardy and Littlewood [**5**, p. 305], it is an easy matter to prove the following result.

2.2. PROPOSITION. *If p is an even integer or ∞ then $L_p(G)$ has the UMP with constant* 1.

Our main result is that the converse is also true.

2.3. MAIN THEOREM. *Suppose that G is a noncompact LCAG and that $1 \leqq p < \infty$. If p is not an even integer, then $L_p(G)$ does not have the UMP.*

To prove this theorem we must show that for every positive constant D, there exist $f, g \in S(G)$ such that g majorizes f and $||f||_p > D||g||_p$.

3. Some reductions. Our proof of 2.3 is based on the structure theorem for LCAG's and in this section we show that it suffices to prove the theorem for certain classes of groups.

3.1. PROPOSITION. *Suppose that G_1 and G_2 are LCAG's and suppose that either $L_p(G_1)$ or $L_p(G_2)$ fails to have the UMP. Then $L_p(G_1 \times G_2)$ does not have the UMP.*

Proof. Recall that if f_i is a function on G_i $(i = 1, 2)$ we define $f_1 \otimes f_2$ on $G_1 \times G_2$ by $f_1 \otimes f_2(x, y) = f_1(x)f_2(y)$ for $(x, y) \in G_1 \times G_2$. If $1 \leqq p < \infty$ and $f_i \in L_p(G_i)$ $(i = 1, 2)$ then $f_1 \otimes f_2 \in L_p(G_1 \times G_2)$ and $||f_1 \otimes f_2||_p = ||f_1||_p||f_2||_p$. For $p = 1$ and $(\gamma_1, \gamma_2) \in \hat{G}_1 \times \hat{G}_2$ we have $(f_1 \otimes f_2)\hat{}(\gamma_1, \gamma_2) = \hat{f}_1 \otimes \hat{f}_2(\gamma_1, \gamma_2)$; consequently, if $f_i \in S(G_i)$ $(i = 1, 2)$, then $f_1 \otimes f_2 \in S(G_1 \times G_2)$.

Suppose, for definiteness, that $L_p(G_1)$ does not have the UMP and let $D > 0$. Let $f_1, g_1 \in S(G_1)$ be such that g_1 majorizes f_1 and $||f_1||_p > D||g_1||_p$. For any nontrivial h belonging to $S(G_2)$ with $\hat{h} \geqq 0$ we set $F = f_1 \otimes h$, $G = g_1 \otimes h$. Then $F, G \in S(G_1 \times G_2)$, G majorizes F and $||F||_p > D||G||_p$, which completes the proof.

We now recall the structure theorem for LCAG's (see [**6**, (24.30)]). This theorem states that any LCAG is of the form $\mathbf{R}^n \times G_0$ where G_0 is an LCAG containing a compact open subgroup. If $\mathbf{R}^n \times G_0$ is infinite, one of the following statements is true:

(a) $n > 0$;
(b) $n = 0$ and G_0 has an infinite compact open subgroup;
(c) $n = 0$ and G_0 is an infinite discrete group.

Applying (3.1) we see that it is enough to consider only the group $\mathbf{R}$ for case (a); in case (b) we will be able to construct examples from those known for the infinite compact open subgroup of G_0.

In case (c) the discrete group G_0 may have an element of infinite order, in which case it contains a copy of $\mathbf{Z}$. Otherwise G_0 is a torsion group. In such a group either we have elements of arbitrarily large order or there is a bound on the orders of all elements. In the latter case G_0 must contain a copy of $\mathbf{Z}(r)^{\omega *}$, the direct sum of countably many copies of $\mathbf{Z}(r)$, $r \geqq 2$ (see [**6**, p. 449]).

We now give a further reduction in the discrete case.

3.2. PROPOSITION. *Let G be an infinite discrete abelian group which contains a subgroup isomorphic to a discrete group H for which $l_p(H)$ does not have the UMP. Then $l_p(G)$ does not have the UMP.*

Proof. For discrete G, $S(G) = l_1(G)$. Let $\phi : H \rightarrow G$ be an embedding of H in G. For a positive constant D there exist $f, g \in l_1(H)$ such that g majorizes f and $||f||_p > D||g||_p$.

Given a function h on H, we define h' on G by $h'(x) = h(\phi^{-1}(x))$ if $x \in \phi(H)$ and let $h'(x) = 0$ otherwise. Then $h' \in l_r(G)$ if $h \in l_r(H)$ and $||h'||_r = ||h||_r$ for any $r \geqq 1$. If f' and g' are constructed thus from f and g respectively it is clear that $||f'||_p > D||g'||_p$. We show now that g' majorizes f'.

If $h \in l_1(H)$ then $h' \in l_1(G)$ is supported by $\phi(H)$ and so $\hat{h}'$ is constant on the cosets of $\phi(H)^{\perp}$, the annihilator of $\phi(H)$ in $\hat{G}$ (see [**12**, p. 96 and p. 118]). Since $\hat{h}'$ can be identified with $\hat{h}$ via $\hat{h} \circ \pi = \hat{h}'$, where π is the canonical projection of $\hat{G}$ on $\hat{G}/\phi(H)^{\perp}$, we have

$$|\hat{f}'| = |\hat{f} \circ \pi| \leqq \hat{g} \circ \pi = \hat{g}'$$

and this completes the proof.

Thus in dealing with discrete groups with an element of infinite order we need only consider the group $\mathbf{Z}$; for infinite discrete groups with a bound on the orders of all elements we need only consider the groups $\mathbf{Z}(r)^{\omega *}$, for $r \geqq 2$.

We now summarize the groups or classes of groups for which we must prove the main theorem: 1) $\mathbf{Z}(r)^{\omega *}$, $r \geqq 2$; 2) G a nondiscrete noncompact LCAG with a compact open subgroup; 3) $\mathbf{Z}$; 4) $\mathbf{R}$; 5) G a discrete abelian torsion group with elements of arbitrarily large order.

4. Examples derived from the compact case. We dispose of cases 1) and 2) of the preceding section.

4.1. THEOREM. *The main theorem holds if G is a nondiscrete, noncompact LCAG with a compact open subgroup.*

Proof. Let G_0 be an (infinite) compact open subgroup of G. Since G_0 is open its Haar measure is the restriction to G_0 of the Haar measure of G.

Suppose D is a positive constant. From [**3**] we have trigonometric polynomials f_1, g_1 on G_0 (hence are members of $S(G_0)$) which satisfy $|\hat{f}_1| \leqq \hat{g}_1$ and $||f_1||_p > D||g_1||_p$. Now any member h_1 of $A(G_0)$ can be extended to a member h of $A(G)$ by letting h agree with h_1 on G_0 and vanish off G_0 (see [**14**, p. 53] and remember G_0 is open). In particular, any member of $S(G_0)$ has such an extension to a member of $S(G)$. Moreover, for $1 \leqq p \leqq \infty$ L_p-norms are preserved.

Let f, g be such extensions to G of f_1 and g_1 respectively. Then $||f||_p > D||g||_p$ and that g majorizes f can be shown as in the proof of 3.2.

We now consider the groups $G = \mathbf{Z}(r)^{\omega *}$, $r \geqq 2$. In this case we can derive our examples from those known for the compact group $X = \mathbf{Z}(r)^{\omega}$, the direct product of countably many copies of $\mathbf{Z}(r)$. Note that $X = \hat{G}$.

4.2. THEOREM. *The main theorem holds if* $G = \mathbf{Z}(r)^{\omega *}$, $r \geqq 2$.

Proof. For a positive constant D let f, g be trigonometric polynomials on X such that g majorizes f and $||f||_p > D||g||_p$. There is a positive integer n for which $\text{supp}\,(\hat{f}) \cup \text{supp}\,(\hat{g}) \subset G_n$, where $G_n = \{(y_j) \in G | y_j = 0 \text{ if } j \geqq n + 1\}$. Note that G_n is isomorphic to $\mathbf{Z}(r)^n$ and thus is self-dual. It follows that f and g are constant on the cosets of $G_n{}^{\perp}$ and can be identified with a pair of functions f', g' on $X/G_n{}^{\perp} = \hat{G}_n = G_n \subset G$. That g' majorizes f' follows as in the proof of 3.2 and since all maps involved in the identification process are positivity preserving. We also have $||f'||_p > D||g'||_p$ since if $h \in L_p(X)$ is constant on the cosets of $G_n{}^{\perp}$ we have $h = H \circ \pi$ where $H \in L_p(X/G_n{}^{\perp})$ (π is the canonical projection of X on $X/G_n{}^{\perp}$) and $||h||_p = r^{-(n/p)}||H||_p$ (see Weil's integration formula in [**12**, p. 70]).

5. The case of the integers. As in the previous cases, our goal is to show that if p is not an even integer or ∞, then for any positive constant D there are functions f, g belonging to $l_1(\mathbf{Z})$ which satisfy $|\hat{f}| \leqq \hat{g}$ on $\mathbf{T}$ and $D||g||_p < ||f||_p$.

Our method, though different in detail, is essentially the same as that used by Bachelis [**1**] and Fournier [**3**]. For a discussion of the origins of this method see Shapiro [**15**]. For the group $\mathbf{T}$, Bachelis proves this result by using a suggestion of Y. Katznelson to show, by an iteration method, that if the UMP fails to hold with $D = 1$, then it fails to hold at all (see [**1**, p. 121]).

We now give an iteration method; it is the dualized version of a special case of Fejér's Lemma (see [**1**, p. 121], and [**16**, p. 49]).

For a function β on $\mathbf{Z}$ we define, for each $n \in \mathbf{N}$, a function β_n by

$$\beta_n(m) = \begin{cases} \beta\left(\dfrac{m}{n}\right) & \text{if } m \in n\mathbf{Z} \\ 0 & \text{otherwise.} \end{cases}$$

5.1. PROPOSITION. *Let* $1 \leqq p < \infty$ *and suppose that* α *is a finitely supported function on* $\mathbf{Z}$. *Then for large* n *we have* $||\alpha * \alpha_n||_p = ||\alpha||_p{}^2$, *and in particular*

$$\lim_{l \to \infty} ||\alpha * \alpha_l||_p = ||\alpha||_p{}^2.$$

This result is easily proved and so we omit the details.

5.2. COROLLARY. *Let $1 \leqq p < \infty$ and suppose p is an exponent for which there is a pair of finitely supported functions f, g on $\mathbf{Z}$ satisfying $|\hat{f}| \leqq \hat{g}$ and $||g||_p < ||f||_p$. Then $l_p(\mathbf{Z})$ fails to have the UMP.*

Proof. If $||g||_p < ||f||_p$, there is a constant $C > 1$ such that $C||g||_p < ||f||_p$. Form f_n and g_n as above. Then, for every $x \in \mathbf{T}$, we have $\hat{f}_n(x) = \hat{f}(nx)$. Similarly for g_n. Since g majorizes f it follows that $g * g_n$ majorizes $f * f_n$ for every positive integer n.

Applying 5.1 to each sequence $(g * g_n)$ and $(f * f_n)$ it follows that there is a positive integer n for which $C^2||g * g_n||_p < ||f * f_n||_p$. Iteration of this procedure shows if $D > 1$ there exist finitely supported functions d, e on $\mathbf{Z}$ with e majorizing d and $D||e||_p < ||d||_p$. Hence $l_p(\mathbf{Z})$ does not have the UMP.

To complete the proof of the main theorem for the integer group we must, for a given finite p which is not an even integer, find a pair of finitely supported functions on $\mathbf{Z}$ which satisfy the hypotheses of Corollary 5.2. We require some preliminaries about the functions which operate on $P_r(\mathbf{Z})$, the set of real-valued positive definite functions on $\mathbf{Z}$.

Recall that if Δ is a subset of the complex plane $\mathbf{C}$ and $F: \Delta \to \mathbf{C}$, then we say *F operates on $P(\mathbf{Z})$* if $F \circ \phi \in P(\mathbf{Z})$ whenever $\phi \in P(\mathbf{Z})$ and range $(\phi) \subset \Delta$. If $\Delta = (-1, 1)$ and F is real-valued then Rudin [**13**] has shown that F must be of the form

$$F(x) = \sum_{n=0}^{\infty} c_n x^n, \quad \text{for } |x| < 1, \quad \text{and}$$
$$c_n \geqq 0 \quad \text{for } n = 0, 1, 2, \ldots .$$

Our next result is stated and proved only for the case at hand. It will be clear that this result can be proved on any LCAG.

5.3. LEMMA. *Let $F: (-1, 1) \to \mathbf{R}$ be continuous and have the property that $F \circ \psi \in P_r(\mathbf{Z})$ whenever ψ is a finitely supported member of $P_r(\mathbf{Z})$ with range $(\psi) \subset (-1, 1)$. Then F operates on $P_r(\mathbf{Z})$.*

Proof. Let $\psi \in P_r(\mathbf{Z})$ have range $(\psi) \subset (-1, 1)$ and let (K_n) be the Fejér kernel in $L_1(\mathbf{T})$. Then $\hat{K}_n \in P_r(\mathbf{Z})$ and is finitely supported for every n. Since $0 \leqq \hat{K}_n \leqq 1$, we have range $(\hat{K}_n\psi) \subset (-1, 1)$. Thus $F \circ (\hat{K}_n\psi) \in P_r(\mathbf{Z})$. Since (K_n) is an approximate identity for $L_1(\mathbf{T})$ we have $\psi = \lim_n \hat{K}_n\psi$ pointwise and thus $F \circ \psi = \lim_n F \circ \hat{K}_n\psi$ pointwise. In particular $F \circ \psi \in P_r(\mathbf{Z})$ and the result is proved.

5.4. PROPSOITION. *Suppose p is finite and not an even integer. Then there exists a finitely supported function ϕ belonging to $P_r(\mathbf{Z})$ for which $|\phi|^p$ does not belong to $P(\mathbf{Z})$.*

Proof. Let $H: (-1, 1) \to \mathbf{R}$ be defined by $H(x) = |x|^p$. It is easy to see that H is not of the form (3) and so cannot operate on $P_r(\mathbf{Z})$. H is continuous and so, applying 5.3, there exists a finitely supported $\phi \in P_r(\mathbf{Z})$ with range $(\phi) \subset (-1, 1)$ and $|\phi|^p \notin P_r(\mathbf{Z})$. This proves the result.

We can now present the necessary examples. The germ of this method is to be found, in a disguised form, in [**3**, p. 163].

5.5. THEOREM. *Suppose that $p \geqq 1$ is finite and not an even integer. Then $l_p(\mathbf{Z})$ does not have the UMP,*

Proof. By 5.2 it is enough to show that there exist a pair, f, g, of finitely supported functions on $\mathbf{Z}$ such that g majorizes f and $||g||_p < ||f||_p$.

Let ϕ be as in 5.4. As ϕ is positive definite it is self-adjoint (that is, $\phi(-n) = \overline{\phi(n)}$ for all $n \in \mathbf{Z}$) and so is $|\phi|^p$. Thus $|\phi|^p$ has a real-valued Fourier transform. Since $|\phi|^p \notin P_r(\mathbf{Z})$, there is an $x \in \mathbf{T}$ for which $(|\phi|^p)\hat{}(x) < 0$, that is

$$\sum_{l=-\infty}^{\infty} |\phi(l)|^p e^{-ilx} < 0.$$

Define $\gamma \in \hat{\mathbf{Z}}$ by $\gamma(l) = e^{ilx}$, for each $l \in \mathbf{Z}$, and for a real parameter t we define functions f_t on $\mathbf{Z}$ by $f_t = (1 + t\gamma)\phi$, where 1 stands for the constant function with value 1. Note that each f_t is finitely supported and $f_t \in P(\mathbf{Z})$ when $t \geqq 0$. Moreover, if $t \geqq 0$ we have $|\widehat{f_{-t}}| \leqq \widehat{f_t}$.

Let the function F be defined on $\mathbf{R}$ by

$$F(t) = ||f_t||_p^p = \sum_{l=-\infty}^{\infty} |\phi(l)|^p |1 + t\gamma(l)|^p.$$

Notice that, for every l, the function $t \to |1 + t\gamma(l)|^p$ is differentiable at 0 and so, since the sum defining F is really finite, we have:

$$\begin{aligned} F'(0) &= \sum_{l=-\infty}^{\infty} |\phi(l)|^p \frac{d}{dt} [|1 + t\gamma(l)|^p]_{t=0} \\ &= p \sum_{l=-\infty}^{\infty} |\phi(l)|^p \operatorname{Re} (\overline{\gamma(l)}) \\ &= p(|\phi|^p)\hat{}(x). \end{aligned}$$

Thus $F'(0) < 0$ and there must be a positive t_0 for which $F(t_0) < F(-t_0)$. If we let $f = f_{-t_0}$ and $g = f_{t_0}$ we obtain the required pair of functions.

5.6. In this section we sketch an alternative proof of 5.5 for p a rational number but not an even integer. This method, except for a minor modification due to $\mathbf{Z}$ not being compact, is directly analogous to the example first produced by Hardy and Littlewood for $L_3(\mathbf{T})$ (see [**5**, p. 305] and [**2**, p. 255]).

Let $1 \leqq p < \infty$ and suppose p is not an even integer. For each positive integer n, let c_n be the binomial coefficient

$$\frac{p/2(p/2 - 1) \cdots (p/2 - n + 1)}{n!}.$$

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Let k be the least positive integer for which $c_n < 0$. We need a function ϕ in $l_1(\mathbf{Z})$ with the following properties:

(a) $\phi = \hat{g}$, for some nontrivial, nonnegative C^∞ function g on $\mathbf{T}$;

(b) for an integer L, to be specified later, the functions $\{\phi^l | 1 \leqq l \leqq L\}$ are mutually orthogonal in $l_2(\mathbf{Z})$. Note that these products are defined by pointwise multiplication.

To obtain such a function ϕ we need a nontrivial, nonnegative C^∞ function g on $\mathbf{T}$ for which the convolution powers $\{g^{*l} | 1 \leqq l \leqq L\}$ are mutually orthogonal in $L_2(\mathbf{T})$. Since the convolution powers are nonnegative they are orthogonal if and only if their supports are disjoint. Suppose g is supported by a small interval $[a, b]$ with $0 < a < b < 2\pi$. These convolution powers will have disjoint supports provided that b is sufficiently small and a is sufficiently close to b.

Now let $\psi = \phi^{2/p}$. Since ϕ is the Fourier transform of a C^∞ function, the same is true of ψ; in particular, $\psi \in l_1(\mathbf{Z})$. Let λ be a real number with $|\lambda| \leqq 1$ and let t be a positive parameter. We define functions f_λ by

$$f_\lambda = \psi(1 + t\phi + \lambda t^k \phi^k),$$

where 1 is the constant function whose sole value is 1. Then f_λ belongs to $l_1(\mathbf{Z})$.

We first examine the $l_p(\mathbf{Z})$-norm of f_λ. We have $||f_\lambda||_p^p = ||f_\lambda^{p/2}||_2^2$ and, if t is small enough, we can apply the binomial theorem to $(1 + t\phi + \lambda t^k\phi^k)^{p/2}$. When we do this we obtain an absolutely convergent double series (in $l_2(\mathbf{Z})$) and thus

$$f_\lambda^{p/2} = \psi^{p/2}(1 + t\phi + \lambda t^k \phi^k)^{p/2} = \phi \sum_{l=0}^{\infty} \beta_l (t\phi)^l,$$

where $\beta_l = c_l$ for $0 \leqq l \leqq k - 1$ and $\beta_k = c_1\lambda + c_k$. If we write

$$f_\lambda^{p/2} = \phi \sum_{l=0}^{2k} \beta_l (t\phi)^l + O(t^{2k+1}),$$

then using (b) with $L = 2k + 1$, we obtain

$$||f_\lambda||_p^p = ||f_\lambda^{p/2}||_2^2 = \sum_{l=0}^{k} |\beta_l|^2 ||\phi^{l+1}||_2^2 \, t^{2l} + o(t^{2k}).$$

Our choice of k shows that $||f_1||_p^p < ||f_{-1}||_p^p$ if t is sufficiently small. This takes care of the norm inequality.

We would like to know if f_1 majorizes f_{-1}. This would certainly be the case if ψ were positive definite. For general p the author does not know if we can assume that $\psi = \phi^{2/p}$ is positive definite in addition to conditions (a) and (b). However, if p is rational this is quite easily done. For suppose $p = m/l$; then $2/p = 2l/m$. If we let $\psi = \hat{g}^{2l}$ and $\phi = \hat{g}^m$ then $\psi^{p/2} = \phi$, ψ is positive definite and ψ belongs to $l_1(\mathbf{Z})$. Hence f_1 majorizes f_{-1}. The only change required for the norm computation is to replace L by mL in condition (b).

6. Examples derived from the integer group. In this section we prove the main theorem for the remaining two cases, namely **R** and discrete torsion groups with elements of arbitrarily large orders.

We first consider **R**, for which a slightly modified form of a device due to de Leeuw (see [**10**, p. 375]) is required. The proof is the same as his and so is omitted. Before giving this lemma we remark that finitely supported functions on **Z** can be identified with finite sums of point masses (concentrated at integer points) on **R**.

6.1. LEMMA. *Let* $1 \leqq p < \infty$. *Suppose that* $\phi \in A_c(\mathbf{R})$ (*the space of compactly supported members of* $A(\mathbf{R})$) *is nonnegative, has nonnegative Fourier transform and satisfies*

(1) $||\phi||_p = 1$
(2) $\text{supp}\ (\phi) \subset [a, a+1]$ *for some real number* a.

Then for any finitely supported function σ *on* **Z** *we have* $\sigma * \phi \in S(\mathbf{R})$ *and* $||\sigma * \phi||_p = ||\sigma||_p$.

Since any interval of length one will work in (2) an example of such a ϕ, except for a normalizing factor required in (1), is $\phi(x) = \max\ (1 - 2|x|, 0)$.

6.2. THEOREM. *Suppose that* $p \geqq 1$ *is not an even integer or* ∞. *Then* $L_p(\mathbf{R})$ *does not have the UMP.*

Proof. Let ϕ be as in 6.1 and suppose that D is a positive constant. From the proof of 5.5 we have finitely supported functions λ, μ on **Z** which satisfy $|\hat{\lambda}| \leqq \hat{\mu}$ on **T** (hence on **R**) and $||\lambda||_p > D||\mu||_p$. Set $f = \lambda * \phi$ at $g = \mu * \phi$. Then g majorizes f and $||f||_p > D||g||_p$.

6.3. THEOREM. *The main theorem holds if* G *is a discrete abelian group containing elements of arbitrarily large order.*

Proof. Let D be a positive constant and suppose that f', g' are finitely supported functions on **Z** which satisfy $|\hat{f'}| \leqq \hat{g'}$ on **T** and $||f'||_p > D||g'||_p$. There is a positive integer n such that

$$\text{supp}\ (f') \cup \text{supp}\ (g') \subset S_n.$$

where $S_n = [-n, n] \cap \mathbf{Z}$.

S_n has $2n + 1$ members and by hypothesis G contains a cyclic subgroup H with $r \geqq 2n + 1$ members. Identify H with the subgroup $\{1, \xi, \cdots, \xi^{r-1}\}$ or **T**, where $\xi = \exp\ (2\pi i/r)$. The map ρ: $\mathbf{Z} \rightarrow G$ defined by $\rho(k) = \xi^k$ is a continuous homomorphism with image contained in H. It is also true that ρ restricted to S_n is injective. Define a function f on G by

$$f(x) = \begin{cases} f'(j) & x = \rho(j), j \in S_n \\ 0 & \text{otherwise;} \end{cases}$$

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define g similarly in terms of g'. Then $f, g \in S(G)$ and $||f||_p > D||g||_p$ since $||f||_p = ||f'||_p$ and $||g||_p = ||g'||_p$.

To see that g majorizes f we note that since f and g are supported by H we can identify $\hat{f}$ and $\hat{g}$ with the functions $\hat{f'}$ and $\hat{g'}$ on $\hat{G}/H^{\perp} = \hat{H} = H$. If $\gamma \in \hat{H}$ corresponds to ξ^k $(0 \leqq k \leqq r-1)$ we have $\hat{f}(\gamma) = \hat{f'}(\xi^k)$ and a similar equality for $\hat{g}$ and $\hat{g'}$. Thus g majorizes f and the proof is complete.

6.4. By combining 4.1, 4.2, 5.5, 6.2 and 6.3 we obtain the main theorem.

7. Final remarks.

7.1. There are other spaces which could be used instead of $S(G)$ in the definition of the UMP. Obvious possibilities are $L_1(G) \cap L_\infty(G)$, $L_1(G) \cap L_p(G)$, $A_c(G)$, the space of compactly supported members of $A(G)$, and $S_c(G)$, the subspace of $S(G)$ consisting of functions with compactly supported Fourier transforms. Standard use of an approximate identity (see [**7**, (33.12)]) for $L_1(G)$ or $L_1(\hat{G})$ shows that the resulting notions of UMP are equivalent to that defined in (2.1).

When $G = \mathbf{R}^n$ we can use the space of compactly supported C^∞ functions. In [**11**] we have shown how the whole space $L_p(G)$ can be used to give an equivalent notion of UMP.

7.2. A property dual to the UMP is the *lower majorant property* (see [**1**; **2**, and **5**]). In [**11**] we have proved an analogue of the Boas duality theorem (see the introduction). That an analogue of the Hardy-Littlewood duality theorem holds on a noncompact LCAG was recently established by E. Lee and G. Sunouchi in [**8**; **9**]. In [**8**] they have also given a proof of our main theorem for the integer group.

References

1. G. F. Bachelis, *On the upper and lower majorant properties in $L_p(G)$*, Quart. J. Math., Oxford (2) *24* (1973), 119–128.

2. R. P. Boas, *Majorant problems for Fourier series*, J. Analyse Math. *10* (1962–3), 253–271.

3. J. Fournier, *Majorants and L_p-norms*, Israel J. Math. *18* (1974), 157–166.

4. G. Gaudry, *Topics in harmonic analysis*, Lecture Notes, Yale University, 1969.

5. G. Hardy and J. Littlewood, *Notes on the theory of series (XIX): a problem concerning majorants of Fourier series*, Quart. J. Math. Oxford (1) *6* (1935), 304–315.

6. E. Hewit and K. Ross, *Abstract harmonic analysis, volume I* (Springer-Verlag, New York, 1963).

7. ——— *Abstract harmonic analysis, volume II* (Springer-Verlag, New York, 1970).

8. E. Lee and G. Sunouchi, *On the majorant properties in l^p*, Bull. Inst. Math. Academia Sinica *4* (1976), 327–336.

9. ——— *On the majorant properties in $L_p(G)$*, preprint, December 1977.

10. K. de Leeuw, *On L_p multipliers*, Ann. of Math. (2) *81* (1965), 364–379.

11. M. Rains, *Majorant problems in harmonic analysis*, Ph.D. Thesis, University of British Columbia, 1976.

12. H. Reiter, *Classical harmonic analysis and locally compact groups*, Oxford Mathematical Monographs (Oxford University Press, Oxford, 1968).

13. W. Rudin, *Positive definite sequences and absolutely monotonic functions*, Duke Math. J. *26* (1959), 617–622.

14. ——— *Fourier analysis on groups* (Wiley Interscience, New York, 1962).

15. H. Shapiro, *Majorant problems for Fourier coefficients*, Quart. J. Math. Oxford (2) *26* (1975), 9–18,

16. A. Zygmund, *Trigonometric series, Volume I* (Cambridge University Press, Cambridge, 1959).

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