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THE POWER STRUCTURE OF FINITE p-GROUPS

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In this survey article we give an exposition of some work on the power structure of p-groups; especially work of the author's. The titles of the three sections are: Section 1. p-groups which have regular power structure; Section 2. Some weaker power structure properties; Section 3. p-central series of p-groups.

Let G be a finite p-group. Assume that $\exp G = p^e$ and $e \geq 1$. For any s with $0 \leqq s \leqq e$, we define a mapping $\pi_s : G \to G$ by the rule

$$g\pi_s = g^{p^s} , \quad \forall g \in G$$

and we call π_s *the sth power mapping of* G. We use $\Lambda_s(G)$ and $V_s(G)$ to denote the kernel and the image of π_s, respectively; that is

$$\Lambda_s(G) = \{g \in G | g^{p^s} = 1\} \text{ and } V_s(G) = \{g^{p^s} | g \in G\} .$$

Setting

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$$\Omega_s(G) = \langle \Lambda_s(G) \rangle \quad \text{and} \quad \mho_s(G) = \langle V_s(G) \rangle ,$$

we get the following two characteristic subgroup series

$$(1) \qquad 1 = \Omega_0(G) < \Omega_1(G) \leqq \dots \leqq \Omega_e(G) = G$$

and

$$(2) \qquad G = \mho_0(G) > \mho_1(G) > \dots > \mho_e(G) = 1 .$$

We call (1) and (2) *the upper and lower power series of* G , respectively. If $\Omega_s(G) = \Lambda_s(G)$ for all s , we say the upper series (1) of G is *normal*. Similarly, (2) is said to be *normal* if $\mho_s(G) = V_s(G)$ for all s .

1. p-groups which have regular power structure

Regular p-groups defined by Hall [3] have very "nice" power structure. He found a lot of properties of them which make regular p-groups very like abelian ones. (See his original paper [3] or Huppert's textbook [5, Chapter III Section 10] for details.)

The main properties of the power structure of a regular p-group G are the following

(i) $\mho_s(G) = V_s(G)$ for all s;

(ii) $\Omega_s(G) = \Lambda_s(G)$ for all s;

(iii) $\pi_s : g\Omega_s(G) \to g^{p^s}$ is a well-defined bijection from $G/\Omega_s(G)$ onto $\mho_s(G)$; in particular we have

(iii') $|G/\Omega_s(G)| = |\mho_s(G)|$ for all s.

However, a p-group with the properties (i)-(iii) need not be regular. We give the following

DEFINITION 1. *If a* p*-group* G *has the properties (i)-(iii), we say that* G *<u>has regular power structure</u>.*

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An interesting problem about the power structure of p-groups is to determine all irregular p-groups which have regular power structure.

In studying this problem we note that the above properties (ii) and (iii) for a given s are equivalent to the following condition:

(iv) $(ab)^{p^s} = 1$ if and only if $a^{p^s} b^{p^s} = 1$ for any $a, b \in G$.

We give the following

DEFINITION 2. *Let s be a positive integer. A p-group is said to be semi-p^s-abelian if it has the property (iv).*

DEFINITION 3. *A p-group is said to be strongly semi-p-abelian if it is semi-p^s-abelian for all s .*

To justify these concepts we have the following.

THEOREM 1. *(Xu and Yang [15]) A p-group is regular if and only if every section of it is semi-p-abelian.*

THEOREM 2. *(Xu [11]) A p-group has regular power structure if and only if it is strongly semi-p-abelian and its lower power series is normal.*

Theorem 1 could be used as another definition for a regular p-group and Theorem 2 is just another way to say that a p-group has regular power structure. However, both theorems show that the concepts of semi-p-abelian and strongly semi-p-abelian p-groups are essential for studying our problem mentioned above.

We studied semi-p-abelian p-groups in [11, 12, 15], and obtained some results about these groups. Tuan has given a nice exposition of our work in those three papers at the Santa Cruz Conference in 1979, (see [8, Section 1]): there is one result which he did not mention, namely Theorem 3 below, and this result gives a partial answer to our problem of determining the irregular p-groups with regular power structure.

THEOREM 3. *(Xu [11, Theorem 4.4]). Any non-abelian 2-generator 2-group $G = \langle a, b\rangle$ which has regular power structure is one of the* *following types:*

(a) *A split metacyclic 2-group with defining relations* $a^{2^m} = b^{2^n} = 1$, $b^{-1}ab = a^{1+2^{m-c}}$, $m,n \geqq 2$, $1 \leqq c < \min\{n, m-1\}$;

(b) *a non-split metacyclic 2-group with defining relations* $a^{2^m} = 1$, $b^{2^n} = a^{2^{m-s}}$, $b^{-1}ab = a^{1+2^{m-c}}$, $m,n \geqq 2$, $1 \leqq c < \min\{n, m-1\}$, $\max\{1, m-n+1\} \leqq s \leqq \min\{c, m-c-1\}$.

The proof of this theorem is quite long. It can be divided into four parts. First, we found a necessary and sufficient condition for a 2-group to be semi-2-abelian, namely

THEOREM 4. *A finite 2-group* G *is semi-2-abelian if and only if* $\Omega_1(G) \leqq Z(G)$ *and none of the subgroups of* G *is isomorphic to either of the following groups:*

(1) *the quaternion group of order* 8;

(2) $\langle a,b \mid a^4 = 1,\ b^{2^n} = 1,\ b^{-1}ab = a^{-1},\ n \geqq 2\rangle$.

Secondly we proved

THEOREM 5. *Semi-2-abelian 2-groups are strongly semi-2-abelian.*

Thirdly we analysed the power structure of semi-2-abelian 2-groups and proved

THEOREM 6. *A 2-generator 2-group* G *has regular power structure if and only if* G *is metacyclic and semi-2-abelian.*

In the last step we used the classification of metacyclic 2-groups due to King [6] and picked out all semi-2-abelian ones from it. This completes the proof of Theorem 3.

I think that the problem of determining all irregular p-groups having regular power structure is very difficult. So I suggest the following

Problem 1. Determine and classify those irregular p-groups G which have regular power structure and satisfy the following extra conditions:

(1) $p = 2$, $d(G) > 2$;

(2) $p = 3$, $d(G) = 2$;

(3) G is metabelian

(Here $d(G)$ denotes the minimum number of generators of G .)

To do this problem for an odd prime p we first have to solve the following problem:

Problem 2. For an odd prime p , are semi-p-abelian p-groups strongly semi-p-abelian?

As a first step can we answer it when $p = 3$ or G is metabelian?

The next problem is about the power structure of a special class of p-groups.

Probelm 3. Let G be a finite p-group with $\Omega_1(G_{p-1}) \leqq Z(G)$ and $p > 2$, where G_{p-1} is the $(p - 1)$-th term of the lower central series of G . (We know that G is strongly semi-p-abelian; see [12].) Does G have regular power structure?

2. Some weaker power structure properties

Several authors have studied some power structure properties of p-groups weaker than having regular power structure, see [1, 7, 9]. Among them, Mann's work [7] is the most remarkable. He defined and studied so-called P_i-groups, $i = 1,2,3$. A p-group G is said to be a P_1- , P_2- , or P_3-group if every section of G has the property (i), (ii), or (iii') listed in Section 1 respectively. The main result of [7] is:

$$G \text{ is a regular group} \Rightarrow G \text{ is a } P_3\text{-group} \Rightarrow$$
$$\Rightarrow G \text{ is a } P_2\text{-group} \Rightarrow G \text{ is a } P_1\text{-group,}$$

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but none of the converses are true. For the case of $p = 2$, he determined all minimal non-P_i-groups, and then he gave a characterization of P_i-groups. He suggested the following problems.

Problem 4. Characterise P_i-groups for $p = 3$.

Problem 5. Characterise metabelian P_i-groups.

In [14], I gave an answer to Problem 4; but I do not know of any work on Problem 5.

Another power structure property is the existence of socalled uniqueness bases of p-groups.

DEFINITION 4. Let G be a finite p-group and $B = (b_1, b_2, \ldots, b_\omega)$ an ordered ω-tuple of elements of G. We call B *a uniqueness basis* of G if every element g of G can be uniquely expressed in the form

$$g = b_1^{\alpha_1} b_2^{\alpha_2} \ldots b_\omega^{\alpha_\omega},$$

where $0 \leqq \alpha_i < o(b_i)$, $i = 1, 2, \ldots, \omega$, and $o(b_i)$ is the order of b_i.

This kind of basis is quite like the basis of abelian p-groups. However, the class of p-groups which have uniqueness bases is much broader than that of abelian ones. Hall [3] proved that every regular p-group has at least one uniqueness basis. In his opinion, this result is the most important property of regular p-groups. His proof is very elegant but rather long. However, I gave another proof in my thesis [10] which is shorter and more natural, but the ideas of it are still his.

Can irregular p-group have a uniqueness basis? Yes! Mann [7] pointed out that every P_3-group has uniqueness bases. If we call a p-group every section of which has a uniqueness basis a UB-group, then P_3-groups are UB-groups. But the converse is not true; the wreath product $Z_p \wr Z_p$ is a counterexample. Naturally, we propose the following

PROBLEM 6. Determine those UB-groups which are not P_3-groups.

I have nothing to say about this problem except that Wang [9] pointed out that a UB-group which is simultaneiously a P_2-group must be a P_3-group. So groups in Problem 6 are not P_2-groups.

To end this section, I should like to suggest the following problem.

Problem 7. Must a p-group having regular power structure have a uniqueness basis?

3. p-central series of p-groups

Hobby [4] introduced the concepts of p-commutators and the p-commutator subgroup in a p-group. Using these concepts he gave a characterization of metacyclic p-groups. Independently, I introduced the concept of the p-centre of a p-group in [13]. Using this I generalized the theorem of Brisley and Macdonald [2] which gives a necessary and sufficient condition for a metabelian p-group to be regular.

I should now like to give an overview of these concepts.

Let G be a p-group and $a,b \in G$. We define *the p-commutator* $[a,b]_p$ *of a and b* by

$$[a,b]_p = b^{-p}a^{-p}(ab)^p .$$

By Hall's collection process (see [3]) it is easy to see that

$$(3) \qquad [a,b]_p = [b,a]^p x ,$$

where x is a product of several commutators in a and b with weights greater than 2.

Let A and B be two normal subgroups of G. We define *the p-commutator subgroup* $[A,B]_p$ *of* A *and* B by

$$[A,B]_p = \langle [a,b]_p,[b,a]_p \mid a \in A,\ b \in B\rangle .$$

We have $[A,B]_p = [B,A]_p$ and, by (3),

$$[A,B]_p \leqq \mho_1([A,B])[A,B,A][A,B,B] \leqq [A,B] .$$

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Now I can imitate the definitions of the derived series and the lower and upper central series of a p-group to give the following

DEFINITION 5. We call $\delta(G) = [G,G]_p$ *the p-commutator subgroup of G* and call the following series *the p-derived series of G :*

$$G = \delta_0(G) > \delta_1(G) > \ldots > \delta_\rho(G) = 1 ,$$

where $\delta_1(G) = \delta(G)$ and $\delta_{i+1}(G) = [\delta_i(G), \delta_i(G)]_p$ for $i > 1$. The number $\rho = \rho(G)$ is called *the length of the p-derived series of G*.

DEFINITION 6. We call the following series *the lower p-central series:*

$$(4) \qquad G = \kappa_1(G) > \kappa_2(G) > \ldots > \kappa_{\gamma+1}(G) = 1 ,$$

where $\kappa_2(G) = \delta(G)$ and $\kappa_{i+1}(G) = [\kappa_i(G),G]_p$ for $i > 1$.

DEFINITION 7. We call the following series *the upper p-central series:*

$$(5) \qquad 1 = \zeta_0(G) < \zeta_1(G) < \ldots < \zeta_\beta(G) = G ,$$

where

$$\zeta_1(G) = \zeta(G) = \{g \in G \mid [g,x]_p = [x,g]_p = 1 , \forall x \in G\}$$

is a characteristic subgroup of G, called *p-centre of G*, and $\zeta_i(G)$ is defined by

$$\zeta_i(G)/\zeta_{i-1}(G) = \zeta(G/\zeta_{i-1}(G))$$

It is easy to prove that series (4) and (5) have the same length; we call their length $\gamma = \gamma(G)$ *the p-class of G.*

I do not know whether these series are useful or not for the study of the power structure of p-groups. But I think the following problem is significant:

Problem 8. Give an estimate of $\gamma(G)$ using the nilpotency class $c(G)$ of G.

This problem is quite a big one. Even for $\gamma(G) = 1$, the existence of an upper bound for $c(G)$ is equivalent to the restricted Burnside problem for exponent p. This was proved by Kostrikin, but up to now we cannot find a general expression for the bound, not even a very rough one.

However, for metabelian p-groups, I proved that $\zeta(G) \leqq Z_p(G)$ (see [13]); so we get the inequality

$$c(G) \geqq \gamma(G) \geqq \left\lceil \frac{1}{p} c(G) \right\rceil . \tag{6}$$

Problem 9. Are those bounds given in (6) the best possible?

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