Canad. J. Math. Vol. **57** (1), 2005 pp. 204–224

# On the Duality between Coalescing Brownian Motions

Jie Xiong and Xiaowen Zhou

*Abstract.* A duality formula is found for coalescing Brownian motions on the real line. It is shown that the joint distribution of a coalescing Brownian motion can be determined by another coalescing Brownian motion running backward. This duality is used to study a measure-valued process arising as the high density limit of the empirical measures of coalescing Brownian motions.

## 1 Introduction

The *duality* concerned in this paper is the duality between the martingale problems for two Markov processes (see section 4.4 of [6]). It can be loosely described as follows: processes $X$ and $Y$ are *dual* to each other with respect to a *dual function* $f$ defined over an appropriate space if $\mathbb{P}[f(X_t;Y_0)] = \mathbb{P}[f(X_0;Y_t)], t > 0$. Under such a dual relationship problems concerning one process can be transformed into problems concerning the other. It has been a powerful technique in the study of certain stochastic systems. For example, the voter model is dual to systems of coalescing random walks. This coalescing duality plays a central role in the analysis of the voter model (see [10]). Dualities are also used intensively in the study of measure-valued stochastic processes such as superprocesses, stepping-stone models and interactive Fleming-Viot processes. A prime application of dualities in those models is to prove the uniqueness of a solution to a desired martingale problem. We refer to [4, 12] for literatures therein on details in this respect.

Coalescing Brownian motion was first studied in [1] and [8], where infinite systems of coalescing Brownian motions later called coalescing Brownian flows were considered. Recently, results were obtained in [13] concerning two families of interacting Brownian motions with one family of Brownian motions running forward and the other running backward. In such a stochastic system coalescing occurs among the Brownian motions running in the same direction and reflecting occurs among those in the opposite direction. The proofs were based on discrete approximations. The duality plays an important role in all the above mentioned work.

Coalescing Brownian motion is interesting because it captures the interactions between Brownian motions. It also serves as the dual process of other measure-valued Markov processes. For example, systems of coalescing Brownian motions are used to

Received by the editors January 28, 2003; revised July 3, 2003.

The first author was supported partially by NSA, Canada Research Chair Program, and NSERC, PIMS, Lockheed Martin Naval Electronics and Surveillance Systems, Lockheed Martin Canada and VisionSmart through a MITACS center of excellence entitled "Prediction in Interacting Systems". The second author was supported by NSERC.

AMS subject classification: 60J65, 60G57.

Keywords: coalescing Brownian motions, duality, martingale problem, measure-valued processes.





https://doi.org/10.4153/CJM-2005-009-2 Published online by Cambridge University Press

define and study the continuous sites stepping-stone models with Brownian migration (see [7, 5]).

The coalescing Brownian motion considered in this paper can be described intuitively as follows. An $n$-dimensional *coalescing Brownian motion* $\hat{\mathbf{X}} = (\hat{X}_1, \dots, \hat{X}_n)$ is a system of $n$ indexed interactive Brownian motions. Each process $\hat{X}_i$ evolves according to a one-dimensional Brownian motion, independent of the others, until two of them meet. Once $\hat{X}_i$ and $\hat{X}_j$, $1 \leq i < j \leq n$, first meet, $\hat{X}_j$ assumes the value of $\hat{X}_i$ after that, and we say the $j$-th process is *attached* to the $i$-th process which is still *free*. The system then evolves in the same fashion. A rigorous definition of the coalescing Brownian motion is postponed to Section 2.

In this paper we are going to propose another duality on the coalescing Brownian motions. Theorem 1.1, the main result of this paper, concerns a duality involving two systems of coalescing Brownian motions. Some notations are needed before we are ready to introduce this result.

Given a positive integer $n$, let $\mathcal{P}_n$ denote the set of *ordered partitions* of $\mathbb{N}_n := \{1, \dots, n\}$. That is, an element $\pi$ of $\mathcal{P}_n$ is a collection $\pi = \{A_1(\pi), \dots, A_h(\pi)\}$ of subsets of $\mathbb{N}_n$ with the property that $\bigcup_i A_i(\pi) = \mathbb{N}_n$ and all the integers contained in $A_i(\pi)$ are smaller than those in $A_j(\pi)$ for $i < j$. The sets $A_1(\pi), \dots A_h(\pi)$ are the *blocks* of the ordered partition $\pi$ and $h$ is the *length* of $\pi$, which is denoted by $l(\pi)$.

***Theorem 1.1*** *Suppose that $\hat{\mathbf{X}} = (\hat{X}_1, \dots, \hat{X}_n)$ is an $n$-dimensional coalescing Brownian motion with $\hat{\mathbf{X}}(0) = (x_1, \dots, x_n)$, $x_1 < \cdots < x_n$. Then for any $\pi \in \mathcal{P}_n$ and any $(y_1, \dots, y_{2l(\pi)})$, $y_1 < \cdots < y_{2l(\pi)}$, we have*

$$(1.1)\quad \mathbb{P}\Big\{\bigcap_{j=1}^{l(\pi)} \bigcap_{i\in A_j} \{\hat{X}_i(t) \in (y_{2j-1}, y_{2j})\}\Big\} = \mathbb{P}\Big\{\bigcap_{j=1}^{l(\pi)} \bigcap_{i\in A_j} \{x_i \in (\hat{Y}_{2j-1}(t), \hat{Y}_{2j}(t))\}\Big\},$$

*where $\hat{\mathbf{Y}} = (\hat{Y}_1, \dots, \hat{Y}_{2l(\pi)})$ is a $2l(\pi)$-dimensional coalescing Brownian motion with $\hat{\mathbf{Y}}(0) = (y_1, \dots, y_{2l(\pi)})$.*

With this duality the joint distribution of one coalescing Brownian motion can be expressed in terms of the distribution of random intervals driven by another coalescing Brownian motion running backward. The coalescing Brownian motion could be thought as self dual in this sense. There is no interaction between the two coalescing Brownian motions. To the authors' knowledge such a duality has not been pointed out even in the setting of coalescing simple random walks.

There are standard ways to show a dual relationship between two martingale problems (*e.g.,* see section 4.4 of [6]). But in general, to be able to apply Ito's formula, only smooth functions are chosen as the dual functions, which is not the case in our duality. To overcome this technical difficulty we adopt a modification of the standard approach. We first obtain approximate dualities with respect to smoothing functions of the original dual function. We then take a limit to reach the exact duality. Such an approach seems to be of independent interest. It could serve as a prototype for showing other dual relationships involving interacting Brownian motions.

A probability-measure-valued process is studied later in this paper. Such a process $Z$ arises as the high density limit of the empirical measures of coalescing Brownian motions. In Theorem 3.5 we are going to show that this $M_1(\mathbb{R})$-valued process $Z$ solves the following Martingale problem:

$$M_t(\phi) := Z_t(\phi) - Z_0(\phi) - \frac{1}{2}\int_0^t Z_s(\phi'')\, ds,\ t \geq 0, \phi \in C^2(\mathbb{R})$$

is a martingale with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t ds \int_\Delta \phi'(x)\phi'(y) Z_s(dx) Z_s(dy),$$

where $\Delta := \{(x,x) : x \in \mathbb{R}\}$. An interesting feature of this martingale problem is that its solution is *not unique*.

Since the above mentioned martingale problem is not well posed, coalescing duality (1.1) plays a crucial role in characterizing and analyzing the process $Z$. The uniqueness of $Z$ is justified in Theorem 3.3 using the coalescing duality (1.1). It can be shown that the marginal distribution of $Z$ is determined by systems of coalescing Brownian motions. The same duality is also applied to show that $Z$ is a Markov process and it only takes values of measures with discrete supports. Some of the arguments there have the flavor of the proof in Theorem 4.4.2 of [6].

The paper is arranged as follows. After an introduction of the main results in Section 1, we proceed to prove the coalescing duality in Section 2. Then the duality is applied to study the measure valued process $Z$ in Section 3.

## 2 Coalescing Duality

The ordered partitions in $\mathcal{P}_n$ can be used to keep track of interactions in a $n$-dimensional coalescing Brownian motion at a time. Given $\pi \in \mathcal{P}_n$, each block in $\pi$ corresponds to a free process. The block contains the index of that free process and indices of all the other processes attached to it. One can think of the ordered partition $\pi$ as an equivalence relation on $\mathbb{N}_n$ and write $i \sim_\pi j$ if $i$ and $j$ belong to the same block of $\pi \in \mathcal{P}_n$. Let

$$a_\pi(i) := \min\{1 \leq j \leq n : i \sim_\pi j\}, \quad 1 \leq i \leq n$$

and

$$a_i(\pi) := \min A_i(\pi), \quad 1 \leq i \leq l(\pi).$$

Then $a_\pi(i)$ is just the index of the free process to which the $i$-th process is attached. $a_1(\pi), \dots, a_{l(\pi)}(\pi)$ are all the indices of the free processes at an increasing order.

For $\pi' \in \mathcal{P}_n$, write $\pi \prec \pi'$ or $\pi' \succ \pi$ if $\pi'$ is obtained by merging some of the blocks in $\pi$. Write $\pi \preceq \pi' (\pi' \succeq \pi)$ if $\pi \prec \pi' (\pi' \succ \pi)$ or $\pi' = \pi$. The history of interactions in a coalescing Brownian can be specified by a sequence of ordered partitions $\pi_1 \prec \pi_2 \prec \cdots \prec \pi_k$.

Given $\pi \in \mathcal{P}_n$, we can define an $l(\pi)$-dimensional subspace $\mathbb{R}^n_\pi$ of $\mathbb{R}^n$ by identifying the coordinates with indices from the same block of $\pi$. More precisely,

$$\mathbb{R}^n_\pi := \{(x_{a_\pi(1)}, \dots, x_{a_\pi(n)}) : x_{a_\pi(i)} \in \mathbb{R}, 1 \leq i \leq n\}.$$

Put

$$\hat{\mathbb{R}}^n_\pi := \mathbb{R}^n_\pi \setminus \bigcup_{\pi' \succ \pi} \mathbb{R}^n_{\pi'}.$$

By definition $\hat{\mathbb{R}}^n_\pi \cap \hat{\mathbb{R}}^n_{\pi'} = \varnothing$ for $\pi \neq \pi'$. $\hat{\mathbb{R}}^n_\pi$ will serve as the effective state space of the coalescing Brownian motion when the coalescing interaction is determined by $\pi$.

When the coalescing interaction is determined by $\pi$, the $n$-dimensional coalescing Brownian motion is essentially an $l(\pi)$-dimensional process. Define $J_\pi(x_1,\dots,x_n) := (x_{a_1(\pi)},\dots,x_{a_{l(\pi)}(\pi)})$, $(x_1,\dots,x_n) \in \mathbb{R}^n_\pi$. Notice that $J_\pi$ is a bijection between $\mathbb{R}^n_\pi$ and $\mathbb{R}^{l(\pi)}$.

For example,

$$\mathbb{R}^4_{\{\{1,2\},\{3,4\}\}} = \{(x_1,x_1,x_2,x_2) : x_1,x_2 \in \mathbb{R}\},$$
$$\hat{\mathbb{R}}^4_{\{\{1,2\},\{3,4\}\}} = \{(x_1,x_1,x_2,x_2) : x_1,x_2 \in \mathbb{R}, x_1 \neq x_2\}$$

and

$$J_{\{\{1,2\},\{3,4\}\}}(x_1,x_1,x_2,x_2) = (x_1,x_2).$$

Write

$$\mathbb{R}^n_0 := \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_1 \le \cdots \le x_n\}$$

and

$$\mathbb{R}^n_1 := \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_1 < \cdots < x_n\}.$$

Given an $n$-dimensional Brownian motion $\mathbf{X} = (X_1,\dots,X_n)$ starting from $(x_1,\dots,x_n) \in \mathbb{R}^n_1$, the $n$-dimensional coalescing Brownian motion $\hat{\mathbf{X}} = (\hat{X}_1,\dots,\hat{X}_n)$ can be constructed from $\mathbf{X}$ inductively as follows. Suppose that times $0 =: \tau_0 \le \cdots \le \tau_k \le \infty$ and partitions $\{\{1\},\dots,\{n\}\} =: \pi_0 \prec \cdots \prec \pi_k \preceq \{\{1,\dots,n\}\}$ have already been defined and $\hat{\mathbf{X}}$ has been defined on $[0,\tau_k)$. If $\pi_k = \{\{1,\dots,n\}\}$, then $\hat{\mathbf{X}}_t = (X_1(t),\dots,X_1(t))$ for $t \ge \tau_k$. Otherwise, let $\pi_k = \{A_1(\pi_k),\dots,A_{l(\pi_k)}(\pi_k)\}$. Put

$$\tau_{k+1} := \inf\{t > \tau_k : \exists i, X_{a_i(\pi_k)}(t) = X_{a_{i+1}(\pi_k)}(t)\}. \tag{2.1}$$

Suppose that $X_{a_i(\pi_k)}(\tau_{k+1}) = X_{a_{i+1}(\pi_k)}(\tau_{k+1})$ for some $1 \le i < l(\pi_k)$; then define

$$\pi_{k+1} := \{A_1(\pi_{k+1}),\dots,A_{l(\pi_k)-1}(\pi_{k+1})\}, \tag{2.2}$$
$$A_r(\pi_{k+1}) := \begin{cases} A_r(\pi_k), & \text{for } 1 \le r < i,\\ A_i(\pi_k) \cup A_{i+1}(\pi_k), & \text{for } r = i,\\ A_{r+1}(\pi_k), & \text{for } i < r \le l(\pi_k) - 1,\end{cases}$$

and $\hat{\mathbf{X}}(t) := (X_{a_{\pi_k}(1)}(t),\dots,X_{a_{\pi_k}(n)}(t))$ for $\tau_k \le t < \tau_{k+1}$.

When the starting point is not in $\mathbb{R}^n_1$, the $n$-dimensional coalescing Brownian motion can be defined similarly.

One can easily see that $\hat{\mathbf{X}}$ is an $n$-dimensional continuous martingale with quadratic variations

$$\langle \hat{X}_i, \hat{X}_j \rangle_t = \begin{cases} t, & i = j,\\ t - T_{ij} \wedge t, & i \neq j,\end{cases}$$

https://doi.org/10.4153/CJM-2005-009-2 Published online by Cambridge University Press

where $T_{ij} := \inf\{t \geq 0 : \hat{X}_i(t) = \hat{X}_j(t)\}$.

$\mathcal{F}_t^{\hat{\mathbf{X}}}$ denotes the $\sigma$-field generated by $\hat{\mathbf{X}}$. Write $C_b^2(\mathbb{R}^n)$ for the collection of functions on $\mathbb{R}^n$ with bounded first and second partial derivatives. Define an operator $L$ on $C_b^2(\mathbb{R}^n)$ by

$$Lf(\mathbf{x}) := \frac{1}{2} \sum_{\pi \in \mathcal{P}_n} 1_{\hat{\mathbb{R}}_\pi^n}(\mathbf{x}) \Delta_{l(\pi)}(f \circ J_\pi^{-1})(J_\pi(\mathbf{x})),\ f \in C_b^2(\mathbb{R}^n), \mathbf{x} \in \mathbb{R}_0^n, \tag{2.3}$$

where $\Delta_d$ denotes the $d$-dimensional Laplacian operator. The next lemma gives the generator of the coalescing Brownian motion.

***Lemma 2.1*** *Let $\hat{\mathbf{X}}$ be an $n$-dimensional coalescing Brownian motion such that $\hat{\mathbf{X}}(0) \in \mathbb{R}_1^n$. Then for any $f \in C_b^2(\mathbb{R}^n)$, $f(\hat{\mathbf{X}}(t)) - \int_0^t Lf(\hat{\mathbf{X}}(s))\,ds$ is an $\mathcal{F}_t^{\hat{\mathbf{X}}}$-martingale.*

**Proof** Given $f \in C_b^2(\mathbb{R}^n)$, applying Ito's formula we have

(2.4)

$$\begin{aligned}
f(\hat{\mathbf{X}}(t)) &- f(\hat{\mathbf{X}}(0)) \\
&= \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(\hat{\mathbf{X}}(s))\, d\langle \hat{X}_i, \hat{X}_j \rangle_s + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(\hat{\mathbf{X}}(s))\, d\hat{X}_i(s) \\
&= \frac{1}{2} \sum_{i,j=1}^n \sum_{\pi \in \mathcal{P}_n} \int_0^t 1_{\hat{\mathbb{R}}_\pi^n}(\hat{\mathbf{X}}(s)) \frac{\partial^2 f}{\partial x_i \partial x_j}(\hat{\mathbf{X}}(s))\, d\langle \hat{X}_i, \hat{X}_j \rangle_s + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(\hat{\mathbf{X}}(s))\, d\hat{X}_i(s) \\
&= \frac{1}{2} \sum_{\pi \in \mathcal{P}_n} \sum_{k=1}^{l(\pi)} \sum_{i,j \in A_k(\pi)} \int_0^t 1_{\hat{\mathbb{R}}_\pi^n}(\hat{\mathbf{X}}(s)) \frac{\partial^2 f}{\partial x_i \partial x_j}(\hat{X}_{a_\pi(1)}(s), \dots, \hat{X}_{a_\pi(n)}(s))\, ds \\
&\quad + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(\hat{\mathbf{X}}(s))\, d\hat{X}_i(s) \\
&= \frac{1}{2} \sum_{\pi \in \mathcal{P}_n} \int_0^t 1_{\hat{\mathbb{R}}_\pi^n}(\hat{\mathbf{X}}(s)) \Delta_{l(\pi)}(f \circ J_\pi^{-1})(J_\pi(\hat{\mathbf{X}}(s)))\, ds + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(\hat{\mathbf{X}}(s))\, d\hat{X}_i(s) \\
&= \int_0^t Lf(\hat{\mathbf{X}}(s))\, ds + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(\hat{\mathbf{X}}(s))\, d\hat{X}_i(s).
\end{aligned}$$

Hence, $f(\hat{\mathbf{X}}(t)) - \int_0^t Lf(\hat{\mathbf{X}}(s))\,ds$ is an $\mathcal{F}_t^{\hat{\mathbf{X}}}$-martingale. ■

Define

$$\mathcal{P}_{2m}^* := \{\pi \in \mathcal{P}_{2m} : a_\pi(2i) = 2i, a_\pi(2i+1) = 2i \text{ or } 2i+1, 1 \leq i \leq m-1\}.$$

https://doi.org/10.4153/CJM-2005-009-2 Published online by Cambridge University Press

*i.e.*, each $\pi^* \in \mathcal{P}^*_{2m}$ is obtained by merging the $\{2i\}$ and the $\{2i+1\}$ blocks in $\pi_0$ for some $1 \le i \le m$. For example,

$$\mathcal{P}^*_6 = \Big\{ \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}, \{\{1\},\{2,3\},\{4\},\{5\},\{6\}\}, \\ \{\{1\},\{2\},\{3\},\{4,5\},\{6\}\}, \{\{1\},\{2,3\},\{4,5\},\{6\}\} \Big\}.$$

Given $\pi = \{A_1(\pi),\dots,A_{l(\pi)}(\pi)\} \in \mathcal{P}_n$, for any $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}_0^{2l(\pi)}$, define

$$f_{\sigma\pi}(\mathbf{x};\mathbf{y}) := \prod_{j=1}^{l(\pi)} \prod_{i\in A_j(\pi)} \int_{y_{2j-1}}^{y_{2j}} \phi_\sigma(z-x_i)\,dz,$$

where $\phi_\sigma$ denotes the density function of the normal distribution with mean 0 and variance $\sigma^2$. Write $\phi := \phi_1$. It is easily seen that $f_{\sigma\pi}(\mathbf{x};\mathbf{y})$ is a smoothing of the function $f(\mathbf{x};\mathbf{y}) := \prod_{j=1}^{l(\pi)} \prod_{i\in A_j} 1_{(y_{2j-1},y_{2j})}(x_i)$.

For any $\theta \in \mathcal{P}_n$, $\vartheta \in \mathcal{P}^*_{2l(\pi)}$ and $\mathbf{x}' \in \mathbb{R}^{l(\theta)}$, $\mathbf{y}' \in \mathbb{R}^{l(\vartheta)}$, let

$$f_{\theta\vartheta}(\mathbf{x}';\mathbf{y}') := f_{\sigma\pi}(J_\theta^{-1}(\mathbf{x}'); J_\vartheta^{-1}(\mathbf{y}'))$$

be a function on $\mathbb{R}^{l(\theta)+l(\vartheta)}$. For $\mathbf{y} \in \hat{\mathbb{R}}_\vartheta^{2l(\pi)}$, define $f_{\theta\mathbf{y}} \in C(\mathbb{R}^{l(\theta)})$ by

$$f_{\theta\mathbf{y}}(\mathbf{x}') := f_{\theta\vartheta}(\mathbf{x}'; J_\vartheta(\mathbf{y})) = f_{\sigma\pi}(J_\theta^{-1}(\mathbf{x}');\mathbf{y}),\ \mathbf{x}' \in \mathbb{R}^{l(\theta)}.$$

Similarly, for $\mathbf{x} \in \hat{\mathbb{R}}^n_\theta$, define $f_{\mathbf{x}\vartheta} \in C(\mathbb{R}^{l(\vartheta)})$ by

$$f_{\mathbf{x}\vartheta}(\mathbf{y}') := f_{\theta\vartheta}(J_\theta(\mathbf{x});\mathbf{y}') = f_{\sigma\pi}(\mathbf{x}; J_\vartheta^{-1}(\mathbf{y}')),\ \mathbf{y}' \in \mathbb{R}^{l(\vartheta)}.$$

Given $\pi = \{A_1, A_2, \dots, A_{l(\pi)}\}, \theta \in \mathcal{P}_n$ and $\vartheta \in \mathcal{P}^*_{2l(\pi)}$, for any $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^{2l(\pi)}$, write

$$G_{\theta\vartheta}(\mathbf{x};\mathbf{y}) := S^1_{\theta\vartheta}(\mathbf{x};\mathbf{y}) + S^2_{\theta\vartheta}(\mathbf{x};\mathbf{y}) + S^3_{\theta\vartheta}(\mathbf{x};\mathbf{y}) + S^4_{\theta\vartheta}(\mathbf{x};\mathbf{y}),$$

where

$$S^1_{\theta\vartheta}(\mathbf{x};\mathbf{y}) := \sum_{i=1}^{l(\theta)} \sum_{j=1}^{l(\pi)} \sum_{\substack{k,l\in A_j, k\neq l \\ a_\theta(k)=a_i(\theta)=a_\theta(l)}} \phi_\sigma(y_{a_\vartheta(2j)} - x_{a_i(\theta)})\phi_\sigma(y_{a_\vartheta(2j-1)} - x_{a_i(\theta)}),$$

$$S^2_{\theta\vartheta}(\mathbf{x};\mathbf{y}) := \sum_{i=1}^{l(\theta)} \sum_{\substack{1\le j,j'\le l(\pi) \\ j\neq j'}} \sum_{\substack{k\in A_j, l\in A_{j'} \\ a_\theta(k)=a_i(\theta)=a_\theta(l)}}$$
$$\Big[ \phi_\sigma(y_{a_\vartheta(2j)} - x_{a_i(\theta)})\phi_\sigma(y_{a_\vartheta(2j')} - x_{a_i(\theta)}) + \phi_\sigma(y_{a_\vartheta(2j-1)} - x_{a_i(\theta)})\phi_\sigma(y_{a_\vartheta(2j'-1)} - x_{a_i(\theta)}) \\ + \phi_\sigma(y_{a_\vartheta(2j)} - x_{a_i(\theta)})\phi_\sigma(y_{a_\vartheta(2j'-1)} - x_{a_i(\theta)}) 1_{\{a_\vartheta(2j)\neq a_\vartheta(2j'-1)\}} \\ + \phi_\sigma(y_{a_\vartheta(2j-1)} - x_{a_i(\theta)})\phi_\sigma(y_{a_\vartheta(2j')} - x_{a_i(\theta)}) 1_{\{a_\vartheta(2j-1)\neq a_\vartheta(2j')\}} \Big],$$

https://doi.org/10.4153/CJM-2005-009-2 Published online by Cambridge University Press

$$S^3_{\theta\vartheta}(\mathbf{x};\mathbf{y}) := \sum_{j=1}^{l(\pi)} \sum_{\substack{k,l\in A_j\\ a_\theta(k)\neq a_\theta(l)}} \big[\phi_\sigma(y_{a_\vartheta(2j)} - x_{a_\theta(k)})\phi_\sigma(y_{a_\vartheta(2j)} - x_{a_\theta(l)}) + \phi_\sigma(y_{a_\vartheta(2j-1)} - x_{a_\theta(k)})\phi_\sigma(y_{a_\vartheta(2j-1)} - x_{a_\theta(l)})\big],$$

and

$$S^4_{\theta\vartheta}(\mathbf{x};\mathbf{y}) := \sum_{\substack{1\leq j\leq l(\pi)-1\\ a_\vartheta(2j)=a_\vartheta(2j+1)}} \sum_{\substack{k\in A_j, l\in A_{j+1}\\ a_\theta(k)\neq a_\theta(l)}} \phi_\sigma(y_{2j} - x_{a_\theta(k)})\phi_\sigma(y_{2j} - x_{a_\theta(l)}).$$

We remark that we have suppressed the dependence of both $\sigma$ and $\pi$ in the definitions of $f_{\theta\vartheta}$, $f_{\mathbf{x}\vartheta}$, $f_{\theta\mathbf{y}}$ and $G_{\theta\vartheta}$.

One can verify that for any $\mathbf{x} \in \hat{\mathbb{R}}^n_\theta$ and $\mathbf{y} \in \hat{\mathbb{R}}^{2l(\pi)}_\vartheta$,

$$\begin{aligned}(2.5)\quad \Delta_{l(\theta)} f_{\theta\mathbf{y}} \circ J_\theta(\mathbf{x}) - \Delta_{l(\vartheta)} f_{\mathbf{x}\vartheta} \circ J_\vartheta(\mathbf{y}) &= \sum_{i=1}^{l(\theta)} \frac{\partial^2 f_{\theta\mathbf{y}}}{\partial x_{a_i(\theta)}{}^2}(x_{a_1(\theta)}, \dots, x_{a_{l(\theta)}(\theta)}) - \sum_{j=1}^{l(\vartheta)} \frac{\partial^2 f_{\mathbf{x}\vartheta}}{\partial y_{a_j(\vartheta)}{}^2}(y_{a_1(\vartheta)}, \dots, y_{a_{l(\vartheta)}(\vartheta)})\\ &= -I_1 + I_2 - I_3 + I_4,\end{aligned}$$

where

$$I_1 := \sum_{i=1}^{l(\theta)} \sum_{j=1}^{l(\pi)} \sum_{\substack{k,l\in A_j, k\neq l\\ a_\theta(k)=a_i(\theta)=a_\theta(l)}} \phi_\sigma(y_{a_\vartheta(2j)} - x_{a_\theta(k)})\phi_\sigma(y_{a_\vartheta(2j-1)} - x_{a_\theta(l)}) \times \prod_{\substack{i'\neq k,l\\ i'\in A_j}} \int_{y_{a_\vartheta(2j-1)}}^{y_{a_\vartheta(2j)}} \phi_\sigma(x - x_{a_\theta(i')})\,dx \prod_{j'\neq j} \prod_{i'\in A_{j'}} \int_{y_{a_\vartheta(2j'-1)}}^{y_{a_\vartheta(2j')}} \phi_\sigma(x - x_{a_\theta(i')})\,dx,$$

$$\begin{aligned} I_2 := \sum_{i=1}^{l(\theta)} \sum_{\substack{1\leq j,j'\leq l(\pi)\\ j\neq j'}} \sum_{\substack{k\in A_j, l\in A_{j'}\\ a_\theta(k)=a_i(\theta)=a_\theta(l)}} \big[&\phi_\sigma(y_{a_\vartheta(2j)} - x_{a_\theta(k)})\phi_\sigma(y_{a_\vartheta(2j')} - x_{a_\theta(k)})\\ &+ \phi_\sigma(y_{a_\vartheta(2j-1)} - x_{a_\theta(l)})\phi_\sigma(y_{a_\vartheta(2j'-1)} - x_{a_\theta(l)})\\ &- \phi_\sigma(y_{a_\vartheta(2j)} - x_{a_\theta(k)})\phi_\sigma(y_{a_\vartheta(2j'-1)} - x_{a_\theta(l)})1_{\{a_\vartheta(2j)\neq a_\vartheta(2j'-1)\}}\\ &- \phi_\sigma(y_{a_\vartheta(2j-1)} - x_{a_\theta(k)})\phi_\sigma(y_{a_\vartheta(2j')} - x_{a_\theta(l)})1_{\{a_\vartheta(2j-1)\neq a_\vartheta(2j')\}}\big]\\ &\times \prod_{\substack{i'\neq k\\ i'\in A_j}} \int_{y_{a_\vartheta(2j-1)}}^{y_{a_\vartheta(2j)}} \phi_\sigma(x - x_{a_\theta(i')})\,dx \prod_{\substack{i'\neq l\\ i'\in A_{j'}}} \int_{y_{a_\vartheta(2j'-1)}}^{y_{a_\vartheta(2j')}} \phi_\sigma(x - x_{a_\theta(i')})\,dx\\ &\times \prod_{j''\neq j,j'} \prod_{i'\in A_{j''}} \int_{y_{a_\vartheta(2j''-1)}}^{y_{a_\vartheta(2j'')}} \phi_\sigma(x - x_{a_\theta(i')})\,dx,\end{aligned}$$

https://doi.org/10.4153/CJM-2005-009-2 Published online by Cambridge University Press

$$I_3 := \sum_{j=1}^{l(\pi)} \sum_{\substack{k,l\in A_j\\ a_\theta(k)\neq a_\theta(l)}} \Big[\phi_\sigma(y_{a_\vartheta(2j)} - x_{a_\theta(k)})\phi_\sigma(y_{a_\vartheta(2j)} - x_{a_\theta(l)})$$
$$+ \phi_\sigma(y_{a_\vartheta(2j-1)} - x_{a_\theta(k)})\phi_\sigma(y_{a_\vartheta(2j-1)} - x_{a_\theta(l)})\Big]$$
$$\times \prod_{\substack{i'\neq k,l\\ i'\in A_j}} \int_{y_{a_\vartheta(2j-1)}}^{y_{a_\vartheta(2j)}} \phi_\sigma(x - x_{a_\theta(i')})\,dx \prod_{j'\neq j}\prod_{i'\in A_{j'}} \int_{y_{a_\vartheta(2j'-1)}}^{y_{a_\vartheta(2j')}} \phi_\sigma(x - x_{a_\theta(i')})\,dx$$

and

$$I_4 := \sum_{\substack{1\leq j\leq l(\pi)-1\\ a_\vartheta(2j)=a_\vartheta(2j+1)}} \sum_{\substack{k\in A_j, l\in A_{j+1}\\ a_\theta(k)\neq a_\theta(l)}} \phi_\sigma(y_{a_\vartheta(2j)} - x_{a_\theta(k)})\phi_\sigma(y_{a_\vartheta(2j+1)} - x_{a_\theta(l)}).$$
$$\times \prod_{\substack{i'\neq k\\ i'\in A_j}} \int_{y_{a_\vartheta(2j-1)}}^{y_{a_\vartheta(2j)}} \phi_\sigma(x - x_{a_\theta(i')})\,dx \prod_{\substack{i'\neq l\\ i'\in A_{j+1}}} \int_{y_{a_\vartheta(2j+1)}}^{y_{a_\vartheta(2j+2)}} \phi_\sigma(x - x_{a_\theta(i')})\,dx$$
$$\times \prod_{j'\neq j,j+1}\prod_{i'\in A_{j'}} \int_{y_{a_\vartheta(2j'-1)}}^{y_{a_\vartheta(2j')}} \phi_\sigma(x - x_{a_\theta(i')})\,dx.$$

It then follows that for any $\mathbf{x}\in\hat{\mathbb{R}}^n_\theta$ and $\mathbf{y}\in\hat{\mathbb{R}}^{2l(\pi)}_\vartheta\cap\mathbb{R}^{2l(\pi)}_0$,

$$|\Delta_{l(\theta)} f_{\theta\mathbf{y}}\circ J_\theta(\mathbf{x}) - \Delta_{l(\vartheta)} f_{\mathbf{x}\vartheta}\circ J_\vartheta(\mathbf{y})| \leq G_{\theta\vartheta}(\mathbf{x};\mathbf{y}). \tag{2.6}$$

***Remark 2.2*** Observe that there are some cancellations occurring in (2.5). Because of these cancellations (2.6) holds. It will lead to (2.7) later, which is crucial to the proof of Theorem 1.1.

**Proof of Theorem 1.1** The outline of the proof is the following. We first pose the martingale problems for both $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ with respect to the test functions $\{f_{\theta\vartheta}\}$. Then we apply the standard approach to set up an approximate duality for $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$. By letting $\sigma\to 0+$ we would eventually obtain the exact duality.

For any $\vartheta\in\mathcal{P}^*_{2l(\pi)}$ and $\mathbf{y}\in\hat{\mathbb{R}}^{2l(\pi)}_\vartheta$, by Lemma 2.1

$$\sum_{\theta\in\mathcal{P}_n} f_{\theta\mathbf{y}}\circ J_\theta(\hat{\mathbf{X}}(t))1_{\{\hat{\mathbf{X}}(t)\in\hat{\mathbb{R}}^n_\theta\}} - \sum_{\theta\in\mathcal{P}_n}\int_0^t \frac{1}{2}\Delta_{l(\theta)} f_{\theta\mathbf{y}}\circ J_\theta(\hat{\mathbf{X}}(s))1_{\{\hat{\mathbf{X}}(s)\in\hat{\mathbb{R}}^n_\theta\}}\,ds$$
$$= f_{\sigma\pi}(\hat{\mathbf{X}}(t);\mathbf{y}) - \int_0^t L_x f_{\sigma\pi}(\hat{\mathbf{X}}(s);\mathbf{y})\,ds$$

is a martingale, where $L_x$ is $L$ acting on the first $n$ variables of $f_{\sigma\pi}$.

https://doi.org/10.4153/CJM-2005-009-2 Published online by Cambridge University Press

Given $\theta \in \mathcal{P}_n$ and $\mathbf{x} \in \hat{\mathbb{R}}^n_\theta$, for any $\vartheta \in \mathcal{P}_{2l(\pi)} - \mathcal{P}^*_{2l(\pi)}$ we have $f_{\sigma\mathbf{x}\vartheta}(\mathbf{y}') = 0,\ \mathbf{y}' \in \hat{\mathbb{R}}^{l(\vartheta)}$. Then

$$\sum_{\vartheta\in\mathcal{P}^*_{2l(\pi)}} f_{\mathbf{x}\vartheta} \circ J_\vartheta(\hat{\mathbf{Y}}(t))1_{\{\hat{\mathbf{Y}}(t)\in\hat{\mathbb{R}}^{2l(\pi)}_\vartheta\}} - \sum_{\vartheta\in\mathcal{P}^*_{2l(\pi)}} \int_0^t \frac{1}{2}\Delta_{l(\vartheta)} f_{\mathbf{x}\vartheta} \circ J_\vartheta(\hat{\mathbf{Y}}(s))1_{\{\hat{\mathbf{Y}}(s)\in\hat{\mathbb{R}}^{2l(\pi)}_\vartheta\}}\, ds$$
$$= f_{\sigma\pi}(\mathbf{x}; \hat{\mathbf{Y}}(t)) - \int_0^t L_y f_{\sigma\pi}(\mathbf{x}; \hat{\mathbf{Y}}(s))\, ds$$

is also a martingale by Lemma 2.1. This time $L_y$ is $L$ acting on the last $2l(\pi)$ variables of $f_{\sigma\pi}$.

Since we can assume that $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ are independent, it follows that

$$\begin{aligned}
&\frac{d}{ds}\mathbb{P}[f_{\sigma\pi}(\hat{\mathbf{X}}(s); \hat{\mathbf{Y}}(t-s))]\\
&\quad= \frac{d}{ds}\mathbb{P}\Big[\sum_{\vartheta\in\mathcal{P}^*_{2l(\pi)}}\sum_{\theta\in\mathcal{P}_n} f_{\sigma\pi}(\hat{\mathbf{X}}(s); \hat{\mathbf{Y}}(t-s)),\ \hat{\mathbf{X}}(s) \in \hat{\mathbb{R}}^n_\theta, \hat{\mathbf{Y}}(t-s) \in \hat{\mathbb{R}}^{2l(\pi)}_\vartheta\Big]\\
&\quad= \frac{d}{ds}\mathbb{P}\Big[\sum_{\vartheta\in\mathcal{P}^*_{2l(\pi)}}\sum_{\theta\in\mathcal{P}_n} f_{\theta\vartheta}(J_\theta(\hat{\mathbf{X}}(s)); J_\vartheta(\hat{\mathbf{Y}}(t-s))),\ \hat{\mathbf{X}}(s) \in \hat{\mathbb{R}}^n_\theta, \hat{\mathbf{Y}}(t-s) \in \hat{\mathbb{R}}^{2l(\pi)}_\vartheta\Big]\\
&\quad= \sum_{\vartheta\in\mathcal{P}^*_{2l(\pi)}}\sum_{\theta\in\mathcal{P}_n}\mathbb{P}\Big[\frac{1}{2}\Delta_{l(\theta)} f_{\theta\hat{\mathbf{Y}}(t-s)} \circ J_\theta(\hat{\mathbf{X}}(s)),\ \hat{\mathbf{X}}(s) \in \hat{\mathbb{R}}^n_\theta, \hat{\mathbf{Y}}(t-s) \in \hat{\mathbb{R}}^{2l(\pi)}_\vartheta\Big]\\
&\qquad- \sum_{\vartheta\in\mathcal{P}^*_{2l(\pi)}}\sum_{\theta\in\mathcal{P}_n}\mathbb{P}\Big[\frac{1}{2}\Delta_{l(\vartheta)} f_{\hat{\mathbf{X}}(s)\vartheta} \circ J_\vartheta(\hat{\mathbf{Y}}(t-s)),\ \hat{\mathbf{X}}(s) \in \hat{\mathbb{R}}^n_\theta, \hat{\mathbf{Y}}(t-s) \in \hat{\mathbb{R}}^{2l(\pi)}_\vartheta\Big].
\end{aligned}$$

Integrating the previous equation from 0 to $t$, by (2.6) we have

(2.7)
$$\begin{aligned}
&\Big|\sum_{\theta\in\mathcal{P}_n}\mathbb{P}\big[f_{\sigma\pi}(\hat{\mathbf{X}}(t); \mathbf{y}),\ \hat{\mathbf{X}}(t) \in \hat{\mathbb{R}}^n_\theta\big] - \sum_{\vartheta\in\mathcal{P}^*_{2l(\pi)}}\mathbb{P}\big[f_{\sigma\pi}(\mathbf{x}; \hat{\mathbf{Y}}(t)),\ \hat{\mathbf{Y}}(t) \in \hat{\mathbb{R}}^{2l(\pi)}_\vartheta\big]\Big|\\
&= \Big|\sum_{\vartheta\in\mathcal{P}^*_{2l(\pi)}}\sum_{\theta\in\mathcal{P}_n}\mathbb{P}\big[f_{\sigma\pi}(\hat{\mathbf{X}}(t); \hat{\mathbf{Y}}(0)),\ \hat{\mathbf{X}}(t) \in \hat{\mathbb{R}}^n_\theta, \hat{\mathbf{Y}}(0) \in \hat{\mathbb{R}}^{2l(\pi)}_\vartheta\big]\\
&\qquad- \sum_{\vartheta\in\mathcal{P}^*_{2l(\pi)}}\sum_{\theta\in\mathcal{P}_n}\mathbb{P}\big[f_{\sigma\pi}(\hat{\mathbf{X}}(0); \hat{\mathbf{Y}}(t)),\ \hat{\mathbf{X}}(0) \in \hat{\mathbb{R}}^n_\theta, \hat{\mathbf{Y}}(t) \in \hat{\mathbb{R}}^{2l(\pi)}_\vartheta\big]\Big|\\
&= \frac{1}{2}\Big|\sum_{\vartheta\in\mathcal{P}^*_{2l(\pi)}}\sum_{\theta\in\mathcal{P}_n}\int_0^t\mathbb{P}\big[\Delta_{l(\theta)} f_{\theta\hat{\mathbf{Y}}(t-s)} \circ J_\theta(\hat{\mathbf{X}}(s)) - \Delta_{l(\vartheta)} f_{\hat{\mathbf{X}}(s)\vartheta} \circ J_\vartheta(\hat{\mathbf{Y}}(t-s)),\\
&\qquad\hat{\mathbf{X}}(s) \in \hat{\mathbb{R}}^n_\theta, \hat{\mathbf{Y}}(t-s) \in \hat{\mathbb{R}}^{2l(\pi)}_\vartheta\big]\, ds\Big|\\
&\le \frac{1}{2}\sum_{\vartheta\in\mathcal{P}^*_{2l(\pi)}}\sum_{\theta\in\mathcal{P}_n}\int_0^t\mathbb{P}\big[G_{\theta\vartheta}(\hat{\mathbf{X}}(s); \hat{\mathbf{Y}}(t-s)),\ \hat{\mathbf{X}}(s) \in \hat{\mathbb{R}}^n_\theta, \hat{\mathbf{Y}}(t-s) \in \hat{\mathbb{R}}^{2l(\pi)}_\vartheta\big]\, ds.
\end{aligned}$$

https://doi.org/10.4153/CJM-2005-009-2 Published online by Cambridge University Press

Now we are left to show that the right hand side of (2.7) goes to 0 as $\sigma \to 0+$. We first show that

$$\lim_{\sigma\to 0+}\int_0^t \mathbb{P}\left[\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_1(s))\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_2(s)),\ T_{12}^x > s\right] ds = 0, \tag{2.8}$$

where $T_{12}^x := \inf\{s \geq 0 : \hat{X}_1(s) = \hat{X}_2(s)\}$.

The heuristic is that $\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_1(s))\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_2(s))1_{\{T_{12}^x>s\}}$ is big only if $\hat{Y}_1(t-s)$, $\hat{X}_1(s)$ and $\hat{X}_2(s)$ are all close to each other. Due to the independence of $\hat{Y}_1$ and $\hat{X}_i$, and the restriction $T_{12}^x > s$, the probability of the latter event is small enough to make (2.8) hold.

We now proceed to carry out the above mentioned heuristic argument. Let $\phi$ denote the density of the standard normal distribution. Given $\epsilon > 0$ small enough, for $\sigma^\epsilon < s < t-\sigma$, by the reflection principle for Brownian motion we can show that

(2.9)

$$\begin{aligned}
&\mathbb{P}\Big[\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_1(s))\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_2(s)),\ T_{12}^x > s\Big]\\
&\quad\leq \frac{1}{2\sigma^2\pi}\mathbb{P}\left\{T_{12}^x > s, \hat{X}_2(s)-\hat{X}_1(s) < \sigma^{1-\epsilon}, |\hat{Y}_1(t-s)-\hat{X}_1(s)| < \sigma^{1-\epsilon}\right\}\\
&\qquad + \mathbb{P}\left[\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_1(s))\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_2(s)), \hat{X}_2(s)-\hat{X}_1(s) \geq \sigma^{1-\epsilon}\right]\\
&\qquad + \frac{1}{\sigma\sqrt{2\pi}}\mathbb{P}\left[\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_1(s)),\ |\hat{Y}_1(t-s)-\hat{X}_1(s)| \geq \sigma^{1-\epsilon}\right]\\
&\quad\leq \frac{\sigma^{1-\epsilon}}{\sigma^2\pi\sqrt{2\pi(t-s)}}\int_0^{\frac{\sigma^{1-\epsilon}}{\sqrt{2\sigma^\epsilon}}} \phi(x-(x_2-x_1)) - \phi(-x-(x_2-x_1))\,dx\\
&\qquad + \frac{1}{\sigma^2\sqrt{2\pi}}\phi(\frac{1}{2\sigma^\epsilon}) + \frac{1}{\sigma^2\sqrt{2\pi}}\phi(\frac{1}{\sigma^\epsilon})\\
&\quad\leq \frac{\sigma^{1-5\epsilon}(x_2-x_1)}{4\pi^2} + \frac{1}{\sigma^2\sqrt{2\pi}}\phi(\frac{1}{2\sigma^\epsilon}) + \frac{1}{\sigma^2\sqrt{2\pi}}\phi(\frac{1}{\sigma^\epsilon}).
\end{aligned}$$

For $0 < s \leq \sigma^\epsilon$, it follows that

$$\begin{aligned}
&\mathbb{P}\Big[\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_1(s))\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_2(s)),\ T_{12}^x > s\Big]\\
&\quad\leq \frac{1}{2\sigma^2\pi}\mathbb{P}\big\{\hat{X}_2(s)-\hat{X}_1(s) < 2\sigma^\epsilon, T_{12}^x > s\big\}\\
&\qquad + \frac{1}{\sigma^2\sqrt{2\pi}}\phi(\sigma^{\epsilon-1})\mathbb{P}\{\hat{X}_2(s)-\hat{X}_1(s) \geq 2\sigma^\epsilon\}\\
&\quad\leq \frac{1}{2\sigma^2\pi}\left(1-\Phi(\frac{x_2-x_1}{2\sqrt{2\sigma^\epsilon}})\right) + \frac{1}{\sigma^2\sqrt{2\pi}}\phi(\sigma^{\epsilon-1}),
\end{aligned} \tag{2.10}$$

https://doi.org/10.4153/CJM-2005-009-2 Published online by Cambridge University Press

where $\Phi$ is the normal distribution function. Moreover, for $t-\sigma \le s < t$,

$$\begin{aligned}
(2.11)\qquad & \mathbb{P}\left[\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_1(s))\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_2(s)),\ T_{12}^x > s\right] \\
&\le \frac{1}{2\pi\sigma^2}\mathbb{P}\left\{|\hat{Y}_1(t-s)-\hat{X}_1(s)| \le \sigma^{1-\epsilon}, |\hat{Y}_1(t-s)-\hat{X}_2(s)| \le \sigma^{1-\epsilon}\right\} \\
&\quad + \frac{1}{\sqrt{2\pi}\sigma}\mathbb{P}\left[\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_1(s)),\ |\hat{Y}_1(t-s)-\hat{X}_1(s)| > \sigma^{1-\epsilon}\right] \\
&\quad + \frac{1}{\sqrt{2\pi}\sigma}\mathbb{P}\left[\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_2(s)),\ |\hat{Y}_1(t-s)-\hat{X}_2(s)| > \sigma^{1-\epsilon}\right] \\
&\le \frac{1}{\pi^2\sigma^{2\epsilon}s} + \frac{1}{\pi\sigma^2}e^{-\frac{1}{2\sigma^\epsilon}}.
\end{aligned}$$

Combining (2.9), (2.10) and (2.11), (2.8) follows readily.

Similarly, we can also show that

$$\lim_{\sigma\to 0+}\int_0^t \mathbb{P}\left[\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_i(s))\phi_\sigma(\hat{Y}_1(t-s)-\hat{X}_j(s)),\ T_{ij}^x > s\right] ds = 0$$

and

$$\lim_{\sigma\to 0+}\int_0^t \mathbb{P}\left[\phi_\sigma(\hat{Y}_i(t-s)-\hat{X}_k(s))\phi_\sigma(\hat{Y}_j(t-s)-\hat{X}_k(s)),\ T_{ij}^y > t-s\right] ds = 0,$$

where $T_{ij}^y := \inf\{t \ge 0 : \hat{Y}_i(t) = \hat{Y}_j(t)\}$ for $1 \le i < j \le n$.

As a result,

$$\begin{aligned}
(2.12)\qquad & \int_0^t \mathbb{P}\left[G_{\theta\vartheta}(\hat{\mathbf{X}}(s);\hat{\mathbf{Y}}(t-s)),\ \hat{\mathbf{X}}(s)\in\hat{\mathbb{R}}_\pi^n, \hat{\mathbf{Y}}(t-s)\in\hat{\mathbb{R}}_\vartheta^{2l(\pi)}\right] ds \\
&\le \int_0^t \mathbb{P}\left[S_{\theta\vartheta}^1(\hat{\mathbf{X}}(s);\hat{\mathbf{Y}}(t-s))\right] ds + \int_0^t \mathbb{P}\left[S_{\theta\vartheta}^2(\hat{\mathbf{X}}(s);\hat{\mathbf{Y}}(t-s))\right] ds \\
&\quad + \int_0^t \mathbb{P}\left[S_{\theta\vartheta}^3(\hat{\mathbf{X}}(s);\hat{\mathbf{Y}}(t-s))\right] ds + \int_0^t \mathbb{P}\left[S_{\theta\vartheta}^4(\hat{\mathbf{X}}(s);\hat{\mathbf{Y}}(t-s))\right] ds \\
&\longrightarrow 0 \text{ as } \sigma \to 0+.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\Big\{\bigcap_{j=1}^{l(\pi)}\bigcap_{i\in A_j}\{\hat{X}_i(t)\in(y_{2j-1},y_{2j})\}\Big\} \\
&\qquad = \lim_{\sigma\to 0+}\sum_{\theta\in\mathcal{P}_n}\mathbb{P}\left[f_{\sigma\pi}(\hat{\mathbf{X}}(t);\mathbf{y}),\ \hat{\mathbf{X}}(t)\in\hat{\mathbb{R}}_\theta^n\right] \\
&\qquad = \lim_{\sigma\to 0+}\sum_{\vartheta\in\mathcal{P}^*_{2l(\pi)}}\mathbb{P}\left[f_{\sigma\pi}(\mathbf{x};\hat{\mathbf{Y}}(t)),\ \hat{\mathbf{Y}}(t)\in\hat{\mathbb{R}}_\vartheta^{2l(\pi)}\right]
\end{aligned}$$

https://doi.org/10.4153/CJM-2005-009-2 Published online by Cambridge University Press

$$= \mathbb{P}\Big\{\bigcap_{j=1}^{l(\pi)}\bigcap_{i\in A_j}\{x_i \in (\hat{Y}_{2j-1}(t), \hat{Y}_{2j}(t))\}, T_{\mathbf{Y}} > t\Big\}$$

$$= \mathbb{P}\Big\{\bigcap_{j=1}^{l(\pi)}\bigcap_{i\in A_j}\{x_i \in (\hat{Y}_{2j-1}(t), \hat{Y}_{2j}(t))\}\Big\},$$

where $T_{\mathbf{Y}} := \inf\{t > 0 : \hat{Y}_{2i-1}(t) = \hat{Y}_{2i}(t), 1 \le i \le l(\pi)\}$. Duality (1.1) then follows. The proof of Theorem 1.1 is thus finished. ■

***Remark 2.3*** One can easily see that (1.1) also holds when $y_1 \le y_2 \le \cdots \le y_{2m}$. Moreover, the following generalized version of Theorem 1.1 also holds.

***Corollary 2.4*** *Given $\pi = \{A_1, \dots, A_{l(\pi)}\} \in \mathcal{P}_n$ and $\mathbf{y} = (y_1, \dots, y_{2l(\pi)})$ such that $y_1 \le y_2 \le \cdots \le y_{2l(\pi)}$, let $\hat{\mathbf{X}}$ be a n-dimensional coalescing Brownian motion with initial distribution $\mu \in M_1(\mathbb{R}^n)$. Let $\hat{\mathbf{Y}}$ be a $2l(\pi)$-dimensional coalescing Brownian motion with $\hat{\mathbf{Y}}(0) = (y_1, \dots, y_{2l(\pi)})$. Then we have*

$$\int \mu(d\mathbf{x})\mathbb{P}\Big\{\bigcap_{j=1}^{l(\pi)}\bigcap_{i\in A_j}\{\hat{X}_i(t) \in (y_{2j-1}, y_{2j})\}\Big\} = \int \mu(d\mathbf{x})\mathbb{P}\Big\{\bigcap_{j=1}^{l(\pi)}\bigcap_{i\in A_j}\{x_i \in (\hat{Y}_{2j-1}(t), \hat{Y}_{2j}(t))\}\Big\}. \tag{2.13}$$

## 3 A Measure-Valued Process

The duality obtained in the previous section can be applied to study a measure-valued Markov process which describes the evolution of the following particle system: particles with possibly different masses evolve according to coalescing Brownian motions. Their respective masses are added together whenever two particles merge into one particle.

In this section we are going to show that the above mentioned model arises from a high density limit of the empirical measures of coalescing Brownian motions. It was expected in [3] that a more complex measure-valued diffusion process (superprocess with coalescing spatial motion) should arise as a high density limit of coalescing-branching particle systems; also see [2]. The superprocess with coalescing spatial motion has been constructed and characterized in [3] without consideration of the particle systems.

$M_1(\mathbb{R})$ denotes the space of probability measures on $\mathbb{R}$ equipped with the topology of weak convergence. As usual, we write $C(M_1(\mathbb{R}))$ as the space of bounded continuous $M_1(\mathbb{R})$-valued functions, and write $D(M_1(\mathbb{R}))$ as the space of cadlag $M_1(\mathbb{R})$-valued functions.

***Definition 3.1*** A collection of processes $\{Z_\alpha, \alpha \in I\}$ with paths in $D(M_1(\mathbb{R}))$ is C-relatively compact in $D(M_1(\mathbb{R}))$ iff it is relatively compact and all weak limit points are continuous a.s..

Let $Z_0$ be a probability measure on $\mathbb{R}$. $(\hat{X}_1, \dots, \hat{X}_n)$ is an $n$-dimensional coalescing Brownian motion such that $\hat{X}_1(0), \dots, \hat{X}_n(0)$ are i.i.d. samples from $Z_0$. Let $Z_t^n := \frac{1}{n}\sum_{i=1}^n \delta_{\{\hat{X}_i(t)\}}$ be the empirical measure of $(\hat{X}_1, \dots, \hat{X}_n)$.

***Lemma 3.2*** *$\{Z_n\}$ is C-relatively compact in $D(M_1(\mathbb{R}))$.*

**Proof** This proof is a modification of the proof of proposition 2.4.2 in [12]. For any $\epsilon > 0$ and $T > 0$, choose a compact set $K_0 \subset D(\mathbb{R})$ such that $\mathbb{P}^{Z_0}\{X \in K_0^c\} < \epsilon^2$. Let $K := \{x_t : t \le T, x \in K_0\}$. Then $K$ is compact in $\mathbb{R}$ and $\mathbb{P}^{Z_0}\{X_t \in K^c, \exists t \le T\} \le \mathbb{P}^{Z_0}\{X \in K_0^c\} < \epsilon^2$. Let $R_n(t) := \frac{1}{n}\sum_{i=1}^n \sup_{0\le s\le t} 1_{K^c}(\hat{X}_i(s))$. Since $R_n(t)$ is increasing in $t$, we then have

$$\begin{aligned}\mathbb{P}\{\sup_{0\le t\le T} Z_t^n(K^c) > \epsilon\} &\le \mathbb{P}\{\sup_{0\le t\le T} R_n(t) > \epsilon\}\\ &\le \frac{1}{\epsilon}\mathbb{P}[R_n(T)]\\ &= \frac{1}{\epsilon}\mathbb{P}\{X(t) \in K^c, \exists t \le T\}\\ &< \epsilon.\end{aligned}$$

Now we are going to show that, for any $f \in C_b^2$, $\{Z_\cdot^n(f)\}$ is C-relatively compact in $D(\mathbb{R}_+, \mathbb{R})$. By Ito's formula,

$$Z_t^n(f) = \frac{1}{n}\sum_{i=1}^n \Big[ f(\hat{X}_i(0)) + \int_0^t f'(\hat{X}_i(s))\, d\hat{X}_i(s) + \int_0^t \frac{1}{2} f''(\hat{X}_i(s))\, ds\Big].$$

$\frac{1}{n}\sum_{i=1}^n \int_0^t \frac{1}{2} f''(\hat{X}_i(s))\, ds$ is C-relatively compact following from the Arzela-Ascoli theorem and Proposition 6.3.26 of [9]. Since

$$\Big\langle \frac{1}{n}\sum_{i=1}^n \int_0^\cdot f'(\hat{X}_i(s))\, d\hat{X}_i(s)\Big\rangle_t = \frac{1}{n^2}\sum_{i,j=1}^n \int_0^t f'(\hat{X}_i(s))^2\, d\langle \hat{X}_i, \hat{X}_j\rangle_s,$$

where $\langle \hat{X}_i, \hat{X}_j\rangle_s = s - T_{ij} \wedge s$, by the Arzela-Ascoli theorem,

$$\Big\{\Big\langle \frac{1}{n}\sum_{i=1}^n \int_0^\cdot f'(\hat{X}_i(s))\, d\hat{X}_i(s)\Big\rangle_t\Big\}$$

is C-relatively compact. Theorem 6.4.13 and Proposition 6.3.26 in [9] then imply that the collection of martingales $\{\frac{1}{n}\sum_{i=1}^n \int_0^\cdot f'(\hat{X}_i(s))\, d\hat{X}_i(s)\}$ is C-relatively compact. Also notice that $\frac{1}{n}\sum_{i=1}^n f(\hat{X}_i(0)) \to Z_0(f)$ a.s. $\{Z_n(f)\}$ is thus C-relatively compact. So by Theorem 2.4.1 in [12] (with $D(M_F(\mathbb{R}))$ replaced by $D(M_1(\mathbb{R}))$ ), we can conclude that $\{Z_n\}$ is C-relatively compact. ∎

Adjoin $\infty$ to $\mathbb{R}$ to make $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ a compact space. Each $M_1(\mathbb{R})^n$-valued process can also be treated as a $M_1(\bar{\mathbb{R}})^n$-valued process. The next theorem characterizes the weak limit of $(Z^n)$ by specifying all its "joint moments".

**Theorem 3.3** *$\{Z^n\}$ has a unique $C(M_1(\mathbb{R}))$-valued weak limit $Z$. $Z$ is a Markov process. It satisfies that for any $a_1 \le \cdots \le a_{2m}$, and any $r_k \in \mathbb{N}_+, k = 1, \dots, m$,*

$$\mathbb{P}\Big[\prod_{k=1}^{m} Z_t([a_{2k-1}, a_{2k}])^{r_k}\Big] = \mathbb{P}\Big[\prod_{k=1}^{m} Z_0([\hat{Y}_{2k-1}(t), \hat{Y}_{2k}(t)])^{r_k}\Big], \tag{3.1}$$

*where $(\hat{Y}_1, \dots, \hat{Y}_{2m})$ is a $2m$-dimensional coalescing Brownian motion starting at $(a_1, \dots, a_{2m})$.*

**Proof** Let $Z$ be any fixed limit point of $\{Z^n\}$, *i.e.*, $Z^{n_q} \xrightarrow{D} Z$ in $D(M_1(\mathbb{R}))$ as $q \to \infty$ for a subsequence $\{n_q\}$. By Theorem 1.1, there is a $2m$-dimensional coalescing Brownian $\hat{\mathbf{Y}}$ independent of $\{\hat{X}_n\}$ such that $\hat{\mathbf{Y}}(0) = (a_1, \dots, a_{2m})$ and for all $n \in \mathbb{N}_+, t > 0$,

$$\begin{aligned}
&\mathbb{P}\Big[\prod_{k=1}^{m} Z_t^n([a_{2k-1}, a_{2k}])^{r_k}\Big] \\
&= \mathbb{P}\Big[\prod_{k=1}^{m}\Big(\frac{1}{n}\sum_{i=1}^{n} 1_{\{\hat{X}_i(t)\in[a_{2k-1},a_{2k}]\}}\Big)^{r_k}\Big] \\
&= \Big(\prod_{k=1}^{m}\frac{1}{n^{r_k}}\Big) \sum_{i_{11},\dots,i_{1r_1},\dots,i_{m1},\dots,i_{mr_m}=1}^{n} \mathbb{P}\Big[\prod_{k=1}^{m}\prod_{j=1}^{r_k} 1_{\{\hat{X}_{i_{kj}}(t)\in[a_{2k-1},a_{2k}]\}}\Big] \\
&= \Big(\prod_{k=1}^{m}\frac{1}{n^{r_k}}\Big) \sum_{i_{11},\dots,i_{1r_1},\dots,i_{m1},\dots,i_{mr_m}=1}^{n} \mathbb{P}\Big[\prod_{k=1}^{m}\prod_{j=1}^{r_k} 1_{\{\hat{X}_{i_{kj}}(0)\in[\hat{Y}_{2k-1}(t),\hat{Y}_{2k}(t)]\}}\Big] \\
&= \mathbb{P}\Big[\prod_{k=1}^{m}\Big(\frac{1}{n}\sum_{i=1}^{n} 1_{\{\hat{X}_i(0)\in[\hat{Y}_{2k-1}(t),\hat{Y}_{2k}(t)]\}}\Big)^{r_k}\Big] \\
&= \mathbb{P}\Big[\prod_{k=1}^{m} Z_0^n([\hat{Y}_{2k-1}(t), \hat{Y}_{2k}(t)])^{r_k}\Big].
\end{aligned} \tag{3.2}$$

Since $\mathbb{P}[Z_t^n([a-\epsilon, a+\epsilon])] = \mathbb{P}[Z_0^n([\hat{Y}^{a-\epsilon}(t), \hat{Y}^{a+\epsilon}(t)])] \le \mathbb{P}\{T_\epsilon > t\}$, where $(\hat{Y}^{a-\epsilon}(t), \hat{Y}^{a+\epsilon}(t))$ is a coalescing Brownian motion with initial value $(a-\epsilon, a+\epsilon)$ and $T_\epsilon := \inf\{t \ge 0 : \hat{Y}^{a-\epsilon}(t) = \hat{Y}^{a+\epsilon}(t)\}$, then uniformly in $n$,

$$\mathbb{P}\Big[\prod_{k=1}^{m} Z_t^n([a_{2k-1}-\epsilon, a_{2k}+\epsilon])^{r_k}\Big] - \mathbb{P}\Big[\prod_{k=1}^{m} Z_t^n([a_{2k-1}, a_{2k}])^{r_k}\Big] \to 0$$

as $\epsilon \to 0+$. Let $n = n_q \to \infty$ and we can obtain (3.1).

Now we are going to show that

$$(Z_{t_1}^n, \dots, Z_{t_m}^n) \xrightarrow{D} (Z_{t_1}, \dots, Z_{t_m}),\ m \in \mathbb{N}_+, 0 \le t_1 < \cdots < t_m, \tag{3.3}$$

https://doi.org/10.4153/CJM-2005-009-2 Published online by Cambridge University Press

from which we will conclude that $Z$ is the unique limit of $\{Z^n\}$.

Observe that the algebra generated by $\{\otimes_{i=1}^m f_i : f_i \in C(\bar{\mathbb{R}})\}$ strongly separates points (see section 3.4 of [6] for the definition). By Theorem 3.4.5 in [6] and standard arguments we only need to show that

(3.4)
$$\lim_{n\to\infty} \mathbb{P}\Big[\prod_{i=1}^m Z_{t_i}^n([a_{2i-1}, a_{2i}])\Big] = \mathbb{P}\Big[\prod_{i=1}^m Z_{t_i}([a_{2i-1}, a_{2i}])\Big], a_{2i-1} < a_{2i}, i = 1, \dots, m.$$

Instead of showing (3.4), we are going to show the apparently stronger result below by induction on $m$.

(3.5)
$$\lim_{n\to\infty} \mathbb{P}\Big[\prod_{i=1}^m Z_{t_i}^n([a_{2i-1}, a_{2i}])^{r_i}\Big] = \mathbb{P}\Big[\prod_{i=1}^m Z_{t_i}([a_{2i-1}, a_{2i}])^{r_i}\Big], r_i \in \mathbb{N}_+, i = 1, \dots, m.$$

The case with $m = 1$ just follows from our proof of (3.1). Suppose that (3.5) hold for $m \le j$. We want to show that it also holds for $m = j + 1$. By the Markov property of $\hat{\mathbf{X}}^n := (\hat{X}_1, \dots, \hat{X}_n)$ and (3.1), we have

$$\begin{aligned}
&\mathbb{P}\Big[\prod_{i=1}^{j+1} Z_{t_i}^n([a_{2i-1}, a_{2i}])^{r_i}\Big] \\
&\qquad = \mathbb{P}\Big[\mathbb{P}\Big[Z_{t_{j+1}-t_j}^n([a_{2j+1}, a_{2j+2}])^{r_{j+1}} | \hat{\mathbf{X}}_{t_j}^n\Big] \prod_{i=1}^{j} Z_{t_i}^n([a_{2i-1}, a_{2i}])^{r_i}\Big] \\
&\qquad = \mathbb{P}\Big[Z_{t_j}^n([\hat{Y}_1(t_{j+1} - t_j), \hat{Y}_2(t_{j+1} - t_j)])^{r_{j+1}} \prod_{i=1}^{j} Z_{t_i}^n([a_{2i-1}, a_{2i}])^{r_i}\Big],
\end{aligned}$$

where $(\hat{Y}_1, \hat{Y}_2)$is a 2-dimensional coalescing Brownian motion starting at $(a_{2j+1}, a_{2j+2})$ and independent of $\{Z^n\}$.

Let $n \to \infty$; by the inductive assumption and the fact that

$$\mathbb{P}[1_{[\hat{Y}_1(t_{j+1}-t_j), \hat{Y}_2(t_{j+1}-t_j)]}(x)]$$

is a continuous function of $x$, through standard arguments we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\Big[\prod_{i=1}^{j+1} Z_{t_i}^n([a_{2i-1}, a_{2i}])^{r_i}\Big] \\
&\qquad = \mathbb{P}\Big[Z_{t_j}([\hat{Y}_1(t_{j+1} - t_j), \hat{Y}_2(t_{j+1} - t_j)])^{r_{j+1}} \prod_{i=1}^{j} Z_{t_i}([a_{2i-1}, a_{2i}])^{r_i}\Big] \\
&\qquad = \lim_{q\to\infty} \mathbb{P}\Big[Z_{t_j}^{n_q}([\hat{Y}_1(t_{j+1} - t_j), \hat{Y}_2(t_{j+1} - t_j)])^{r_{j+1}} \prod_{i=1}^{j} Z_{t_i}^{n_q}([a_{2i-1}, a_{2i}])^{r_i}\Big]
\end{aligned}$$

$$= \lim_{q\to\infty} \mathbb{P}\Big[\prod_{i=1}^{j+1} Z_{t_i}^{n_q}([a_{2i-1}, a_{2i}])^{r_i}\Big]$$

$$= \mathbb{P}\Big[\prod_{i=1}^{j+1} Z_{t_i}([a_{2i-1}, a_{2i}])^{r_i}\Big].$$

Hence, (3.5) holds for $m = j + 1$.

We are left to show that $Z$ has the Markov property. For any $0 \le s_1 < \cdots < s_k \le s < t$, any $k \in \mathbb{N}_+, m_i \in \mathbb{N}_+, i = 1, \ldots, k+1$, any $a_{2i-1,1} < a_{2i,1} < a_{2i-1,2} < a_{2i,2} < \cdots < a_{2i-1,m_i} < a_{2i,m_i}, i = 1, \ldots, k+1$ and any $r_{i,j} \in \mathbb{N}_+, j = 1, \ldots, m_i, i = 1, \ldots, k+1$, it follows from the Markov property of $\hat{\mathbf{X}}^n$ and the coalescing duality (1.1) that

$$\mathbb{P}\Big[\prod_{j=1}^{m_{k+1}} Z_t([a_{2k+1,j}, a_{2k+2,j}])^{r_{k+1,j}} \prod_{i=1}^{k}\prod_{j=1}^{m_i} Z_{s_i}([a_{2i-1,j}, a_{2i,j}])^{r_{i,j}}\Big]$$

$$= \lim_{n\to\infty} \mathbb{P}\Big[\prod_{j=1}^{m_{k+1}} Z_t^n([a_{2k+1,j}, a_{2k+2,j}])^{r_{k+1,j}} \prod_{i=1}^{k}\prod_{j=1}^{m_i} Z_{s_i}^n([a_{2i-1,j}, a_{2i,j}])^{r_{i,j}}\Big]$$

$$= \lim_{n\to\infty} \mathbb{P}\Big[\mathbb{P}\Big[\prod_{j=1}^{m_{k+1}} Z_t^n([a_{2k+1,j}, a_{2k+2,j}])^{r_{k+1,j}} |\hat{\mathbf{X}}_s^n\Big] \prod_{i=1}^{k}\prod_{j=1}^{m_i} Z_{s_i}^n([a_{2i-1,j}, a_{2i,j}])^{r_{i,j}}\Big]$$

$$= \lim_{n\to\infty} \mathbb{P}\Big[\prod_{j=1}^{m_{k+1}} Z_s^n([\hat{Y}_{2j-1}(t-s), \hat{Y}_{2j}(t-s)])^{r_{k+1,j}} \prod_{i=1}^{k}\prod_{j=1}^{m_i} Z_{s_i}^n([a_{2i-1,j}, a_{2i,j}])^{r_{i,j}}\Big]$$

$$= \mathbb{P}\Big[\prod_{j=1}^{m_{k+1}} Z_s([\hat{Y}_{2j-1}(t-s), \hat{Y}_{2j}(t-s)])^{r_{k+1,j}} \prod_{i=1}^{k}\prod_{j=1}^{m_i} Z_{s_i}([a_{2i-1,j}, a_{2i,j}])^{r_{i,j}}\Big],$$

where $(\hat{Y}_1, \ldots, \hat{Y}_{2m_{k+1}})$ is a two-dimensional coalescing Brownian motion starting from $(a_{2k+1,1}, a_{2k+2,1}, \ldots, a_{2k+1,m_{k+1}}, a_{2k+2,m_{k+1}})$. By the Stone-Weierstrass theorem and standard arguments again, we have, for any $g \in C(M_1(\bar{\mathbb{R}}))$,

$$\mathbb{P}[g(Z_t)|Z_{s'}, s' \le s] = g^*(Z_s),$$

where $g^*$ is a function on $M_1(\bar{\mathbb{R}})$). $Z$ is thus a Markov process. ■

***Remark 3.4*** Notice that $Z$ never charges $\{\infty\}$. But we need the one-point compactification to be able to apply the Stone-Weierstrass theorem.

Since

$$M_t^n(f) = Z_t^n(f) - Z_0^n(f) - \int_0^t Z_s^n(\frac{1}{2}f'')\,ds, \;\; f \in C_b^2,$$

is a martingale with quadratic variation process

$$\begin{aligned}\langle M^n(f)\rangle_t &= \frac{1}{n^2}\sum_{i,j=1}^n \int_{\tau_{ij}\wedge t}^t f'(\hat{X}_i(s))f'(\hat{X}_j(s))\,ds\\ &= \int_0^t ds\int_\Delta f'(x)f'(y)Z_s^n(dx)Z_s^n(dy),\end{aligned}$$

where $\Delta := \{(x,y)\in\mathbb{R}^2 : x = y\}$. It is not surprising that $Z$ solves the following martingale problem.

***Theorem 3.5*** *The process $Z$ in Theorem 3.3 solves the following martingale problem:*

$$M_t(f) = Z_t(f) - Z_0(f) - \int_0^t Z_s(\frac{1}{2}f'')\,ds,\ f\in C_b^2, \tag{3.6}$$

*is a martingale with quadratic variation process*

$$\langle M(f)\rangle_t = \int_0^t ds\int_\Delta f'(x)f'(y)Z_s(dx)Z_s(dy). \tag{3.7}$$

**Proof** Let $T_{12}$ be the first time that $\frac{\hat{X}_1(t)-\hat{X}_2(t)}{\sqrt{2}}$ hits 0. Before $T_{12}$, $\frac{\hat{X}_1(t)-\hat{X}_2(t)}{\sqrt{2}}$ and $\frac{\hat{X}_1(t)+\hat{X}_2(t)}{\sqrt{2}}$ are independent Brownian motions.Therefore,

$$\begin{aligned}&\mathbb{P}[f'(\hat{X}_1(s))f'(\hat{X}_2(s))1_{\{T_{12}<s\}}]\\ &\quad= \int_0^s dF_{\frac{x_1-x_2}{\sqrt{2}}}(r)\int_\mathbb{R} dx p_r(x-\frac{x_1+x_2}{\sqrt{2}})\int_\mathbb{R} dy f'(y)^2 p_{s-r}(y-\frac{x}{\sqrt{2}})\\ &\quad= \int_0^s dF_{\frac{x_1-x_2}{\sqrt{2}}}(r)\int_\mathbb{R} dx p_r(x-\frac{x_1+x_2}{\sqrt{2}})\int_\mathbb{R} dy f'(y)^2\sqrt{2}p_{2s-2r}(\sqrt{2}y-x)\\ &\quad= \int_0^s dF_{\frac{x_1-x_2}{\sqrt{2}}}(r)\int_\mathbb{R} dy f'(y)^2\sqrt{2}p_{2s-r}(\sqrt{2}y-\frac{x_1+x_2}{\sqrt{2}})\\ &\quad= \int_0^s dF_{\frac{x_1-x_2}{\sqrt{2}}}(r)\int_\mathbb{R} dy f'(y)^2 p_{s-r/2}(y-\frac{x_1+x_2}{2}),\end{aligned}$$

where $F_x$ is the distribution of the hitting time of 0 by a Brownian motion starting at $x$, *i.e.*,

$$\frac{F_x(dr)}{dr} = \frac{|x|}{\sqrt{2\pi r^3}}e^{-\frac{x^2}{2r}}.$$

Hence,

$$\begin{aligned}&\mathbb{P}[\langle M^n(f)\rangle_t - \langle M^n(f)\rangle_s|\mathcal{F}_s]\\ &\quad= \mathbb{P}[\langle M^n(f)\rangle_t - \langle M^n(f)\rangle_s|Z_s^n]\\ &\quad= \int_{\mathbb{R}^2} Z_s^n(dx_1)Z_s^n(dx_2)\int_0^{t-s} dl\int_0^l dF_{\frac{x_1-x_2}{\sqrt{2}}}(r)\int_\mathbb{R} dy f'(y)^2 p_{l-\frac{r}{2}}(y-\frac{x_1+x_2}{2}).\end{aligned}$$

https://doi.org/10.4153/CJM-2005-009-2 Published online by Cambridge University Press

Similar to the proof of the tightness of $Z^n$, we can prove that $\{\langle M^n(f)\rangle\}$ is tight in $C([0,\infty),\mathbb{R})$. We assume that $(Z^n,\langle M^n(f)\rangle)\to(Z,\Lambda)$. Further, it is easy to show that for fixed $t>0$ and $f\in C_b^2(\mathbb{R})$, $Z_t^n(f)^2$ and $\langle M^n(f)\rangle_t$ are uniformly integrable.

Then it is easy to see that $M_t(f)$ is a square integrable martingale with quadratic variation process $\Lambda(t)$. Then

$$
\begin{aligned}
&\mathbb{P}[\Lambda(t)-\Lambda(s)|\mathcal{F}_s]\\
&\quad=\lim_{n\to\infty}\mathbb{P}[\langle M^n(f)\rangle_t-\langle M^n(f)\rangle_s|\mathcal{F}_s]\\
&\quad=\lim_{n\to\infty}\int_{\mathbb{R}^2}Z_s^n(dx_1)Z_s^n(dx_2)\int_0^{t-s}dl\int_0^l dF_{\frac{x_1-x_2}{\sqrt{2}}}(r)\int_{\mathbb{R}}dyf'(y)^2p_{s-\frac{r}{2}}(y-\frac{x_1+x_2}{2})\\
&\quad=\int_{\mathbb{R}^2}Z_s(dx_1)Z_s(dx_2)\int_0^{t-s}dl\int_0^l dF_{\frac{x_1-x_2}{\sqrt{2}}}(r)\int_{\mathbb{R}}dyf'(y)^2p_{l-\frac{r}{2}}(y-\frac{x_1+x_2}{2}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(3.8)\qquad &\int_0^t ds\frac{1}{\delta}\mathbb{P}[\Lambda(s+\delta)-\Lambda(s)|\mathcal{F}_s]\\
&=\int_0^t ds\frac{1}{\delta}\int_0^\delta dl\int_{\mathbb{R}^2}Z_s(dx_1)Z_s(dx_2)\int_0^l dF_{\frac{x_1-x_2}{\sqrt{2}}}(r)\int_{\mathbb{R}}dyf'(y)^2p_{l-\frac{r}{2}}(y-\frac{x_1+x_2}{2}).
\end{aligned}
$$

By Theorem 37 of ([11, p. 126]) and the continuity of $\Lambda$, the right hand side of (3.8) converges to $\Lambda(t)$ as $\delta\to 0$. On the other hand, the other side of (3.8) is

$$
\begin{aligned}
&\lim_{\delta\to 0}\int_0^t ds\frac{1}{\delta}\int_0^\delta dl\int_{\mathbb{R}^2}f'(\frac{x_1+x_2}{2})^2F_{\frac{x_1-x_2}{\sqrt{2}}}([0,l])Z_s(dx_1)Z_s(dx_2)\\
&\qquad=\int_0^t ds\int_\Delta f'(\frac{x_1+x_2}{2})^2Z_s(dx_1)Z_s(dx_2)\\
&\qquad\quad+\lim_{\delta\to 0}\int_0^t ds\frac{1}{\delta}\int_0^\delta dl\int_{\mathbb{R}^2\setminus\Delta}f'(\frac{x_1+x_2}{2})^2F_{\frac{x_1-x_2}{\sqrt{2}}}([0,l])Z_s(dx_1)Z_s(dx_2)
\end{aligned}
$$

while the second term is bounded by

$$
\lim_{\delta\to 0}\int_0^t ds\int_{\mathbb{R}^2\setminus\Delta}f'(\frac{x_1+x_2}{2})^2F_{\frac{x_1-x_2}{\sqrt{2}}}([0,\delta])Z_s(dx_1)Z_s(dx_2)=0.
$$

Therefore, $\Lambda(t)$ equals the right hand side of (3.7) and hence, $Z$ solves the martingale problem (3.6)–(3.7). ■

***Remark 3.6*** The solution to the martingale problem (3.6)–(3.7) is not unique. For example, if the initial measure is $\frac{1}{2}(\delta_{\{0\}}+\delta_{\{1\}})$, then for a two-dimensional coalescing Brownian motion $(\hat{X}_t^1,\hat{X}_t^2)$ starting at $(0,1)$, the process $\frac{1}{2}(\delta_{\{\hat{X}_t^1\}}+\delta_{\{\hat{X}_t^2\}})$ is a solution. On the other hand, $\frac{1}{2}(\delta_{\{B_t^1\}}+\delta_{\{B_t^2\}})$ is also a solution where $(B_t^1,B_t^2)$ is a standard two-dimensional Brownian motion starting at $(0,1)$. The martingale problem does not appear to be the best way to characterize the desired model.

Let $S_t$ be the support of $Z_t$. The coalescing duality (1.1) can be used to derive another seemingly different duality.

***Proposition 3.7*** *Given $y_1 < \cdots < y_{2m}$ and a $2m$-dimensional coalescing Brownian motion $\hat{\mathbf{Y}} = (\hat{Y}_1, \ldots, \hat{Y}_{2m})$ with initial value $(y_1, \ldots, y_{2m})$, then*

(3.9)
$$\mathbb{P}\Big\{\bigcap_{i=1}^{n}\{\hat{X}_i(t) \in \bigcup_{j=1}^{m}[y_{2j-1}, y_{2j}]\}\Big\} = \mathbb{P}\Big\{\bigcap_{i=1}^{n}\{\hat{X}_i(0) \in \bigcup_{j=1}^{m}[\hat{Y}_{2j-1}(t), \hat{Y}_{2j}(t)]\}\Big\}.$$

*Consequently,*

$$\mathbb{P}\Big\{S_t \subset \bigcup_{j=1}^{m}[y_{2j-1}, y_{2j}]\Big\} = \mathbb{P}\Big\{S_0 \subset \bigcup_{j=1}^{m}[\hat{Y}_{2j-1}(t), \hat{Y}_{2j}(t)]\Big\}. \tag{3.10}$$

**Proof** By (3.2) we can show that

$$\mathbb{P}\Big[\Big(\sum_{j=1}^{m} Z_t^n([y_{2j-1}, y_{2j}])\Big)^k\Big] = \mathbb{P}\Big[\Big(\sum_{j=1}^{m} Z_0^n([\hat{Y}_{2j-1}(t), \hat{Y}_{2j}(t)])\Big)^k\Big].$$

Let $k \to \infty$; we have

$$\mathbb{P}\Big\{Z_t^n\Big(\bigcup_{j=1}^{m}[y_{2j-1}, y_{2j}]\Big) = 1\Big\} = \mathbb{P}\Big\{Z_0^n(\bigcup_{i=1}^{m}[\hat{Y}_{2j-1}(t), \hat{Y}_{2j}(t)]) = 1\Big\}.$$

Then (3.9) follows, and (3.10) follows by letting $n \to \infty$ in (3.9). ∎

***Remark 3.8*** It is not hard to see that (3.9) is essentially the continuous space counterpart of the duality in the one-dimensional nearest neighbor voter model.

The coalescing duality can also be used to show that $S_t$ is discrete for $t > 0$. Write $|A|$ for the cardinality of set $A \subset \mathbb{R}$.

***Proposition 3.9*** *For each $t > 0$, $\mathbb{P}\big[\big|S_t \cap [-M, M]\big|\big] \le \frac{2M}{\sqrt{\pi t}}$, $M > 0$.*

**Proof** By (3.10) and the reflection principle of Brownian motion,

$$\begin{aligned} \mathbb{P}\{S_t \cap (a,b) \neq \varnothing\} &= 1 - \mathbb{P}\{S_t \subset (-\infty, a] \cup [b, \infty)\} \\ &\le 1 - \mathbb{P}\{T_{a,b} < t\} \\ &= \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-\left(x - \frac{b-a}{\sqrt{2}}\right)^2/2t} - e^{-\left(x + \frac{b-a}{\sqrt{2}}\right)^2/2t}\, dx, \end{aligned} \tag{3.11}$$

https://doi.org/10.4153/CJM-2005-009-2 Published online by Cambridge University Press

where $T_{a,b}$ is the time when two independent Brownian motions starting at $a$ and $b$ respectively first meet. Therefore, for any $M > 0$,

(3.12)

$$\begin{aligned}\mathbb{P}\big[\,\big|S_t \cap [-M, M]\big|\,\big] &= \lim_{n\to\infty} \mathbb{P}\Big[\sum_{i=-n}^{n-1} 1_{\{S_t \cap (\frac{iM}{n}, \frac{(i+1)M}{n}) \neq \varnothing\}}\Big] \\ &\leq \lim_{n\to\infty} \sum_{i=-n}^{n-1} \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-\left(x - \frac{M}{\sqrt{2}n}\right)^2/2t} - e^{-\left(x + \frac{M}{\sqrt{2}n}\right)^2/2t}\, dx \\ &= \frac{2M}{\sqrt{\pi t}}.\end{aligned}$$

■

***Remark 3.10*** The bound in Proposition 3.9 is rather crude since it holds for processes with any initial value $S_0$. A sharper result can be obtained if the configuration of $S_0$ is known. One would certainly expect that, almost surely, $S_t$ is finite over any compact set for all $t > 0$. But we do not have a proof yet.

**Acknowledgment**: The authors are grateful to Donald Dawson for raising a question which resulted in the coalescing duality in this paper. We thank Zenghu Li for some useful discussions. We also thank an anonymous referee for helpful suggestions.

## References


[1] R. A. Arratia, *Coalescing Brownian motions on the line.* PhD thesis, University of Wisconsin-Madison, 1979.

[2] D. A. Dawson, Z. H. Li and H. Wang, *Superprocesses with dependent spatial motion and general branching densities.* Electron. J. Probab. **6**(2001), 1–33.

[3] D. A. Dawson, Z. H. Li and X. Zhou, *Superprocesses with coalescing Brownian spatial motion as large scale limits,* to appear in J. Theo. Probab.

[4] D. A. Dawson and E. Perkins, *Measure-valued processes and renormalization of branching particle systems.* In: Stochastic Partial Differential Equations: Six Perspectives (R. A. Carmona and B. Rozovskii, eds.), Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1999, pp. 45–106.

[5] P. Donnelly, S. N. Evans, K. Fleischmann, T. G. Kurtz and X. Zhou, *Continuum-sites stepping-stone models, coalescing exchangeable partitions, and random trees.* Ann. Probab. **28**(2000), 1063–1110.

[6] S. N. Ethier and T. G. Kurtz, *Markov Processes: Characterization and Convergence*. Wiley, New York, 1986.

[7] S. N. Evans, *Coalescing Markov labeled partitions and a continuous sites genetics model with infinitely many types.* Ann. Inst. H. Poincaré Probab. Statist. **33**(1997), 339–358.

[8] T. E. Harris, *Coalescing and noncoalescing stochastic flows in* $\mathbb{R}^1$. Stoch. Process. Appl. **17**(1984), 187–210.

[9] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes.* Springer-Verlag, Berlin, 1987.

[10] T. M. Liggett, *Interacting Particle Systems*. Springer-Verlag, New York, 1985.

[11] P. A. Meyer, *Probability and Potentials*. Blaisdell Publishing Company, Toronto, 1966.

[12] E. Perkins, *Dawson-Watanabe Superprocesses and Measure-valued Diffusions.* Lectures on probability theory and statistics, Lecture Notes in Math., 1781, Springer-Verlag, Berlin, 2002, pp. 125–324.

[13] F. Soucaliuc, B. Tóth and W. Werner, *Reflection and coalescence between independent one-dimensional Brownian paths.* Ann. Inst. H. Poincaré Probab. Statist. **36**(2000) 509–545 .

*Department of Mathematical and Statistical Sciences*
*University of Alberta*
*Edmonton, Alberta*
*T6G 2G1*
*and*
*Department of Mathematics*
*University of Tennessee*
*Knoxville, TN 37996-1300*
*U.S.A.*

*Department of Mathematics and Statistics*
*Concordia University*
*7141 Sherbrooke Street West*
*Montreal, Quebec*
*H4B 1R6*
*e-mail: xzhou@mathstat.concordia.ca*