PARTITION RELATIONS FOR ORDINAL NUMBERS

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1. Introduction. Capital letters denote sets and the cardinal of A is $|A|$. Greek letters always denote ordinal numbers and, unless stated otherwise, small latin letters denote non-negative integers. The symbol $[A]^r$ is used to denote the set $\{X: X \subset A;\ |X| = r\}$ of all subsets of A with r elements. If A is a simply ordered set with the order relation $<$, then the order type of A with this ordering is written as $\mathrm{tp}_< A$ or simply as $\mathrm{tp}\, A$ when there is no ambiguity about the intended order relation. A graph $G = (A, E)$ is an ordered pair with A as the set of vertices and $E \subset [A]^2$ as the set of edges. In particular, if A is simply ordered, we call G a graph of type $\mathrm{tp}\, A$. A *complete subgraph* of $G = (A, E)$ is a set $B \subset A$ such that $[B]^2 \subset E$; a set $C \subset A$ is *independent* if $[C]^2 \cap E = \emptyset$.

The partition symbol

$$\alpha \rightarrow (\alpha_\nu)^r_{\nu<\kappa}, \tag{1.1}$$

by definition, means that the following statement is true. *If A is an ordered set of type α and if*

$$[A]^r = \bigcup_{\nu<\kappa} E_\nu$$

is any partition of $[A]^r$ into disjoint sets E_ν $(\nu < \kappa)$, then there are $\nu < \kappa$ and $A' \subset A$ such that $\mathrm{tp}\, A' = \alpha_\nu$ *and* $[A']^r \subset E_\nu$. The negation of (1.1) is written as

$$\alpha \not\rightarrow (\alpha_\nu)^r_{\nu<\kappa}.$$

In this paper, we are mainly concerned with a simple form of (1.1) in which $r = \kappa = 2$ and $\alpha, \alpha_0, \alpha_1$ are denumerable and we write, instead of (1.1),

$$\alpha \rightarrow (\alpha_0, \alpha_1)^2. \tag{1.2}$$

There is an obvious graph-theoretic interpretation of (1.2). For, if $G = (A, E)$ is any graph of type α, then (1.2) asserts that either there is a complete subgraph of type α_0, or there is an independent set of type α_1.

Partition relations were first introduced by Erdős and Rado in (**3**), and in (**4**) they established a wide variety of such relations and studied certain generalizations. Since then, Erdős, Hajnal, and Rado (**7**) have given a remarkably comprehensive analysis of relations (1.1) when α, α_ν $(\nu < \kappa)$ are *cardinal numbers* (i.e., initial ordinals) which are not all finite; comparatively few open questions remain in this case; see (**7**, Problem 2). Compared with this, little

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seems to be known about such relations when the ordinals are not initial numbers.

It is clear from the definition that, if (1.1) holds and $\alpha' \geqq \alpha$, then $\alpha' \to (\alpha_\nu)_{\nu<\kappa}{}^r$ also holds and the problem is to determine the least α which satisfies (1.1) when r, κ, and α_ν $(\nu < \kappa)$ are given. The simplest case is when $r = 1$ and $\kappa, \alpha_0, \ldots, \alpha_{\kappa-1}$ are finite as this corresponds to the ordinary pigeon-hole principle of elementary arithmetic. The extension of this to the case when the ordinals κ and α_ν $(\nu < \kappa)$ are arbitrary has been completely analyzed in (**11**) so that interest remains only for the case $r > 1$. A well-known theorem of Ramsey (**12**) states that if $r, k, m_1, \ldots, m_k$ are finite, then there is a (least) integer $q = q(r, m_1, \ldots, m_k)$ such that $q \to (m_1, \ldots, m_k)^r$. The problem of determining the finite Ramsey function $q(r, m_1, \ldots, m_k)$ is an extremely difficult one; a few special values are known (see **8**; **9**) and various estimates of the magnitude have been given (see **5–8**). In view of this difficulty, there can be little hope of providing a complete analysis for (1.1) with arbitrary ordinals (and $r > 1$). It is just possible, however, that the least ordinal satisfying (1.1), when r, κ, and α_ν $(\nu < \kappa)$ are given, is expressible in terms of the unknown Ramsey function and certain other finite combinatorial rules. A few simple results in this direction are established here (see the remarks following Theorems 4 and 5 and also Theorem 7).

The convenient symbolism of the partition calculus was developed in order to study various extensions and analogues of another, transfinite, theorem of Ramsey (**12**) which states that[1]

$$\omega \to (\omega, \ldots, \omega)_k{}^r. \tag{1.3}$$

In one sense, (1.3) cannot be extended for *denumerable* ordinals since it has been proved (see **4** and **14**) that

$$\alpha \not\to (\omega, \omega + 1)^2 \quad \textit{if } \alpha < \omega_1.$$

Furthermore (see **10**),

$$\alpha \not\to (r + 1, \omega + 1)^r \quad \textit{if } \alpha < \omega_1 \textit{ and } r \geqq 3.$$

From these negative results it follows that the only relations of the form (1.2) which are of interest in the case of denumerable ordinals are those of the form

$$\alpha \to (m, \gamma)^2 \tag{1.4}$$

with $3 \leqq m < \omega < \gamma \leqq \alpha < \omega_1$.

Previously known relations of the form (1.4) are the following. In (**4**) it is proved that

$$\omega m \to (m, \omega + p)^2, \tag{1.5}$$

$$\omega m \not\to (m + 1, \omega + 1)^2. \tag{1.6}$$

[1]ω is the first infinite ordinal and ω_1 is the first non-denumerable ordinal.

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Furthermore, a finite rule is given in (**4**) for calculating an integer $l = l(m, n)$ such that

(1.7) $$\omega l \to (m, \omega n)^2$$

and this result is best possible in the sense that

$$\alpha \not\to (m, \omega n)^2 \quad \textit{if } \alpha < \omega l.$$

In (**14**), Specker proved that (1.7) remains valid in the limiting case when $l = n = \omega$, i.e.

(1.8) $$\omega^2 \to (m, \omega^2)^2 \quad \textit{for any } m < \omega.$$

Specker (**14**) also noted that ω^2 cannot be replaced by any higher finite power of ω in (1.8), i.e.

(1.9) $$\omega^k \not\to (3, \omega^k)^2 \quad \textit{if } 2 < k < \omega.$$

An interesting unresolved question is whether or not the relation

(?) $$\omega^\omega \to (m, \omega^\omega)^2$$

holds for any $m > 2$.

It is proved in Theorem 1 that

(1.10) $$\omega^4 \to (3, \omega^3)^2,$$

and Theorem 2 (which generalizes (1.9)) shows that this is best possible in the following two ways:

(1.11) $$\omega^4 \not\to (3, \omega^3 + 1)^2,$$

(1.12) $$\alpha \not\to (3, \omega^3)^2 \quad \textit{if } \alpha < \omega^4.$$

The method used to prove (1.10) can be modified to determine some $\alpha < \omega^\omega$ so that (1.4) holds for given $m < \omega$ and $\gamma < \omega^\omega$. However, stronger results than these (except for (1.10) itself) can be obtained by a different type of argument. It will be shown elsewhere (**2**) that

(1.13) $$\omega^{1+\nu h} \to (2^h, \omega^{1+\nu})^2 \quad \textit{if } h < \omega \textit{ and } \nu < \omega_1.$$

For example, (1.13) yields

$$\omega^{4k+1} \to (4, \omega^{2k+1})^2$$

and this should be contrasted with the negative relation

$$\omega^{3k} \not\to (3, \omega^{2k+1})^2$$

given by Theorem 2. The simplest problem we have been unable to solve is whether (1.10) is best possible in a third sense, namely whether[2]

(?) $$\omega^4 \to (4, \omega^3)^2.$$

[2]Since this paper was written, A. Hajnal (and later F. Galvin) proved that $\omega^4 \to (4, \omega^3)^2$ and $\omega^4 \not\to (5, \omega^3)^2$ (written communications).

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2. Additional notation. The obliterator sign $\wedge$ above any symbol indicates that that symbol is to be disregarded. Thus, we sometimes write $A_0 \cup \ldots \cup \hat{A}_\alpha$ instead of $\bigcup_{\nu<\alpha} A_\nu$. The symbols $\{x_0, \ldots, \hat{x}_\alpha\}_<$ or $\{x_\nu: \nu < \alpha\}_<$ are both used to denote the set $S = \{x_\nu: \nu < \alpha\}$ and at the same time express the fact that S is ordered by $<$ so that $x_\nu < x_\mu$ whenever $\nu < \mu < \alpha$. Similarly, $\{x_0, \ldots, \hat{x}_\alpha\}_{\neq}$ means that $x_\nu \neq x_\mu$ if $\nu < \mu < \alpha$. If S is a simply ordered set, we write

$$S = S_0 \cup \ldots \cup \hat{S}_\alpha \text{ (tp)} \quad \text{or} \quad S = \bigcup_{\nu<\alpha} S_\nu \text{ (tp)}$$

to indicate that S is the union of disjoint sets S_ν $(\nu < \alpha)$ and that the elements of S_ν precede the elements of S_μ in the ordering of S if $\nu < \mu < \alpha$. In this case, $\operatorname{tp} S = \sum_{\nu<\alpha} \operatorname{tp} S_\nu$.

If $G = (A, E)$ is a graph and $x \in A$, we let $E(x)$ denote the set $\{y \in A: \{x, y\} \in E\}$ of vertices joined to x by edges of the graph. Similarly, we define $E'(x) = \{y \in A: \{x, y\}_{\neq} \notin E\}$. If $\emptyset \neq X \subset A$, we write

$$E(X) = \bigcap_{x\in X} E(x) \quad \text{and} \quad E'(X) = \bigcap_{x\in X} E'(x).$$

It is proved in (**12**) that $\omega^\alpha \to (\omega^\alpha, \omega^\alpha)^1$ for any $\alpha \geqq 0$. From this, it follows that $T = \{X: X \subset S;\ \operatorname{tp} X < \omega^\alpha\}$ is a proper ideal of $\mathscr{P}(S)$, the set of all subsets of S, which is contained in a maximal ideal T^*. The function m defined on $\mathscr{P}(S)$ by putting $m(X) = 0$ $(X \in T^*)$, $m(X) = 1$ $(X \notin T^*)$ is a non-trivial, two-valued finitely additive measure (see, e.g., **14**).

3. Theorem 1. $\omega^4 \to (3, \omega^3)^2$.

Proof. We shall assume that $G = (S, E)$ is a graph of type ω^4 which contains no triangle and no independent set of type ω^3, and deduce a contradiction.

Since $\operatorname{tp} S = \omega^4$, we may write $S = \bigcup_{\nu<\omega^2} S_\nu$ (tp), where $\operatorname{tp} S_\nu = \omega^2$ $(\nu < \omega^2)$. Let m_ν be a two-valued measure function defined on $\mathscr{P}(S_\nu)$ such that $m_\nu(S_\nu) = 1$ and $m_\nu(X) = 0$ whenever $X \subset S_\nu$ and $\operatorname{tp} X < \omega^2$. Since the sets S_ν are mutually disjoint, we shall henceforth omit the suffix from m_ν and simply write m.

For short, we write $N = \{\nu: \nu < \omega^2\}$ and we define

$$A(\alpha, \beta) = \{x \in S_\alpha: m(E(x) \cap S_\beta) = 1\} \qquad (\alpha, \beta \in N).$$

Suppose that $\alpha_1, \alpha_2, \alpha_3 \in N$ and that

$$m(A(\alpha_i, \alpha_j)) = 1 \qquad (1 \leqq i < j \leqq 3). \tag{3.1}$$

Then elements x_1, x_2, and x_3 can be chosen successively so that

$$x_1 \in A(\alpha_1, \alpha_2) \cap A(\alpha_1, \alpha_3), \quad x_2 \in E(x_1) \cap A(\alpha_2, \alpha_3), \quad x_3 \in E(\{x_1, x_2\}) \cap S_{\alpha_3}.$$

This is possible since the right-hand sides of these relations are sets of measure 1, and thus non-empty. This implies that $\{x_1, x_2, x_3\}_{\neq}$ is a complete triangle of the graph G, contrary to our initial assumption. This proves that at least one of the equations in (3.1) is false. In particular, since we did not assume that the α_i are distinct, it follows that

(3.2) $$m(A(\alpha, \alpha)) = 0 \qquad (\alpha \in N).$$

Consider the partition $[N]^2 = L_0 \cup L_1$, where $\{\alpha, \beta\}_{\neq} \in L_0$ if and only if $m(A(\alpha, \beta)) + m(A(\beta, \alpha)) \neq 0$. By the special case

$$\omega^2 \rightarrow (4, \omega^2)^2$$

of Specker's theorem (1.8), it follows that either

(3.3) *there is* $\{\beta_1, \beta_2, \beta_3, \beta_4\}_{\neq} \subset N$ *such that* $\{\beta_i, \beta_j\}_{\neq} \in L_0$

or

(3.4) *there is* $N_0 \subset N$ *such that* $\operatorname{tp} N_0 = \omega^2$ *and* $[N_0]^2 \subset L_1$.

Suppose that (3.3) holds. Put $\rho_{ij} = m(A(\beta_i, \beta_j))$. Then $\rho_{ij} \in \{0, 1\}$ and

$$\rho_{ij} + \rho_{ji} \neq 0 \quad \text{if } 1 \leqq i < j \leqq 4.$$

By symmetry we can assume that $\rho_{12} = \rho_{13} = \rho \in \{0, 1\}$ and that $\rho_{23} = 1$. If $\rho = 1$, then (3.1) holds with $\alpha_i = \beta_i$ $(i = 1, 2, 3)$ and this contradicts the falsity of (3.1). A similar contradiction follows if $\rho = 0$ for, in this case, (3.1) holds with $\alpha_1 = \beta_2$, $\alpha_2 = \beta_3$, and $\alpha_3 = \beta_1$. Therefore, (3.4) must hold. Combining this with (3.2) we deduce that

(3.5) $$m(A(\alpha, \beta)) = 0 \quad \textit{for } \alpha, \beta \in N_0.$$

Since N_0 is a subset of N of type ω^2 we may write

$$N_0 = \{\alpha_\nu^{(0)}: \nu < \omega^2\}_<.$$

Let $(\gamma_\nu)_{\nu<\omega}$ be any sequence which repeats every ordinal $\gamma < \omega^2$ infinitely often, i.e. $\gamma_\nu < \omega^2$ $(\nu < \omega^2)$ and

(3.6) $$|\{\nu: \nu < \omega; \gamma_\nu = \gamma\}| = \aleph_0 \quad \textit{for any } \gamma < \omega^2.$$

Let $\ll$ be any well-ordering of N as an ω-sequence which is such that

(3.7) $$\tau \ll \pi \quad \textit{whenever } \tau < \pi < \tau + \omega < \omega^2.$$

For example, such an ordering is defined by the relation $\ll$, where

$$\omega m + n \ll \omega m' + n'$$

if and only if $m, n, m', n' < \omega$ and $(m + n, m)$ alphabetically precedes $(m' + n', m')$. For $i < \omega$ we define $\gamma(i)$ to be the least ordinal γ which has the property that

$$\gamma_j \ll \gamma \quad \textit{for all } j \leqq i.$$

It is easy to see that $\gamma(i) < \omega^2$ for $i < \omega$.

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We shall show that elements $x_i \in S$ and sets $N_i \subset N_0$ can be chosen for $i < \omega$ so that the following conditions are satisfied.

(3.8) $$N_{i+1} = \{\alpha_\nu^{(i+1)}: \nu < \omega^2\}_< \subset N_i,$$

(3.9) $$\alpha_\nu^{(i+1)} = \alpha_\nu^{(i)} \quad \textit{for } \nu \ll \gamma(i),$$

(3.10) $$x_i \in S_{\alpha_i}, \quad \textit{where } \alpha_i = \alpha_{\gamma_i}^{(i)},$$

(3.11) $$[\{x_0, \ldots, x_i\}_{\neq}]^2 \cap E = \emptyset.$$

Suppose that this has already been done and that x_i, N_i have been defined so that (3.8)–(3.11) all hold for $i < \omega$. Let $X = \{x_0, \ldots, \hat{x}_\omega\}$ and $N^* = \{\alpha_0, \ldots, \hat{\alpha}_\omega\}$. If $i, j < \omega$ and $\gamma_i \leqq \gamma_j$, then by (3.8), (3.9), and the definition of α_i, we have that

(3.12) $$\alpha_i = \alpha_{\gamma_i}^{(i)} = \alpha_{\gamma_i}^{(i+j)} \leqq \alpha_{\gamma_j}^{(i+j)} = \alpha_{\gamma_j}^{(j)} = \alpha_j,$$

and there is strict inequality if $\gamma_i < \gamma_j$. It follows from this that

$$\text{tp}_< N^* = \text{tp}_<\{\gamma_i: i < \omega\} = \omega^2.$$

If $\alpha \in N^*$, then $\alpha = \alpha_k$ for some $k < \omega$. It follows from (3.12) that $\alpha_i = \alpha_k = \alpha$ if $\gamma_i = \gamma_k$, and therefore, by (3.6),

$$|\{i < \omega: \alpha_i = \alpha\}| = \aleph_0.$$

This, together with (3.10), implies that $|X \cap S_\alpha| = \aleph_0$ $(\alpha \in N^*)$. Since the x_i are distinct by (3.11), it follows that

$$\text{tp}\, X \geqq \omega\, \text{tp}\, N^* = \omega^3.$$

This yields the desired contradiction since (3.11) also implies that X is an independent set.

To complete the proof of the theorem it remains to show that x_i and N_i can be chosen so that (3.8)–(3.11) hold for all $i < \omega$.

If $x \in S$, then there is a unique $\alpha = \alpha(x) < \omega^2$ so that $x \in S_\alpha$ and we define

$$M(x) = \{\beta \in N: x \in A(\alpha, \beta)\}.$$

Suppose that $|M(x)| = \aleph_0$ for some $x \in S$. Then the set

$$T = \bigcup_{\beta \in M(x)} E(x) \cap S_\beta$$

has order type greater than or equal to ω^2 tp $M(x) \geqq \omega^3$ since $m(E(x) \cap S_\beta) = 1$ for $\beta \in M(x)$. The initial assumption that there is no independent set of type ω^3 implies that there is an edge of the graph G in T. This edge, together with x, forms a complete triangle in the graph, contrary to our assumption. This proves that

(3.13) $$|M(x)| < \aleph_0 \qquad (x \in S).$$

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We shall define the x_i, N_i so that, in addition to (3.8)–(3.11),

$$N_{i+1} \cap M(x_i) = \emptyset \tag{3.14}$$

also holds for $i < \omega$. Let $j < \omega$ and suppose that $x_0, \ldots, \hat{x}_j$ and $N_0, \ldots, N_j$ have already been defined so that (3.8)–(3.11) and (3.14) hold for $i < j$. Let

$$Y_j = \bigcup_{\nu \ll \gamma(j)} A(\alpha_j, \alpha_\nu^{(j)}) \cup \bigcup_{i<j} (S_{\alpha_j} \cap E(x_i)).$$

If $i < j$, then $\alpha_j \in N_j \subset N_{i+1}$ by (3.8) and $\alpha_j \notin M(x_i)$ by (3.14). Therefore,

$$m(S_{\alpha_j} \cap E(x_i)) = 0 \qquad (i < j).$$

Since $m(A(\alpha_j, \alpha_\nu^{(j)})) = 0$ by (3.5) and there are only finitely many $\nu \ll \gamma(j)$, it follows that Y_j is a subset of S_{α_j} of zero measure. Hence, x_j may be chosen so that

$$x_j \in S_{\alpha_j} - Y_j \cup \{x_0, \ldots, \hat{x}_j\}. \tag{3.15}$$

Now, define $N_{j+1} = N_j - M(x_j)$.

With these definitions of x_j and N_{j+1} it is immediately clear that (3.10), (3.11), and (3.14) remain true with $i = j$. Furthermore, (3.8) holds with $i = j$ since tp $N_j = \omega^2$ and $M(x_j)$ is a finite set. Thus, it only remains to verify that, if $N_{j+1} = \{\alpha_\nu^{(j+1)}: \nu < \omega^2\}_<$, then

$$\alpha_\nu^{(j+1)} = \alpha_\nu^{(j)} \quad \textit{for } \nu \ll \gamma(j).$$

If this is false, then there is a least ordinal number π ($<\omega^2$) such that $\pi \ll \gamma(j)$ and $\alpha_\pi^{(j+1)} \neq \alpha_\pi^{(j)}$. By (3.15) and the definition of Y_j, we see that $x_j \notin A(\alpha_j, \alpha_\pi^{(j)})$, and hence $\alpha_\pi^{(j)} \in N_j - M(x_j) = N_{j+1}$. Since $M(x_j)$ is finite, it follows that there is $\tau < \omega^2$ such that

$$\alpha_\pi^{(j)} = \alpha_\tau^{(j+1)} \quad \text{and} \quad \tau < \pi < \tau + \omega.$$

Hence, $\tau \ll \pi$ by (3.7) and, by the minimal property of π, we deduce that $\alpha_\tau^{(j)} = \alpha_\tau^{(j+1)} = \alpha_\pi^{(j)}$. This is a contradiction since $\tau < \pi$ implies that $\alpha_\tau^{(j)} < \alpha_\pi^{(j)}$. This completes the inductive definitions of x_i and N_i so that (3.8)–(3.11) and (3.14) all hold for $i < \omega$.

4. A negative result. Theorem 2 generalizes Specker's theorem (1.9).

THEOREM 2. *If $k, l < \omega$ and $\rho \in \{1, 2\}$, then*

$$\omega^{3k} l \nrightarrow (3, \omega^{2k+1})^2 \tag{4.1}$$

and

$$\omega^{3k+\rho} \nrightarrow (3, \omega^{2k+\rho} + 1)^2. \tag{4.2}$$

Remark. Formula (4.2) is valid for any *positive* integer ρ; however, (4.2) is weaker than (4.1) if $2 < \rho < \omega$; (4.2) is false when $\rho = 0$ since we have the positive relation $\omega^2 2 + 1 \to (3, \omega^2 + 1)^2$ (see Theorem 3).

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Proof of Theorem 2. The proofs of (4.1) and (4.2) are very similar and we shall prove these results simultaneously and distinguish between them, where necessary, by referring to cases (i) and (ii), respectively.

We define δ and λ in the two cases as follows:

Case (i). $\delta = 0$ and $\lambda = l$;

Case (ii). $\delta = \rho - 1$ and $\lambda = \omega$.

Since (4.1) holds trivially when $l = 0$, we may assume that $\lambda > 0$. Furthermore, we may assume that $k > 0$.

Let $S = \bigcup_{m<\lambda} S_m$, where S_m is the set of all $(3k + \delta + 1)$-tuples of the form $(m, x_1, \ldots, x_{3k+\delta})$ with $x_\nu < \omega$ $(0 < \nu \leqq 3k + \delta)$. If S is ordered alphabetically, then $\operatorname{tp} S_m = \omega^{3k+\delta}$ $(m < \lambda)$ and $\operatorname{tp} S = \omega^{3k+\delta}\lambda$. The typical element $(x_0, \ldots, x_{3k+\delta})$ of S is denoted by $\mathbf{x}$ and we write $\mathbf{x}' = (x_0', \ldots, x_{3k+\delta}')$, etc. We shall consider the graph $G = (S, E)$ defined on S in which $\{\mathbf{x}, \mathbf{x}'\}_< \in E$ if and only if

$$(4.3) \qquad \sum_{\nu=k+1}^{2k} x_\nu < \sum_{\nu=1}^{k} x_\nu' < \sum_{\nu=2k+1}^{3k+\delta} x_\nu < \sum_{\nu=k+1}^{2k} x_\nu'.$$

Suppose that $[\{\mathbf{x}, \mathbf{x}', \mathbf{x}''\}_<]^2 \subset E$. Then (4.3) holds and so also do the inequalities

$$\sum_{\nu=1}^{k} x_\nu'' < \sum_{\nu=2k+1}^{3k+\delta} x_\nu \quad \text{and} \quad \sum_{\nu=k+1}^{2k} x_\nu' < \sum_{\nu=1}^{k} x_\nu'',$$

which are inconsistent with the final inequality of (4.3). This shows that the graph G contains no complete triangle.

Now, let X be any subset of S such that

Case (i). $\operatorname{tp} X = \omega^{2k+1}$;

Case (ii). $\operatorname{tp} X = \omega^{2k+\rho} + 1$.

The proof of the theorem will be complete if we show that X is not an independent set; i.e., there is an edge of the graph joining two elements of X.

We first show that (in both cases (i) and (ii)) there are $m_0 < \lambda$ and $X_0 \subset X \cap S_{m_0}$ such that

$$(4.4) \qquad \operatorname{tp} X_0 = \omega^{2k+\delta+1}.$$

Case (i). Since λ is finite, it follows that $\omega^{2k+1} \rightarrow (\omega^{2k+1})_\lambda^1$ (see, e.g., **11**), and hence there is $m_0 < \lambda$ so that $\operatorname{tp} X \cap S_{m_0} = \omega^{2k+1}$. Then (4.4) holds with $X_0 = X \cap S_{m_0}$ since, in this case, $\delta = 0$.

Case (ii). Since $\operatorname{tp} X = \omega^{2k+\rho} + 1$, X has a last element, i.e., $X = X' \cup \{\mathbf{x}^*\}$ and $\mathbf{x} < \mathbf{x}^*$ for all $\mathbf{x} \in X'$. There is $m_1 < \lambda$ such that $\mathbf{x}^* \in S_{m_1}$, and hence $X' \subset \bigcup_{m \leqq m_1} S_{m_1}$. It follows in the same way as in Case (i) that there is $m_0 \leqq m_1$ such that $\operatorname{tp} X' \cap S_{m_0} = \omega^{2k+\rho}$. Then (4.4) holds with $X_0 = X' \cap S_{m_0}$.

For fixed integers $a_1, \ldots, a_k$ we put

$$A(a_1, \ldots, a_k) = \{\mathbf{x} \in X_0 \colon \mathbf{x} = (x_0, \ldots, x_{3k+\delta}); x_\nu = a_\nu \quad (1 \leqq \nu \leqq k)\}.$$

It is clear that $\operatorname{tp} A(a_1, \ldots, a_k) \leqq \omega^{2k+\delta}$ for any $a_1, \ldots, a_k$ (for if

$(x_0, \ldots, x_{3k+\delta}) \in X_0$, then $x_0 = m_0$). We shall show that the set of k-tuples

$$V = \{(a_1, \ldots, a_k) \colon a_\nu < \omega \ (1 \leqq \nu \leqq k); \ \mathrm{tp}\, A(a_1, \ldots, a_k) \geqq \omega^{k+\delta+1}\}$$

is infinite. Let

$$X_1 = \bigcup_{(a_1, \ldots, a_k) \in V} A(a_1, \ldots, a_k) \quad \text{and} \quad X_2 = \bigcup_{(a_1, \ldots, a_k) \notin V} A(a_1, \ldots, a_k).$$

Then $X_0 = X_1 \cup X_2$. For fixed integers $a_1, \ldots, a_k$ with $(a_1, \ldots, a_k) \notin V$, we have that

$$\mathrm{tp}\{\mathbf{x} \in X_2 \colon \mathbf{x} = (m_0, a_1, \ldots, a_k, x_{k+1}, \ldots, x_{3k+\delta})\} < \omega^{k+\delta+1}.$$

Therefore, since S is ordered alphabetically,

$$\mathrm{tp}\, X_2 = \textstyle\sum_{\mu < \omega^k} \delta_\mu$$

for certain $\delta_\mu < \omega^{k+\delta+1}$. For $\mu < \omega^k$ there is $t_\mu < \omega$ such that $\delta_\mu < \omega^{k+\delta} t_\mu$, and hence

$$\mathrm{tp}\, X_2 \leqq \textstyle\sum_{\mu < \omega^k} \omega^{k+\delta} t_\mu \leqq \omega^{2k+\delta} < \mathrm{tp}\, X_0.$$

If V is a finite set with n elements, then

$$\mathrm{tp}\, X_1 \leqq \omega^{2k+\delta} n < \mathrm{tp}\, X_0.$$

This is a contradiction since $\mathrm{tp}\, X_0 = \omega^{2k+\delta+1} \to (\mathrm{tp}\, X_0, \mathrm{tp}\, X_0)^1$. This proves that

$$(4.5) \qquad |V| = \aleph_0.$$

For given integers $a_1, \ldots, a_{2k}$, we define

$$B(a_1, \ldots, a_{2k}) = \{\mathbf{x} \in X_0 \colon \mathbf{x} = (m_0, a_1, \ldots, a_{2k}, x_{2k+1}, \ldots, x_{3k+\delta})\}.$$

Furthermore, for fixed $(a_1, \ldots, a_k)$, we define

$$W(a_1, \ldots, a_k) = \{(a_{k+1}, \ldots, a_{2k}) \colon \mathrm{tp}\, B(a_1, \ldots, a_{2k}) \geqq \omega^{\delta+1}\}.$$

We shall show that

$$(4.6) \qquad |W(a_1, \ldots, a_k)| = \aleph_0 \quad \textit{if } (a_1, \ldots, a_k) \in V.$$

Suppose that this is false and that there is $(a_1, \ldots, a_k) \in V$ such that $W = W(a_1, \ldots, a_k)$ is a finite set containing q elements. Let

$$A_1 = \bigcup_{(a_{k+1}, \ldots, a_{2k}) \in W} B(a_1, \ldots, a_{2k}) \quad \text{and} \quad A_2 = \bigcup_{(a_{k+1}, \ldots, a_{2k}) \notin W} B(a_1, \ldots, a_{2k}).$$

By reasoning in a similar way as in the proof of (4.5), we see that

$$\mathrm{tp}\, A_1 \leqq \omega^{k+\delta} q < \omega^{k+\delta+1} \quad \text{and} \quad \mathrm{tp}\, A_2 = \textstyle\sum_{\mu < \omega^k} \epsilon_\mu$$

for certain $\epsilon_\mu < \omega^{\delta+1}$, and hence $\mathrm{tp}\, A_2 < \omega^{k+\delta+1}$. We now have a contradiction since $A_1 \cup A_2 = A(a_1, \ldots, a_k)$ and

$$\mathrm{tp}\, A(a_1, \ldots, a_k) \geqq \omega^{k+\delta+1} \to (\omega^{k+\delta+1})_2^1.$$

This proves (4.6). Let $(x_1^{(0)}, \ldots, x_k^{(0)})$ be the *first* element of V (ordered alphabetically). By (4.6) there is $(x_{k+1}^{(0)}, \ldots, x_{2k}^{(0)}) \in W(x_1^{(0)}, \ldots, x_k^{(0)})$.

Since V is infinite, there is $(x_1^{(1)}, \ldots, x_k^{(1)}) \in V$ different from $(x_1^{(0)}, \ldots, x_k^{(0)})$ such that

(4.7) $$\sum_{\nu=1}^{k} x_\nu^{(1)} > \sum_{\nu=k+1}^{2k} x_\nu^{(0)}.$$

The set $B(x_1^{(0)}, \ldots, x_{2k}^{(0)})$ is infinite, and hence there is

$$\mathbf{x}^{(0)} = (m_0, x_1^{(0)}, \ldots, x_{3k+\delta}^{(0)}) \in B(x_1^{(0)}, \ldots, x_{2k}^{(0)}) \subset X_0$$

such that

(4.8) $$\sum_{\nu=2k+1}^{3k+\delta} x_\nu^{(0)} > \sum_{\nu=1}^{k} x_\nu^{(1)}.$$

Again, since $W(x_1^{(1)}, \ldots, x_k^{(1)})$ is infinite, it follows that there is $(x_{k+1}^{(1)}, \ldots, x_{2k}^{(1)}) \in W(x_1^{(1)}, \ldots, x_k^{(1)})$ so that

(4.9) $$\sum_{\nu=k+1}^{2k} x_\nu^{(1)} > \sum_{\nu=2k+1}^{3k+\delta} x_\nu^{(0)}.$$

Since $B(x_1^{(1)}, \ldots, x_{2k}^{(1)}) \neq \emptyset$, there is some element

$$\mathbf{x}^{(1)} = (m_0, x_1^{(1)}, \ldots, x_{3k+\delta}^{(1)})$$

in this set.

From the construction, $\mathbf{x}^{(0)}, \mathbf{x}^{(1)} \in X_0 \subset X$ and $\mathbf{x}^{(0)} < \mathbf{x}^{(1)}$ since $(x_1^{(0)}, \ldots, x_k^{(0)})$ alphabetically precedes $(x_1^{(1)}, \ldots, x_k^{(1)})$. The inequalities (4.7), (4.8), and (4.9) now imply that $\{\mathbf{x}^{(0)}, \mathbf{x}^{(1)}\}$ is an edge of the graph. This shows that X is not an independent set and completes the proof of Theorem 2.

5. Two lemmas on matrices. We call the $n \times k$ matrix $B = (b_{ij})$ *nulliplex* if $b_{ij} \in \{0, 1\}$ and the columns of B are different from the null vector. Corresponding to such a matrix B, we define

$$F_B(i_1, \ldots, i_r) = \{j: 1 \leqq j \leqq k; b_{i_\rho j} = 0 \ (1 \leqq \rho < r); b_{i_r j} = 1\}$$

for $1 \leqq r \leqq n$ and $\{i_1, \ldots, i_r\}_{\neq} \subset \{i: 1 \leqq i \leqq n\}$.

LEMMA 1. *If B is a nulliplex $n \times k$ matrix, then*

$$\sum_{r=1}^{n} |F_B(1, 2, \ldots, r)| = k.$$

Proof. The n sets $F_B(1, 2, \ldots, r)$ $(1 \leqq r \leqq n)$ are mutually disjoint and, if $1 \leqq j \leqq k$, then there is some r such that $1 \leqq r \leqq n$ and $b_{ij} = 0$ $(1 \leqq i < r)$, $b_{rj} = 1$.

The next lemma is a result in the opposite direction.

LEMMA 2. *If the integers $k_1, \ldots, k_n$ are strongly monotonic, i.e.,*

(5.1) $$l_r = \sum_{s=0}^{r-1} (-1)^s \binom{r-1}{s} k_{r-s} \geqq 0 \qquad (1 \leqq r \leqq n),$$

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then there is a nulliplex $n \times (k_1 + \ldots + k_n)$ matrix B such that

$$|F_B(i_1, \ldots, i_r)| = k_{n-r+1} \tag{5.2}$$

whenever $1 \leqq r \leqq n$ and $\{i_1, \ldots, i_r\}_{\neq} \subset \{i: 1 \leqq i \leqq n\}$.

Proof. Consider a nulliplex matrix $B = (b_{ij})$ which contains exactly l_r columns of the form $(\delta_1, \ldots, \delta_n)'$ (the transpose of $(\delta_1, \ldots, \delta_n)$) whenever $\delta_1, \ldots, \delta_n \in \{0, 1\}$ and $1 \leqq \delta_1 + \ldots + \delta_n = r \leqq n$. The matrix B is uniquely determined apart from interchange of columns, and the total number of columns is

$$k = \sum_{r=1}^{n} \binom{n}{r} l_r = \sum_{r=1}^{n} \binom{n}{r} \sum_{s=0}^{r-1} (-1)^s \binom{r-1}{s} k_{r-s}$$
$$= \sum_{r=1}^{n} k_r \sum_{t=0}^{n-r} (-1)^t \binom{r-1+t}{t} \binom{n}{r+t} = \sum_{r=1}^{n} k_r.$$

We now verify that (5.2) holds. If $A \subset \{1, \ldots, n\}$, let

$$g_B(A) = \{j: 1 \leqq j \leqq k; b_{ij} = 0 \ (i \notin A); b_{ij} = 1 \ (i \in A)\}.$$

The definition of B implies that

$$|g_B(A)| = l_r \qquad (A \subset [\{1, \ldots, n\}]^r; 1 \leqq r \leqq n).$$

If $1 \leqq r \leqq n$ and $C = \{i_1, \ldots, i_r\}_{\neq} \subset \{1, 2, \ldots, n\}$, then

$$|f_B(i_1, \ldots, i_r)| = \left| \bigcup_{A \subset \{1,\ldots,n\}-C} g_B(A \cup \{i_r\}) \right|$$
$$= \sum_{s=0}^{n-r} \binom{n-r}{s} l_{s+1}$$
$$= \sum_{s=0}^{n-r} \binom{n-r}{s} \sum_{\sigma=0}^{s} (-1)^\sigma \binom{s}{\sigma} k_{s+1-\sigma}$$
$$= \sum_{s=0}^{n-r} k_{s+1} \sum_{t=0}^{n-r-s} (-1)^t \binom{s+t}{t} \binom{n-r}{s+t}$$
$$= \sum_{s=0}^{n-r} k_{s+1} \binom{n-r}{s} \sum_{t=0}^{n-r-s} (-1)^t \binom{n-r-s}{t} = k_{n-r+1}.$$

This proves Lemma 2.

6. It is easy to verify the following generalization of (1.6)

$$\omega^\alpha n \nrightarrow (n+1, \omega^\alpha + 1)^2 \tag{6.1}$$

by considering the n-partite graph $G = (S, E)$, where $S = S_0 \cup \ldots \cup \hat{S}_n$ (tp), tp $S_\nu = \omega^\alpha$ $(\nu < n)$ and $\{x, y\} \in E$ if and only if $x \in S_\mu$, $y \in S_\nu$ and $\mu < \nu < n$.

Also, by using the same argument used in (**4**) to prove (1.5), one can show more generally that

(6.2) *if* $\omega^\alpha \to (n+1, \omega^\alpha)^2$, *then* $\omega^\alpha(n+1) \to (n+1, \omega^\alpha + p)^2$.

A stronger result than (6.2) is established in Theorem 3.

We first prove a simple lemma.

LEMMA 3. *If* $\gamma \geqq 2$ *and* $\alpha \nrightarrow (\beta, \gamma)^2$, *then* $\alpha + 1 \nrightarrow (\beta + 1, \gamma)^2$.

Proof. Let $A = \{\nu : \nu < \alpha\}$ and $A' = A \cup \{\alpha\}$. The hypothesis implies that there is a graph $G = (A, E)$ which contains no complete subgraph of type β and no independent set of type γ. The graph $G' = (A', E')$, where $E' = E \cup \{\{\nu, \alpha\} : \nu < \alpha\}$, clearly contains no complete subgraph of type $\beta + 1$ and no independent set of type γ (since $\gamma > 1$).

THEOREM 3. *Suppose that* $\alpha \geqq 0$, $1 \leqq n$, $p < \omega$,

(6.3) $$k_i + 1 \to (i+1, p)^2 \qquad (1 \leqq i \leqq n)$$

and

(6.4) $$\omega^\alpha \to (n+1, \omega^\alpha)^2.$$

Then

(6.5) $$\omega^\alpha n + k_1 + \ldots + k_n + 1 \to (n+1, \omega^\alpha + p)^2.$$

Remark. If $p = 1$, then (6.3) holds with $k_i = 0$ and Specker's theorem (1.8), together with Theorem 3, yields

(6.6) $$\omega^2 n + 1 \to (n+1, \omega^2 + 1)^2.$$

This justifies the remark after the statement of Theorem 2.

Proof of Theorem 3. Let $k < \omega$ and suppose that

(6.7) $$\omega^\alpha n + k \nrightarrow (n+1, \omega^\alpha + p)^2.$$

We shall deduce that

(6.8) $$k \leqq k_1 + \ldots + k_n.$$

Let $S = A_1 \cup \ldots \cup A_n \cup B$ (tp), tp $A_\nu = \omega^\alpha$ $(1 \leqq \nu \leqq n)$, $B = \{b_1, \ldots, b_k\}_<$. By (6.7) there is a graph $G = (S, E)$ which contains no complete subgraph with $n+1$ elements and no independent subset of type $\omega^\alpha + p$. From the hypothesis (6.4) it follows that there are independent sets $A_\nu' \subset A_\nu$ such that tp $A_\nu' = \omega^\alpha$ $(1 \leqq \nu \leqq n)$. Let m_ν be a two-valued measure defined on the subsets of A_ν' such that $m_\nu(A_\nu') = 1$ and $m_\nu(X) = 0$ if $X \subset A_\nu'$ and tp $X < \omega^\alpha$. Let

$$C_{\nu\mu} = \{x : x \in A_\nu' ; m_\mu(A_\mu' \cap E'(x)) = 1\} \qquad (1 \leqq \mu < \nu \leqq n).$$

If X is a finite subset of $C_{\nu\mu}$, then $(A_\mu' \cap E'(X)) \cup X$ is an independent set of order type $\omega^\alpha + \text{tp}\, X$. Therefore, by our assumption,

$$|C_{\nu\mu}| < p \qquad (1 \leqq \mu < \nu \leqq n). \tag{6.9}$$

It follows from the definition of the $C_{\nu\mu}$ that the sets

$$A_\nu'' = A_\nu' - \bigcup_{1 \leqq \mu < \nu} C_{\nu\mu} \qquad (1 \leqq \nu \leqq n)$$

satisfy

$$m_\mu(A_\mu'' \cap E'(x)) = 0 \quad \textit{if } x \in A_\nu'' \textit{ and } 1 \leqq \mu < \nu \leqq n. \tag{6.10}$$

If $\alpha > 0$, it follows immediately from (6.9) that

$$m_\nu(A_\nu'') = 1 \qquad (1 \leqq \nu \leqq n). \tag{6.11}$$

Now suppose that $\alpha = 0$, in which case $|A_\nu| = 1$ $(1 \leqq \nu \leqq n)$. By the finite Ramsey theorem there is some integer q such that $q \to (n + 1, 1 + p)^2$, and therefore we can assume in this case that the integer k which satisfies (6.7) is maximal. Then, by Lemma 3, $n + k \to (n, 1 + p)^2$, and therefore the graph G contains a complete subgraph of n elements. Since S is finite we may, in this case, re-order the elements of S so that $A_1 \cup \ldots \cup A_n$ is a complete subgraph of G. Now (6.11) holds in this case also since $C_{\nu\mu} = \emptyset$ $(1 \leqq \mu < \nu \leqq n)$ and $A_\nu'' = A_\nu' = A_\nu$.

Consider the $n \times k$ matrix $B = (b_{ij})$, where

$$b_{ij} = m_i(A_i'' \cap E'(b_j)) \qquad (1 \leqq i \leqq n;\, 1 \leqq j \leqq k).$$

Let $b_{n+1,j} = 1$ $(1 \leqq j \leqq k)$ and let

$$F_r = \{j\colon 1 \leqq j \leqq k;\, b_{ij} = 0 \ (1 \leqq i < r);\, b_{rj} = 1\} \qquad (1 \leqq r \leqq n + 1).$$

We shall prove that

$$|F_r| \leqq k_{n-r+1} \qquad (1 \leqq r \leqq n + 1), \tag{6.12}$$

where $k_0 = 0$.

Suppose that (6.12) is false for some r $(1 \leqq r \leqq n + 1)$. Since $k_{n-r+1} + 1 \to (n - r + 2, p)^2$, it follows that either (i) there is $F' \subset F_r$ such that $|F'| = n - r + 2$ and $\{b_j\colon j \in F'\}$ is a complete subgraph of G, or (ii) $r \leqq n$ and there is $F'' \subset F_r$ such that $|F''| = p$ and $\{b_j\colon j \in F''\}$ is an independent set. If (i) holds, then by (6.10) and the fact that $b_{ij} = 0$ $(1 \leqq i < r;\, j \in F')$ we can choose $x_1, \ldots, \hat{x}_r$ successively so that

$$x_i \in A_{r-i}'' - \left(\bigcup_{1 \leqq \lambda < i} E'(x_\lambda) \cup \bigcup_{j \in F'} E'(b_j)\right) \qquad (1 \leqq i < r),$$

for the set on the right-side of this relation is a subset of A_{r-i}'' of measure 1. This yields a contradiction since the set $\{x_1, \ldots, \hat{x}_r\} \cup \{b_j\colon j \in F'\}$ is a complete subgraph with $n + 1$ elements. Now, suppose that (ii) holds. Since

$F'' \subset F_r$, we have that $b_{rj} = 1$ $(j \in F'')$ and (since $r \leqq n$) $m_r(Y) = 1$, where

$$Y = \bigcap_{j \in F''} A_r'' \cap E'(b_j).$$

We now have the contradiction that $Y \cup \{b_j: j \in F''\}$ is an independent subset of type $\omega^\alpha + p$. This proves (6.12).

Since $F_{n+1} = \emptyset$ by (6.12), it follows that B is a nulliplex. Therefore, by Lemma 1 and (6.12), it follows that $k \leqq k_1 + \ldots + k_n$. This proves (6.8) and Theorem 3 follows.

Condition (6.4) of Theorem 3 is known to hold in the following cases: (1) $\alpha = 0$ (trivially); (2) $\alpha = 1$ (by Ramsey's theorem (1.5)); (3) $\alpha = 2$ (Specker's theorem (1.8)); (4) $\alpha = \omega_\beta$ $(\beta > 0)$, the initial ordinal of cardinal $\aleph_\beta$, since $\omega^{\omega_\beta} = \omega_\beta \to (\omega_0, \omega_\beta)^2$ by a theorem of Dushnik and Miller (**1**). The case $\alpha = 0$ is not without interest since, in this case, Theorem 3 immediately yields the inequality

$$q(2, n+1, p+1) \leqq \sum_{i=1}^{n+1} q(2, i, p),$$

where $q(r, m_1, \ldots, m_k)$ is the finite Ramsey function mentioned in § 1. A simple induction argument on this yields the result of Erdős and Szekeres (**6**)

(6.13) $$q(2, s, t) \leqq \binom{s+t-2}{t-1}.$$

It is known (**9**) that (6.13) does not give best possible results; e.g., it is easy to verify that

(6.14) $$9 \to (4, 3)^2.$$

Conditions (6.3) and (6.4) are satisfied, in particular, when $\alpha = 0$, $n = 3$, $p = 2$, and $k_i = i$ $(1 \leqq i \leqq 3)$ so that (6.5) yields $10 \to (4, 3)^2$. Thus, Theorem 3 does not yield the best result in this case by (6.14). However, in view of Lemma 2, it may be that Theorem 3 does yield best possible partition relations when $\alpha > 0$. That is to say, *if* $\alpha > 0$ *and*

(6.15) $$k_i \not\to (i + 1, p)^2,$$

does it follow that

(6.16) $$\omega^\alpha n + k_1 + \ldots + k_n \not\to (n+1, \omega^\alpha + p)^2?$$

By (6.1), we see that this conjecture is true if (1) $p = 1$ and $k_i = 0$ $(1 \leqq i \leqq n)$; it is also true (trivially) when (2) $n = 1$ and $k_1 = p - 1$. In Theorems 4 and 5 we shall show that the conjecture also holds in the cases (3) $p = 2$ and (4) $n = 2$.

THEOREM 4. *If* $\alpha > 0$, *then* $\omega^\alpha n + \frac{1}{2}n(n+1) \not\to (n+1, \omega^\alpha + 2)^2$.

Remark. If $p = 2$ and $k_i = i$ $(1 \leqq i \leqq n)$, then (6.3) holds, and consequently

$$\omega^\alpha n + \tfrac{1}{2}n(n+1) + 1 \to (n+1, \omega^\alpha + 2)^2,$$

provided (6.4) also holds. In this case, the strong monotonicity condition (5.1) is satisfied trivially.

Proof of Theorem 4. Let $S = S_0 \cup \ldots \cup \hat{S}_n \cup X \cup Y$ (tp), $\operatorname{tp} S_\nu = \omega^\alpha$ $(\nu < n)$, $X = \{x_0, \ldots, \hat{x}_n\}_{\neq}$, $Y = \{y_{ij}: i < j < n\}_{\neq}$. Then $\operatorname{tp} S = \omega^\alpha n + \frac{1}{2}n(n+1)$. For $k < n$, let $Y_k = \{y_{ij}: k \in \{i, j\}\}$. Consider the graph $G = (S, E)$ in which $\{\lambda, \mu\}_< \in E$ if and only if one of the following holds:

(i) $\lambda \in S_i$, $\mu \in S_j$, and $i < j < n$;
(ii) $\lambda \in S_i$, $\mu \in (X - \{x_i\}) \cup (Y - Y_i)$, $i < n$;
(iii) $\lambda = x_i$, $\mu \in Y_i$, $i < n$;
(iv) $\lambda, \mu \in Y_i$, $i < n$.

Suppose that $A = \{\alpha_0, \ldots, \alpha_n\}_< \subset S$ is a complete subgraph of G. Then $|A \cap S_i| \leqq 1$ $(i < n)$ and there are $r < n$ and $i_0, \ldots, \hat{\imath}_r$ such that $i_0 < \ldots < \hat{\imath}_r < n$ and $\alpha_\rho \in S_{i_\rho}$ $(\rho < r)$ and

$$(6.17) \qquad \{\alpha_r, \ldots, \alpha_n\} \subset \left(X - \bigcup_{\rho<r} \{x_{i_\rho}\}\right) \cup \left(Y - \bigcup_{\rho<r} Y_{i_\rho}\right).$$

Also, from the definition of G, there is $k < n$ such that

$$(6.18) \qquad A \cap (X \cup Y) \subset \{x_k\} \cup Y_k.$$

From (6.17) and (6.18), it follows that $k \neq i_\rho$ $(\rho < r)$ and

$$(6.19) \qquad \{\alpha_r, \ldots, \alpha_n\} \subset \{x_k\} \cup Y_k - \bigcup_{\rho<r} Y_{i_\rho}.$$

Since the r sets $Y_k \cap Y_{i_\rho}$ are mutually disjoint and each is a singleton, the set on the right-hand side of (6.19) contains only $n - r$ elements and (6.19) is impossible.

Now, suppose that the graph contains an independent set, B, of type $\omega^\alpha + 2$. Since $\alpha > 0$, there is $i < n$ such that $\operatorname{tp} B \cap S_i = \omega^\alpha$. Then (i) implies that $B \cap S_j = \emptyset$ $(j \neq i)$, and hence $|B \cap (X \cup Y)| = 2$. From (ii) and (iii), $B \cap (X \cup Y) \subset Y_i$ and this is impossible by (iv).

THEOREM 5. *If* $\alpha > 0$, *and* $q \nrightarrow (3, p)^2$, *then*

$$(6.20) \qquad \omega^\alpha 2 + p + q - 1 \nrightarrow (3, \omega^\alpha + p)^2.$$

Remark. If (6.4) holds and if q is the largest integer consistent with the hypothesis, i.e.

$$(6.21) \qquad q + 1 \rightarrow (3, p)^2,$$

then, by Theorem 3,

$$(6.22) \qquad \omega^\alpha 2 + p + q \rightarrow (3, \omega^\alpha + p)^2.$$

Proof of Theorem 5. The hypothesis $q \nrightarrow (3, p)^2$ implies that $p > 0$. If $p = 1$, then $q = 0$ and (6.20) is a special case of (6.1). We now assume that

$p > 1$ and also that q is the largest integer satisfying the hypothesis, i.e. (6.21) holds. Then, by Lemma 3,

(6.23) $$q \to (3, p-1)^2.$$

Let $S = S_0 \cup S_1 \cup A$ (tp), $\text{tp}\, S_\nu = \omega^\alpha$ $(\nu < 2)$, $A = \{a_1, \ldots, \hat{a}_p, b_1, \ldots, b_q\}_<$. Then $\text{tp}\, S = \omega^\alpha 2 + p + q - 1$. There is a partition

$$[\{1, \ldots, q\}]^2 = L_0 \cup L_1$$

such that, if $T \subset \{1, \ldots, q\}$, then

(6.24) $$[T]^2 \subset L_0 \textit{ implies } |T| < 3$$

and

(6.25) $$[T]^2 \subset L_1 \textit{ implies } |T| < p.$$

From (6.23) and (6.24) it follows that there is $U \subset \{1, \ldots, q\}$ such that $|U| = p - 1$ and $[U]^2 \subset L_1$, and there is no loss of generality if we assume that $U = \{1, \ldots, \hat{p}\}$.

Consider the graph $G = (S, E)$ in which $\{x, y\}_< \subset E$ if and only if one of the following holds:

(i) $x \in S_0$, $y \in S_1 \cup \{a_1, \ldots, \hat{a}_p\}$;
(ii) $x \in S_1$, $y \in \{b_1, \ldots, \hat{b}_p\}$;
(iii) $x = a_\lambda$, $y = b_\mu$, $\{\lambda, \mu\}_{\neq} \in L_0$;
(iv) $x = b_\lambda$, $y = b_\mu$, $\{\lambda, \mu\}_{\neq} \in L_0$.

Suppose that $X = \{x, y, z\}_<$ is a complete subgraph of G. Then $|X \cap S_\nu| \leqq 1 (\nu < 2)$. If $x \in S_0$, $y \in S_1$, then (i) and (ii) imply the contradiction that $z \in \{a_1, \ldots, \hat{a}_p\} \cap \{b_1, \ldots, \hat{b}_p\} = \emptyset$. If $x \in S_0$ and $y \notin S_1$, then $\{y, z\} \subset \{a_1, \ldots, \hat{a}_p\}$ and is not an edge of the graph. If $x \in S_1$, then $y = b_\lambda$, $z = b_\mu$ and $1 \leqq \lambda < \mu < p$; this also implies that $\{y, z\}_<$ is not an edge of the graph since $[U]^2 \subset L_1$. If $x = a_\lambda$, then by (iii) and (iv) $y = b_\mu$, $z = b_\nu$, and $[\{\lambda, \mu, \nu\}_{\neq}]^2 \subset L_0$, a contradiction against (6.24). Finally, if $x = b_\lambda$, $y = b_\mu$, $z = b_\nu$, then by (iv) $[\{\lambda, \mu, \nu\}_{\neq}]^2 \subset L_0$ and this again contradicts (6.24).

Suppose that Y is an independent subset of type $\omega^\alpha + p$. Then $Y = Z \cup P(\text{tp})$, where $\text{tp}\, Z = \omega^\alpha$ and $\text{tp}\, P = p$. Since $\alpha > 0$, there is $\pi < 2$ such that $\text{tp}\, Z \cap S_\pi = \omega^\alpha$ and $P \cap S_\pi = \emptyset$. If $\pi = 0$, then by (i), $P = \{b_i : i \in T\}$, where $T \subset \{1, \ldots, q\}$ and $|T| = p$. Since P contains no edge of the graph, it follows from (iv) that $[T]^2 \subset L_1$ and this contradicts (6.25). Now suppose that $\pi = 1$. Then by (ii), $P = \{a_i : i \in V\} \cup \{b_j : j \in W\}$, where $V \subset \{1, \ldots, \hat{p}\}$, $W \subset \{p, \ldots, q\}$, and $|V \cup W| = p$. Since P is an independent set and $[\{1, \ldots, \hat{p}\}]^2 \subset L_1$, it follows from (iii) and (iv) that $[V \cup W]^2 \subset L_1$, and this again contradicts (6.25).

In contrast to (6.22), we establish the following negative result.

THEOREM 6. *If* $0 < \beta \leqq \alpha < \omega_1$ *and* $\gamma < \omega^\beta$, *then*

$$\omega^\alpha 3 + \gamma \not\to (3, \omega^\alpha + \omega^\beta)^2.$$

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Proof. Let $S = S_0 \cup S_1 \cup S_2$ (tp), tp $S_\rho = \omega^\alpha$ ($\rho < 2$), and tp $S_2 = \omega^\alpha + \gamma$. Since $\alpha > 0$, we may write $S_\rho = \bigcup_{\nu<\omega} S_{\rho\nu}$ (tp) ($\rho < 2$), where $0 <$ tp $S_{\rho\nu} < \omega^\alpha$. Let $S_2 = \{x_0, \ldots, \hat{x}_\omega\}_{\neq}$. Consider the graph $G = (S, E)$, with $\{u, v\}_< \in E$ if and only if either (i) $u \in S_{0\lambda}$, $v \in S_{1\mu}$, and $\mu < \lambda < \omega$, or (ii) $v = x_\nu$ and $u \in \bigcup_{\lambda<\nu} S_{0\lambda} \cup \bigcup_{\nu<\mu} S_{1\mu}$.

If $U = \{u, v, w\}_<$ is a complete subgraph of G, then $|U \cap S_\rho| = 1$ ($\rho \leqq 2$) and there are λ, μ, $\nu < \omega$ such that $u \in S_{0\lambda}$, $v \in S_{1\mu}$, $w = x_\nu$. By (i), $\mu < \lambda$, and by (ii), $\lambda < \nu < \mu$, a contradiction.

Suppose that V is an independent subset of S of type $\omega^\alpha + \omega^\beta$. Since $\gamma < \omega^\beta$, there is $\pi < 2$ such that tp$V \cap S_\pi = \omega^\alpha$, and hence there is $L = \{\lambda_0, \ldots, \hat{\lambda}_\omega\}_< \subset \{\lambda : \lambda < \omega\}$ such that $V \cap S_{\pi\lambda} \neq \emptyset$ ($\lambda \in L$). If $\pi = 0$, then (i) implies that $V \cap S_1 = \emptyset$, and hence $V \cap S_2$ is infinite and there is $\nu > \lambda_0$ such that $x_\nu \in V$. Condition (ii) implies that there is an edge of the graph joining x_ν and a point of $V \cap S_{\pi\lambda}$; i.e., V is not independent. It follows that tp$V \cap S_0 < \omega^\alpha$, and therefore $\pi = 1$ and $V \cap S_2 \neq \emptyset$. Hence, there are ν and n such that $x_\nu \in V$ and $\nu < \lambda_n$, and it follows from (ii) that V is not independent.

We conclude by establishing certain extensions of (1.10) and (1.11) which are analogous to (6.20) and (6.21).

THEOREM 7. *If* $q \not\to (3, p)^2$ *and* $q + 1 \to (3, p)^2$, *then*

$$\omega^4 + q + 1 \to (3, \omega^3 + p)^2, \tag{6.26}$$

and

$$\omega^4 + q \not\to (3, \omega^3 + p)^2. \tag{6.27}$$

Proof. Let $S = A \cup B$ (tp), tp $A = \omega^4$, tp $B = q + 1$. Suppose that $G = (S, E)$ is a graph containing no complete subgraph with three elements and no independent subset of type $\omega^3 + p$. Then for any $x \in S$, $E(x)$ is an independent set and tp $E(x) < \omega^3 + p < \omega^4$. Since $\omega^4 \to (\omega^4)_{|B|}{}^1$, it follows that $A' = A - \bigcup_{x\in B} E(x)$ has order type ω^4. By Theorem 1, there is an independent set $A'' \subset A'$ of type ω^3. Furthermore, since $q + 1 \to (3, p)^2$, there is an independent set $B' \subset B$ such that $|B'| = p$. Then $A'' \cup B'$ is an independent set of type $\omega^3 + p$. This contradiction proves (6.26).

Now consider the set $A \cup C$ (tp), where tp $A = \omega^4$ and tp $C = q$. From the special case (1.11) of Theorem 2 and the fact that $q \not\to (3, p)^2$, it follows that there are graphs $G_1 = (A, E_1)$ and $G_2 = (C, E_2)$ neither of which contain complete subgraphs of three elements, G_1 contains no independent set of type $\omega^3 + 1$, and G_2 contains no independent set of p elements. Let G be the graph $(A \cup C, E_1 \cup E_2)$.

If X is a complete subgraph of G, then either $X \subset A$ or $X \subset C$, and X is a complete subgraph of G_1 or G_2. Hence, $|X| < 3$. If Y is an independent subset of the graph G, then $Y \cap A$ and $Y \cap C$ are independent subsets of G_1 and G_2, respectively. Therefore, tp$Y$ = tp$(Y \cap A)$ + tp$(Y \cap C) \leqq \omega^3 + (p - 1)$. This proves (6.27).

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