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REPRESENTATION OF p-LATTICE SUMMING OPERATORS

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ABSTRACT. In this paper we study some aspects of the behaviour of p-lattice summing operators. We prove first that an operator T from a Banach space E to a Banach lattice X is p-lattice summing if and only if its bitranspose is. Using this theorem we prove a characterization for 1-lattice summing operators defined on a $C(K)$ space by means of the representing measure, which shows that in this case 1-lattice and ∞-lattice summing operators coincide. We present also some results for the case $1 \leq p < \infty$ on $C(K, E)$.

1. **Introduction.** In this paper we present some results concerning the behaviour of p-lattice summing operators on spaces of continuous functions. In the first section, using the Local Reflexivity Principle and some properties of Banach lattices, we prove that an operator is p-lattice summing if and only if its bitranspose is p-lattice summing also. This is an important result related to the representation of p-lattice summing operators defined on spaces of continuous functions by means of a vector measure. The relation between an operator and its representing measure has been considered by many authors (see for instance [2], [3], [4], [5] and [6]).

In the second section we obtain some results in the case where the operators are defined on a $C(K)$ space (space of real continuous functions on a compact Hausdorff space K) by means of the representing measure: we characterize 1-lattice summing operators and prove that they coincide with ∞-lattice summing operators. For $1 < p < \infty$ we show necessary conditions for an operator on $C(K)$ and $C(K, E)$ to be p-lattice summing. We give also some examples and partial results for operators defined on a $C(K, E)$ space (space of vector-valued continuous functions, from a compact Hausdorff space K to a Banach space E).

Throughout this paper E and F will be Banach spaces, and X, Y will be Banach lattices. We will denote by E' the topological dual of E, B_E the closed unit ball in E, and $J_E\colon E \longrightarrow E''$ will be the natural inclusion. We will consider only real vector spaces.

For $p \in R$, let $q = p/(p-1)$, so that $\frac{1}{p} + \frac{1}{q} = 1$. An operator T (linear and continuous) from a Banach space E to a Banach lattice X is p-lattice summing (p l. s.) if there is a constant $K > 0$ such that for each finite family $\{x_1, \ldots, x_n\}$ in E we have:

$$\Big\|(\sum_{i=1}^{n} |Tx_i|^p)^{\frac{1}{p}}\Big\|_X \leq K\omega_p((x_i)_{i=1}^n) \text{ if } 1 \leq p < \infty$$

$$\Big\|\bigvee_{i=1}^{n} |Tx_i|\Big\|_X \leq K \max\{\|x_i\|, 1 \leq i \leq n\} \text{ if } p = \infty$$

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where

$$\omega_p\big((x_i)_{i=1}^n\big) = \sup\{(\sum_{i=1}^n |\langle x_i, x'\rangle|^p)^{\frac{1}{p}}, x' \in B_{E'}\}$$

$$= \sup\{\|\sum_{i=1}^n a_i x_i\|_E, a_i \in R\ , 1 \le i \le n, \sum_{i=1}^n |a_i|^q \le 1\}$$

and

$$(\sum_{i=1}^n |Tx_i|^p)^{\frac{1}{p}} = \sup\{\sum_{i=1}^n a_i Tx_i,\ a_i \in R\ ,\ 1 \le i \le n,\ \sum_{i=1}^n |a_i|^q \le 1\}$$

in the form described by Krivine's calculus for 1-homogeneous continuous functions ([8]).

The smallest constant K satisfying the inequalities above is denoted by $\lambda_p(T)$ (or $\lambda_\infty(T)$), and defines a Banach norm in the space $\Lambda_p(E,X)$ of p l. s. operators from E to X. Operators in $\Lambda_\infty(E,X)$ are called also majorizing operators ([12]).

Some general results about these classes of operators can be found in Nielsen and Szulga's papers [9], [15] (in which they compare these operators with the absolutely summing operators defined by Pietsch ([10], [11])), or in Schaefer's book [12].

If T is an operator between two Banach lattices X and Y, T is called order-bounded if it maps order bounded sets of X into order bounded sets of Y. T is called regular if there exist positive operators T_1 and T_2 from X to Y such that $T = T_1 - T_2$. If Y is an order complete Banach lattice (for example the dual of a Banach lattice), regular and order bounded operators are the same for all lattices X, and it is possible to define the modulus of T:

$$|T|(x) = \sup\{|Tz|, |z| \le x\} \text{ for any } x \ge 0 \text{ in X.}$$

The class of regular operators between X and Y, $L^r(X,Y)$, is a Banach space with respect to the norm

$$\|T\|_r = \inf\{\|T_1 + T_2\|, T_1, T_2 \ge 0, T_1 - T_2 = T\}$$

and in the case where Y is an order complete Banach lattice, $L^r(X,Y)$ is a Banach lattice too, and $\|T\|_r = \|\,|T|\,\|$ for all T in $L^r(X,Y)$.

Other properties of these operators are given in references [1], and [12].

Now let K be a compact Hausdorff space, $\beta_0(K)$ the Borel σ-algebra of K, and $C(K)$ the space of real-valued continuous functions on K. The representation theorem shows that each operator $T: C(K) \longrightarrow F$ (F a Banach space) determines a unique vector measure $m: \beta_0(K) \longrightarrow F''$, such that for each $f \in C(K)$, $T(f) = \int f\,dm$, with some special regularity properties. This measure is defined by $m(A) = T''\big(\chi(A)\big)$, where $\chi(A)$ is the characteristic function of A ($A \in \beta_0(K)$), and is called the representing measure of T.

And there is also a representation theorem for operators defined on a space of vector valued continuous functions: if E and F are Banach spaces, and $T: C(K,E) \longrightarrow F$ is an operator, there is a vector measure m on $\beta_0(K)$, with values in $L(E,F'')$ such that for each $f \in C(K,E)$, $T(f) = \int f\,dm$. In this case the measure is defined by $m(A)(x) = T''\big(\chi(A)\cdot x\big)$ for each $x \in E$ and each Borel set A in K, and has also some regularity properties which make it unique. m is called again the representing measure of T.

Classical references for this theory are [2], [3] and [5].

2. **Bitranspose of p-lattice summing operators.** For the proof of the main result in this section, we use the Local Reflexivity Principle, in a version of Pietsch ([11]). And we shall use also two standard properties of Banach lattices. As a consequence of Krivine's calculus for 1-homogeneous continuous functions in Banach lattices we have the following property: let X be a closed sublattice of a Banach lattice Y, and $J: X \longrightarrow Y$ the inclusion; for each 1-homogeneous continuous function $f: R^n \longrightarrow R$ and for each finite family $\{x_1, \ldots, x_n\}$ in X, we have that

$$J\big(f(x_1, \ldots, x_n)\big) = f\big(J(x_1), \ldots, J(x_n)\big)$$

because of the unicity of Krivine's calculus.

The second property is called the Fatou property: let X' be the topological dual of a Banach lattice X, and let $\{y_\alpha\}$ be an upwards directed family in X' with $y_\alpha \geq 0$ and $\|y_\alpha\| \leq M$; then there exists $y = \sup\{y_\alpha\}$ in X', $\|y\| = \sup\{\|y_\alpha\|\}$ and y_α converges to y in the weak* topology.

Classical references for the theory of Banach lattices are [8] and [12].

THEOREM 1. *Let E be a Banach space and $T: E \longrightarrow F$ an operator. Then, given p, $1 \leq p < \infty$, T is p l. s. if and only if $T'': E'' \longrightarrow F''$ is. Moreover, $\lambda_p(T) = \lambda_p(T'')$.*

PROOF. Using the remarks above it is obvious that if $T'': E'' \longrightarrow F''$ is p l. s., T must be p l. s. too, and $\lambda_p(T) \leq \lambda_p(T'')$.

So, take $x_1'', \ldots, x_n''$ in E''. We have to find a suitable estimate for the norm $\big\|(\sum_{i=1}^n |T''x_i''|^p)^{\frac{1}{p}}\big\|$.

For each finite set C in the closed unit ball B_q of $(R^n, \|.\|_q)$, we define

$$y_C'' = \sup\{\sum_{i=1}^n a_i T''x_i'', a = (a_1, \ldots, a_n) \in C \cup \{0\}\}$$

Then

$$(\sum_{i=1}^n |T''x_i''|^p)^{\frac{1}{p}} = \sup_C y_C''$$

The family $\{y_C''\}_C$ is an increasing net of positive vectors in X'', bounded in norm (by the norm of the supremum), and therefore by the Fatou property there exists $y_0'' = \sup_C y_C''$ and $\langle y_C'', y'\rangle$ converges to $\langle y_0'', y'\rangle$ for all y' in X'.

So given $\epsilon > 0$ there exists y' in X, $y' \geq 0$, with $\|y'\| \leq 1$ and such that

$$\Big\|(\sum_{i=1}^n |T''x_i''|^p)^{\frac{1}{p}}\Big\| = \|y_0''\| \leq \langle y_0'', y'\rangle + \epsilon \leq \langle y_{C_0}'', y'\rangle + 2\epsilon$$

for some finite set $C_0 = \{a^1, \ldots, a^m\}$ in B_q, with $0 \in C_0$. By [12] II.5.5. and II.4.2,

$$\begin{aligned}\langle y_0'', y'\rangle &= \langle \sup\{\sum_{i=1}^n a_i^j T''x_i'', 1 \leq j \leq m\}, y'\rangle \\ &= \sup\{\sum_{j=1}^m \Big(\sum_{i=1}^n a_i^j \langle T''x_i'', y_j'\rangle\Big), \sum_{j=1}^m y_j' = y', y_j' \geq 0\} \\ &\leq \sum_{j=1}^m (\sum_{i=1}^n a_i^j \langle x_i'', T'y_j'\rangle) + \epsilon\end{aligned}$$

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for some $y'_1, \dots, y'_m$ in X', with $y'_j \geq 0$, $1 \leq j \leq m$, and $\sum_{j=1}^m y'_j = y'$.

We use now the Local Reflexivity Principle: given the linear spans $G = [x''_1, \dots, x''_n]$ in E'', and $H = [T'y'_1, \dots, T'y'_m]$ in E', and given $t > 0$, there is an operator $R: G \longrightarrow E$ such that $\|R\| \leq 1 + t$, and $\langle x''_i, x' \rangle = \langle Rx''_i, x' \rangle$ for every x' in H and $1 \leq i \leq n$ ([11]).

Writing $x_i = Rx''_i$, we have for any $1 \leq i \leq n$ and any $1 \leq j \leq m$

$$\langle x''_i, T'y'_j \rangle = \langle x_i, T'y'_j \rangle = \langle Tx_i, y'_j \rangle$$

and

$$\begin{aligned} \|y''_0\|_{X''} &\leq \sum_{j=1}^m \Big(\sum_{i=1}^n a^j_i \langle Tx_i, y'_j \rangle\Big) + 3\epsilon \\ &\leq \Big\langle \sup\{\sum_{i=1}^n a^j_i Tx_i,\ 1 \leq j \leq m\}, y' \Big\rangle + 3\epsilon \\ &\leq \Big\|\Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac{1}{p}}\Big\|_X + 3\epsilon \end{aligned}$$

As by hypothesis T is p l. s.,

$$\|y''_0\|_{X''} \leq \lambda_p(T) \cdot \omega_p((x_i)_{i=1}^n) + 3\epsilon$$

Finally

$$\begin{aligned} \omega_p\big((x_i)_{i=1}^n\big) &= \sup\Big\{\|\sum_{i=1}^n a_i x_i\|_E,\ a_i \in R,\ \sum_{i=1}^n |a_i|^q \leq 1\Big\} \\ &= \sup\Big\{\|R(\sum_{i=1}^n a_i x''_i)\|_E,\ a_i \in R,\ \sum_{i=1}^n |a_i|^q \leq 1\Big\} \\ &\leq \|R\| \cdot \omega_p((x''_i)_{i=1}^n) \leq (1+t) \cdot \omega_p\big((x''_i)_{i=1}^n\big). \end{aligned}$$

Consequently, we have obtained that for every $\epsilon > 0$ and every $t > 0$

$$\Big\|\Big(\sum_{i=1}^n |T''x''_i|^p\Big)^{\frac{1}{p}}\Big\|_{X''} \leq \lambda_p(T) \cdot (1+t) \cdot \omega_p((x''_i)_{i=1}^n) + 3\epsilon.$$

Thus T'' is p l. s. Also we have obtained that $\lambda_p(T) \geq \lambda_p(T'')$, and we have finished the proof. ∎

REMARKS. 1. The Local Reflexivity Principle can be used in the same way to give a direct proof of the well known result of Pietsch which asserts that for every operator T between two Banach spaces E and F, T is absolutely (p, r)-summing if and only if T'' from E'' to F'' is: one has just to bound the expression $(\sum_{i=1}^n \|T''x''_i\|^r)^{\frac{1}{r}}$ with another one of the form $(\sum_{i=1}^n |\langle T''x''_i, y'_i \rangle|^r)^{\frac{1}{r}} + \epsilon$ with $y'_i \in F'$, $\|y'_i\| \leq 1$, $1 \leq i \leq n$ ([11], [13]).

2. With the same technique we can give a direct proof of Theorem 1 in the case $p = \infty$. This case is proved in [12] using the duality between ∞-lattice summing operators and cone-absolutely summing operators.

3. **p-Lattice summing operators on spaces of continuous functions.** In this section we are going to study the behaviour of p l. s. operators defined on $C(K)$ and $C(K,E)$ spaces. We shall denote by $S(K)$ the space of simple Borel functions on K, and by $B(K)$ the space of functions which are uniform limits of simple functions, with the supremum norm. $B(K)$ is a closed subspace of $C(K)''$. By considering $C(K)$ as a Banach lattice (with the pointwise order), $C(K)$ is a closed sublattice of $B(K)$, and $B(K)$ is a closed sublattice of $C(K)''$.

It is easy to see that Theorem 1 in the first section can be slightly modified in the case of operators defined on $E = C(K)$, in this form:

THEOREM 2. *Let E be $C(K)$, X a Banach lattice and $T: E \longrightarrow X$ an operator. Then, given $1 \leq p \leq \infty$, T is p l. s. if and only if $\bar{T} = T''|_{B(K)}$ is p l. s. too. Moreover, $\lambda_p(T) = \lambda_p(\bar{T}) = \lambda_p(T'')$.*

Beside this theorem, we shall use two lemmas, which are essentially known:

LEMMA 1. *Let E be a Banach space, X be a Banach lattice and T from E to X an operator. Then the following are equivalent:*

1. *T is ∞ l. s.*
2. *T is p l. s. for every $p \geq 1$, and there exists a constant $L \geq 0$ such that $\lambda_p(T) \leq L$ for every $p \geq 1$.*

LEMMA 2. *Let K be a compact Hausdorff space, X a Banach lattice and T from $B(K)$ to X an operator. If T is positive, then T is ∞ l. s. The same result is true for positive operators $T: C(K) \longrightarrow X$. And it is also true for regular operators. For positive operators we have $\lambda_\infty(T) = \|T\|$, and for regular operators we have $\lambda_\infty(T) \leq \|T\|_r$.*

Nielsen and Szulga proved in [9] that it is always true that given E a Banach space and X a Banach lattice,

$$\Lambda_\infty(E,X) \subseteq \Lambda_p(E,X) \subseteq \Lambda_2(E,X)$$

for any p, $1 \leq p \leq \infty$. In the case where $E = C(K)$ we have obtained the next result, using only lattice techniques and representation theory, which is the main theorem in this section. As a consequence it holds that 1-lattice summing operators and majorizing operators on $C(K)$ coincide.

THEOREM 3. *Let K be a compact Hausdorff space, X be a Banach lattice, T an operator from $C(K)$ to X, and m the representing measure of T. Then the following statements are equivalent:*

1. *T is 1 l. s.*
2. *There is a constant $L \geq 0$ such that for any finite family $\{B_1, \ldots, B_n\}$ of pairwise disjoint Borel subsets of K,*

$$\Big\| \sum_{i=1}^{n} |m(B_i)| \Big\|_{X''} \leq L$$

3. $\bar{T} = T''|_{B(K)}: B(K) \longrightarrow X''$ *is order-bounded (or regular since* X'' *is order complete).*

PROOF. 1) $\Rightarrow$ 2) Assume that T is 1 l. s. Then $T'': C(K)'' \longrightarrow X''$ is also 1 l. s. Let us take $B_1, \ldots, B_n$ in $\beta_0(K)$, pairwise disjoint. Then

$$\begin{aligned}\| \sum_{i=1}^{n} |m(B_i)| \|_{X''} &= \| \sum_{i=1}^{n} |T''(\chi(B_i))| \| \\ &\leq \lambda_1(T'') \sup\{ \sum_{i=1}^{n} |\langle \chi(B_i), \nu \rangle|, \nu \in B(K)', \|\nu\| \leq 1 \} \\ &\leq \lambda_1(T'') \sup\{ \mathrm{var}(\nu), \nu \in B(K)', \|\nu\| \leq 1 \} \leq \lambda_1(T''),\end{aligned}$$

so $L = \lambda_1(T'')$.

2) $\Rightarrow$ 3) We have now to prove that T'' maps order intervals of $B(K)$ into order intervals of X''. But any order interval of $B(K)$ is contained in a scalar multiple of the unit ball $[-\chi(K), \chi(K)]$, and so it is enough to prove that the image of this interval is order bounded in X''.

Let $f = \sum_{i=1}^{n} a_i \chi(B_i)$ be a simple function, where $\{B_i, 1 \leq i \leq n\}$ is a family of pairwise disjoint Borel sets of K, and such that f belongs to $[-\chi(K), \chi(K)]$. Taken for each i, $t_i \in B_i$, we have $|a_i| = |f(t_i)| \leq 1$ for all i, $1 \leq i \leq n$, and thus

$$|\bar{T}(f)| = |T''(f)| = |\sum_{i=1}^{n} a_i T''(\chi(B_i))| \leq \sum_{i=1}^{n} |T''(\chi(B_i))|.$$

Consider Φ the class of all finite disjoint partitions of K by sets in $\beta_0(K)$, ordered by inclusion, and define for each $P \in \Phi$

$$y_P = \sum_{B \in P} |T''(\chi(B))| \in X''$$

$\{y_P\}_{P \in \Phi}$ is an increasing net in X'', and $\|y_P\| = \| \sum_{B \in P} |m(B)| \|$, less or equal than L by Hypothesis 2.

Then using the Fatou property, there exists $y_0'' = \sup\{y_P, P \in \Phi\}$ which is the weak* limit of $\{y_P, P \in \Phi\}$.

Thus, there exists $y_0'' \in X''$ such that for any simple function f in the unit ball $[-\chi(K), \chi(K)]$, $\|\bar{T}(f)\| \leq y_0''$. But $\bar{T}$ is continuous, simple functions are dense in $[-\chi(K), \chi(K)]$ (in $B(K)$) and $[-y_0'', y_0'']$ is closed in X_0'', so we have $\bar{T}([\chi(K), \chi(K)]) \subseteq [-y_0'', y_0'']$.

Hence, $\bar{T}$ is order-bounded.

Observe that, moreover, $\|y_0''\| \leq L$.

3) $\Rightarrow$ 1) We suppose now that $T: B(K) \longrightarrow X''$ is order-bounded; then $\bar{T}$ is regular since X'' is order complete, and by Lemma 2 $\bar{T}$ is ∞ l. s., and also 1 l. s. Now Theorem 2 proves that T is 1 l. s. ∎

We have the equivalence of 1), 2) and 3). Now we study the norms: following the steps of the proof above

$$\lambda_\infty(\bar{T}) \leq \|\bar{T}\|_r = \| |\bar{T}| \| = \sup\{ \| |\bar{T}|(f)\|, \|f\| \leq 1, f \geq 0\}$$

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with $|\bar{T}|(f) = \sup\{|\bar{T}(g)|, |g| \leq f\}$. Then $\|g\| \leq \|f\| \leq 1$ and $|\bar{T}(g)| \leq y_0''$.

So $|\bar{T}|(f) \leq y_0''$ and $\|\bar{T}\|_r \leq \|y_0''\| \leq L = \lambda_1(T'') = \lambda_1(T)$.

So that,

$$\lambda_\infty(T) = \lambda_\infty(\bar{T}) \leq \|\bar{T}\|_r \leq \lambda_1(T) \leq \lambda_\infty(T),$$

and all are the same.

As a consequence, if E is an AM space, 1 l. s. operators from E to X and ∞ l. s., or majorizing, operators are the same for any Banach lattice X. This result somehow reminds us of the result of Pietsch ([10]) which asserts that for any $C(K)$ space absolutely summing and dominated operators coincide; and, as in that case, it is not possible to extend the result to operators on spaces of vector valued continuous functions, as we shall see later.

If X has the property P ([12]), i. e. there exists a continuous positive (and contractive) projection from X'' to X, any 1 l. s. operator from an AM space to X is regular, and $\lambda_1(T) = \lambda_\infty(T) = \|T\|_r$. Lattices with this property are, for example, those which do not contain c_0.

Reasoning in a similar way as in the first part of the theorem, for $1 < p < \infty$ we have:

PROPOSITION 4. *Let K be a compact Hausdorff space, X a Banach lattice and T an operator from $C(K)$ to X. Let m be the representing measure of T, and let $1 < p < \infty$. If T is p-lattice summing, there exist a constant $L > 0$ such that for every finite family of Borel sets, $A_1, \ldots, A_n$, pairwise disjoint,*

$$\Big\|\Big(\sum_{i=1}^{n} |m(A_i)|^p\Big)^{\frac{1}{p}}\Big\|_{X''} \leq L$$

Another example of the particular behaviour of $C(K)$ is the following:

If X and Y are Banach lattices, Krivine's generalization of Grothendieck's inequality for lattices shows that for any operator $T: X \longrightarrow Y$ and any finite set $\{x_1, \ldots, x_n\}$ in X

$$\Big\|\Big(\sum_{i=1}^{n} |Tx_i|^2\Big)^{\frac{1}{2}}\Big\|_Y \leq K_G \cdot \|T\| \cdot \Big\|\Big(\sum_{i=1}^{n} |x_i|^2\Big)^{\frac{1}{2}}\Big\|_X$$

where K_G is Grothendieck's constant ([8]).

So that we have readily:

PROPOSITION 5. *Any operator from a $C(K)$ space to a Banach lattice is 2 l. s.*

Let us show now some results for operators defined on spaces of vector valued contінous functions $C(K, E)$ (space of E-valued continuous functions on K, where E is a Banach space, with the supremum norm).

First of all, as it occurs in the scalar case, an operator T from $C(K, E)$ to a Banach lattice X is p l. s. for some p, $1 \leq p \leq \infty$, if and only if the restriction $\bar{T} = T''|_{B(\beta(K),E)}$ is p l. s. too, where $B(\beta(K), E)$ is the space of functions which are uniform limit of E-valued measurable simple functions on K.

We shall use also a general result on p l.s. operators:

LEMMA 3. *Let E be a Banach space, X be a Banach lattice, and T an operator from E to X. If there exists a dense subspace F of E such that the restriction $T|_F$ is p l. s., then T is p l. s. too.*

PROOF. Let $x_1, \ldots x_n$ be in E, and let $K = \omega_p((x_i)_{i=1}^n)$. We choose for each i, $1 \leq i \leq n$, $z_i \in F$ such that

$$\|z_i - x_i\| \leq \frac{K}{(1+\|T\|)2^i}$$

Then,

$$\begin{aligned}\Big\|\Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac{1}{p}}\Big\| &\leq \Big\|\Big(\sum_{i=1}^n |Tx_i - Tz_i|^p\Big)^{\frac{1}{p}}\Big\| + \Big\|\Big(\sum_{i=1}^n |Tz_i|^p\Big)^{\frac{1}{p}}\Big\| \\ &\leq \Big\|\sum_{i=1}^n |Tx_i - Tz_i|\Big\| + \Big\|\Big(\sum_{i=1}^n |Tz_i|^p\Big)^{\frac{1}{p}}\Big\| \\ &\leq \sum_{i=1}^n \|Tx_i - Tz_i\| + \lambda_p(T|_F)\cdot\omega_p((z_i)) \\ &\leq K + \lambda_p(T|_F)\cdot\omega_p((z_i))\end{aligned}$$

Now, for any $x' \in B_{F'}$, let $\hat{x}'$ be an extension to E with the same norm. Then

$$\begin{aligned}\Big(\sum_{i=1}^n |< z_i, x' >|^p\Big)^{\frac{1}{p}} &= \Big(\sum_{i=1}^n |< z_i, \hat{x}' >|^p\Big)^{\frac{1}{p}} \\ &\leq \Big(\sum_{i=1}^n |< z_i - x_i, \hat{x}' >|^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^n |< x_i, \hat{x}' >|^p\Big)^{\frac{1}{p}} \\ &\leq \sum_{i=1}^n |< z_i - x_i, \hat{x}' >| + \Big(\sum_{i=1}^n |< x_i, \hat{x}' >|^p\Big)^{\frac{1}{p}} \\ &\leq \frac{K}{1+\|T\|} + K \leq 2K \leq 2\omega_p((x_i))\end{aligned}$$

And thus,

$$\Big\|\Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac{1}{p}}\Big\| \leq [1 + 2\lambda_p(T|_F)]\cdot\omega_p((x_i))$$

so T is p l.s. ∎

Now, we can prove the following theorem, which gives sufficient conditions on the representing measure of T to be p l. s.

THEOREM 6. *Let K be a compact Hausdorff space, E be a Banach space, X a Banach lattice and $T: C(K,E) \longrightarrow X$ an operator. Let m be the representing measure of T, and let be $1 \leq p < \infty$. Assume that*

1. *For every Borel subset A of K the operator $m(A)$ from E to X'' is p-lattice summing.*
2. *The variation of m with the norm λ_p is finite. That is, there is a constant $L > 0$ such that for every finite family of pairwise disjoint Borel subsets $B_1, \ldots, B_n$ in $\beta_0(K)$*

$$\sum_{i=1}^n \lambda_p(m(B_i)) \leq L$$

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Then T is p-lattice summing.

PROOF. Using the remarks above, it is enough to prove that $T''|_{S(\beta(K),E)}$ is p l. s. So, take $f_1, \ldots, f_k$ in $S(\beta(K), E)$. There is a finite family of pairwise disjoint Borel subsets of K, $B_1, \ldots, B_n$, and a finite family of vectors in E, $\{x_{ij}\}$, $1 \leq i \leq k$, $1 \leq j \leq n$, such that

$$f_i = \sum_{j=1}^{k} x_{ij}\chi(B_j), \text{ for all } 1 \leq i \leq k$$

Then

$$\Big\|\Big(\sum_{i=1}^{k} |T''f_i|^p\Big)^{\frac{1}{p}}\Big\| = \Big\|\Big(\sum_{i=1}^{k} |\sum_{j=1}^{n} m(B_j)x_{ij}|^p\Big)^{\frac{1}{p}}\Big\|$$

where

$$\begin{aligned}\Big(\sum_{i=1}^{k} |\sum_{j=1}^{n} m(B_j)x_{ij}|^p\Big)^{\frac{1}{p}} &= \sup\Big\{ \sum_{i=1}^{k} a_i\big(\sum_{j=1}^{n} m(B_j)x_{ij}\big),\ a = (a_i)_{i=1}^{k} \in B_{p'} \Big\}\\ &= \sup\Big\{ \sum_{j=1}^{n}\sum_{i=1}^{k} a_i m(B_j)x_{ij},\ a = (a_i)_{i=1}^{k} \in B_{p'} \Big\}\\ &\leq \sum_{j=1}^{n}\big(\sum_{i=1}^{k} |m(B_j)x_{ij}|^p\big)^{\frac{1}{p}}\end{aligned}$$

Thus

$$\begin{aligned}\Big\|\Big(\sum_{i=1}^{k} |T''f_i|^p\Big)^{\frac{1}{p}}\Big\| &\leq \Big\|\sum_{j=1}^{n}\Big(\sum_{i=1}^{k} |m(B_j)x_{ij}|^p\Big)^{\frac{1}{p}}\Big\|\\ &\leq \sum_{j=1}^{n} \big\|\big(\sum_{i=1}^{k} |m(B_j)x_{ij}|^p\big)^{\frac{1}{p}}\big\|\\ &\leq \sum_{j=1}^{n} \lambda_p(m(B_j)) \cdot \omega((x_{ij})_{i=1}^{k})\end{aligned}$$

Now, for each $j \in \{1, \ldots, n\}$ fixed, taken $t_j \in B_j$ we have

$$\begin{aligned}\omega_p((x_{ij})_{i=1}^{k}) &= \sup\Big\{ \big(\sum_{i=1}^{k} | < x_{ij}, x' > |^p\big)^{\frac{1}{p}},\ x' \in B_{E'} \Big\}\\ &= \sup\Big\{ \big(\sum_{i=1}^{k} | < f_i, x' \circ \delta_{t_j} > |^p\big)^{\frac{1}{p}},\ x' \in B_{E'} \Big\}\\ &\leq \omega_p((f_i)_{i=1}^{k})\end{aligned}$$

As a consequence

$$\Big\|\Big(\sum_{i=1}^{k} |T''f_i|^p\Big)^{\frac{1}{p}}\Big\| \leq L \cdot \omega_p((f_i)_{i=1}^{k})$$

from where we deduce that $T''|_{S(\beta(K),E)}$ is p l. s. Then $\hat{T} = T''|_{B(\beta(K),E)}$ and T are p l. s. too. ∎

REMARKS. 1—For $p \neq 2$, there are examples that prove that there are operators defined in $C(K)$ spaces which are not p l. s. Concretely, in [9] Nielsen and Szulga, for $1 < p < 2$, construct operators defined on c_0 , with image in $L^1[0,1]$, which are not p l. s.: knowing that for $1 < p < 2$, ℓ^p is not of p-stable type, given a sequence $(f_n)_n$ of p-stable independent randon variables in $[0,1]$, with the Lebesgue measure, there is a sequence $(a_n)_n$ in ℓ^p such that $\sum_{i=1}^{\infty} |a_n f_n(t)|^p$ diverges in a set with strictly positive measure. We can assume that $a_n \geq 0$ and $\|f_n\| = 1$ for all $n \in N$, so that the map

$$\ell^p \longrightarrow L^1[0,1]: (b_n)_n \longrightarrow \sum b_n f_n$$

is an isometry. If we define

$$T: c_0 \longrightarrow L^1[0,1] : T(e_n) = a_n f_n$$

(where $(e_n)_n$ is the canonical basis of c_0), T is not p l. s., since given any $m \in N$, $\omega_p((e_n)_{n=1}^m) \leq 1$ but

$$\Big\|\Big(\sum_{n=1}^m a_n^p |f_n|^p\Big)^{\frac{1}{p}}\Big\|_{L^1[0,1]} = \int_0^1 \Big(\sum_{n=1}^m |a_n f_n|^p\Big)^{\frac{1}{p}} dt$$

diverges when m tends to infinity.

As for any metrizable infinite compact space K, $C(K)$ contains a complemented subspace isomorphic to c_0 , at least in these cases it is possible to define operators on $C(K)$ which are not p l. s.

For $p > 2$, in the same paper it is shown that, as a consequence of Kwapien's theorem ([Kwapien]), there are operators from ℓ^∞ to ℓ^p which are not p l. s..

2—The results obtained for $C(K)$ spaces are not true in general for $C(K,E)$ spaces, when E is a Banach space.

In fact, if E is a Banach lattice and every positive operator from $C(K,E)$ to any Banach lattice is 2 l. s., this is equivalent to saying that the identity map $I: E \longrightarrow E$ is 2 l. s. So, taking $E = \ell^p$, with $1 < p < 2$, there has to be an operator $C(K,E) \longrightarrow \ell^p$ which is not 2 l. s.

Neither is it true in general that on $C(K,E)$ 1 l. s. operators and ∞ l. s. coincide: just choose a Banach space E, a Banach lattice X and an operator $T: E \longrightarrow X$ that is 1 l. s. but not ∞ l. s. and extend T to $C(K,E)$ by means of a continuous projection from $C(K,E)$ to E.

3—The converse of theorem 5 is not true: It is obvious that when T is a p l. s. operator from $C(K,E)$ to X, for every Borel subset A of K the operator $m(A)$ from E to X'' is p l. s., since $m(A)(x) = T''(\chi(A)\cdot x)$ for all $x \in E$. But m may not satisfy condition 2: consider the operator T from $C(N^*, \ell^1)$ to c_0 (N^* is the Alexandroff compactification of N) associated with the measure m defined by $m(A)(x) = \sum_{n \in A} x_n e_n \in \ell^\infty$ for any $x = (x_n)_n$ in ℓ^1 and any subset A of N not empty, and $m(\emptyset) = 0 = m(\infty)$. T is p-lattice summing for every p, $1 \leq p \leq \infty$, since ℓ^∞ is an AM space (see [12]), but it does not satisfy condition 2.

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