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ANOTHER CLASS OF CYCLICLY EXTENSIBLE AND REDUCIBLE PROPERTIES

BY

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ABSTRACT. A space S has property P_{-1} if S is nonempty. For $n > -1$, S has property P_n if it is locally connected, has property P_{n-1} and if whenever it is written as a union, $S = A \cup B$ where each of A and B is closed and has property P_{n-1}, then $A \cap B$ also has property P_{n-1}. The purpose of this paper is to establish that for locally compact spaces, each of the properties P_n is both cyclicly extensible and reducible.

Introduction. In [3], it was shown that k-coherence is, under certain conditions, both cyclicly extensible and reducible. In this paper, we define a weaker class of properties similar to properties defined by Vietoris in 1932 [14]. We show that for all connected, locally connected and locally compact Hausdorff spaces, these properties are all cyclicly extensible and reducible.

1. PRELIMINARIES. Throughout this paper, M denotes a connected and locally connected Hausdorff space.

A point p of M is an *end point* of M if p has a neighbourhood base of open sets having singleton boundaries. p is a *cut* point of M if $M - p$ is disconnected. Two points a and b of M are said to be *conjugate in M* if and only if no point of M separates a and b in M. $E(a,b)$ denotes the collection of all points of M which separate a and b in M and "$<$" denotes the *cut point order* on $E(a,b) \cup \{a,b\}$ – i.e. $a < x$, $x < b$ for all $x \in E(a,b)$; $a < b$, and if $x, y \in E(a,b)$ then $x < y$ if and only if $x \in E(a,y)$. (It has been established, ([15 and [2]), that $E(a,b) \cup \{a,b\}$ is closed and compact and that the subspace topology on $E(a,b) \cup \{a,b\}$ is the order topology relative to the cut point order.) An A-set of M is a closed subset of M such that every component of $M - A$ has singleton boundary. If a, $b \in M$, then $C(a,b) = \cap \{A$: A is an A-set of M and a, $b \in A\}$ and is called the *cyclic chain in M from a to b*. If $E \subset M$, then E is an E_0-set of M if E is nondegenerate, connected, contains no cut point of itself and is maximal with respect to these properties. A *cyclic element* of M is a subset of M which is an E_0-set of M or is a singleton cut point or end point of M. A property is *cyclicly reducible* if whenever a space M has the property, then every cyclic element of M has the property; a property is *cyclicly extensible* if whenever every cyclic element of M has the property, then M does also.

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2. Definition. *A space has property P_{-1} if S is non-empty. S has property P_n, $n > -1$ if S is locally connected, has property P_{n-1} and whenever $S = A \cup B$ is a division of S into closed sets each having property P_{n-1}, then $A \cap B$ also has property P_{n-1}. We note that P_0 is connectedness plus local connectedness and P_1 is unicoherence plus local connectedness.*

3. Notation. In the rest of this paper, if $S \subset A$ for some A-set A, then S^* denotes the union of S with all components C of $M - A$ such that the boundary of C belongs to S.

4. Lemma. *If M is locally compact, then for each nonnegative integer n, if M has property P_{n-1}, then M has property P_n if and only if every cyclic element of M has property P_n.*

Proof. We state an induction hypothesis $I(n)$: If M is locally compact, then

(a) if M has property P_n, then every cyclic element of M has property P_n;

(b) if M has property P_{n-1} and every cyclic element of M has property P_n; then M has property P_n;

(c) if M has property P_n, then every A-set of M has property P_n; and if A is an A-set of M and Z is any locally compact subset of M such that $A \cap Z \neq \phi$ and Z has property P_n, then $A \cap Z$ has property P_n;

(d) if M has property P_n, A is an A-set of M, and $A = S \cup T$ is a division of M into closed sets each having property P_n, and if $L = S^*$ and $N = T \cup (A - S)^*$, then $M = L \cup N$ is a division of M into closed sets each having property P_n.

$I(0) - a$ and b are immediate and $I(0) - c$ and d have been established elsewhere, ([5], Theorem 5.3 and [3], Theorem 3, respectively).

Assume that $I(n - 1)$ has been established, $n \geq 1$. Suppose that M has property P_n and that E is an E_0-set of M. Then M has property P_{n-1}, so by $I(n - 1)$, E has property P_{n-1}. Suppose $E = S \cup T$ is a division of E into closed sets each having property P_{n-1}. Let $L = S^*$, $N = T \cup (E - S)^*$. By $I(n - 1)$, $M = L \cup N$ is a division of M into closed sets each having property P_{n-1}. Since M has property P_n, $L \cap N$ has property P_{n-1}. Further, it was established in [3], (Theorem 3) that $L \cap N = S \cap T$. Thus E has property P_n, and it follows that every cyclic element of M has property P_n. Thus $I(n - 1)$ implies $I(n) - a$.

Now assume that M has property P_{n-1} and that every cyclic element of M has property P_n. Suppose $M = S \cup T$ is a division of M into closed sets each having property P_{n-1}. Then each of S and T is connected and locally connected and $S \cap T$ is locally compact. We now show that $S \cap T$ is connected and locally connected.

Suppose that $S \cap T = W \cup Z$ where W and Z are disjoint closed sets. If E is an E_0-set of M, then $E = (E \cap S) \cup (E \cap T)$ and since S and T are each connected, it follows from Theorem 5.3 of [6] that $E \cap S$ and $E \cap T$ are each connected and locally connected. Since E has property P_n, $E \cap S \cap T$ is connected. Thus $E \cap S \cap T \subset W$ or $E \cap S \cap T \subset Z$. Let $w \in W$, $z \in Z$. Since no E_0-set of M meets both W and Z, $E(w, z) \neq \phi$. Since w, $z \in S \cap T$, and S and T are each connected, $E(w, z) \subset S \cap$

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T. Let t_1 be the last point in $(E(w, z) \cup \{w, z\}) \cap W$ and t_2 be the first point in $(E(w, z) \cup \{w, z\}) \cap Z$. Then $t_1 \neq t_2$ and t_1 and t_2 are conjugate in M. It follows ([5], Theorem 6.2) that $C(t_1, t_2)$ is an E_0-set of M that meets both W and Z. Since this is a contradiction, $S \cap T$ is connected.

Suppose now that $S \cap T$ is not locally connected. Then there is an open set of M and a point $p \in S \cap T$ such that p lies on a continuum of convergence $D = \lim_\alpha D_\alpha$, where for each α, D_α is *the closure* of a component C_α of $O \cap S \cap T$ and the components of $O \cap S \cap T$ containing D and D_α are distinct, ([1], Theorem 8). Since D is a continuum of convergence of M, there is an E_0-set E of M such that $D = \lim_\alpha E \cap D_\alpha$ ([2], Theorem 5.13). This implies that $E \cap S \cap T$ is nondegenerate. Now $E \cap S$ and $E \cap T$ are connected and locally connected and for each α, $E \cap D_\alpha$ is contained in a component of $O \cap S \cap T \cap E$ distinct from the component of $O \cap S \cap T \cap E$ containing $E \cap D$. But this implies that $E \cap S \cap T$ is not locally connected, which is a contradiction since E has property P_n. Thus $S \cap T$ is locally connected.

Now let E be an E_0-set of $S \cap T$. $E \subset \tilde{E}$ for some E_0-set $\tilde{E}$ of M. Since M has property P_{n-1}, $I(n - 1)$ implies that each of $\tilde{E}$, $S \cap \tilde{E}$ and $T \cap \tilde{E}$ has property P_{n-1}. Further since $\tilde{E}$ has property P_n, $\tilde{E} \cap S \cap T$ has property P_{n-1}. Now E is an E_0-set of $\tilde{E} \cap S \cap T$, so by $I(n - 1)$, E has property P_{n-1}. It follows that evey cyclic element of $S \cap T$ has property P_{n-1}, so again by the induction hypothesis, $S \cap T$ has property P_{n-1}. Thus M has property P_n and $I(n) - b$ is established.

Now assume that M has property P_n and that A is an A-set of M. By $I(n - 1)$, A has property P_{n-1} (since M does). If E is a cyclic element of A, then E is a cyclic element of M. It follows, since $I(n - 1)$ implies $I(n) - a$, that E has property P_n. Thus every cyclic element of A as property P_n and since $I(n - 1)$ implies $I(n - 1) - b$, A has property P_n. Further, if Z is any locally compact subset of M such that $A \cap Z \neq \phi$ and Z has property P_n, then since $A \cap Z$ is an A-set of Z, it follows from what we have just proved that $A \cap Z$ has property P_n. Thus $I(n) - c$ is proved.

Finally, assume that M has property P_n, that A is an A-set of M and that $A = S \cup T$ is a division of A into closed sets each having property P_n. Let $L = S^*$, $N = T \cup (A - S)^*$. By $I(n - 1)$, L and N each have property P_{n-1}. Let E be an E_0-set of L. Then either $E \subset A$ or $E \subset \bar{C}$ for some component C of $M - A$ such that $C \subset L$. If the latter, then E is an E_0-set of $\bar{C}$ and therefore of M, and it follows from what has already been proved that E has property P_n. If $E \subset A$, let $\tilde{E}$ be the E_0-set of M such that $E \subset \tilde{E}$. Then $\tilde{E}$ has property P_n. Further, since S is a locally compact subset of M and has property P_n, it again follows from what has already been proved that $S \cap \tilde{E}$ has property P_n. Since E is an E_0-set of $S \cap E$, E has property P_n. Thus every cyclic element of L has property P_n, so L has property P_n. The case for N is similar, so $I(n) - d$ is established and the Lemma is proved.

The following corollary is immediate.

5. Corollary. *If M is locally compact, then for every nonnegative integer n, if M has property P_n, and A is an A-set of M, then*

(a) A has property P_n;

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(b) if Z is a locally compact subset of M that meets A and has property P_n, then $A \cap Z$ has property P_n;

(c) if $A = S \cup T$ is a division of A into closed sets each having property P_n and $L = S^*$, $N = T \cup (A - S)^*$, then $M = L \cup N$ is a division of M into closed sets having property P_n.

6. THEOREM. *If M is locally compact and n is an integer ≥ -1, then M has property P_n if and only if every cyclic element of M has property P_n.*

PROOF. If $n = -1$ or $n = 0$, the result is immediate, so we assume $n > 0$. If M has property P_n, then from our Lemma, every cyclic element of M has property P_n. Suppose that every cyclic element of M has property P_n and that M does not. Since M is connected and locally connected, M has property P_0. Let n^* be the first integer ($n^* > 0$) such that M does not have property P_{n^*}. Then $0 < n^* \leq n$ and M has property P_{n^*-1}. Since every cyclic element of M has property P_n, evey cyclic element of M has property P_{n^*} so from our Lemma, M has property P_{n^*}, contrary to the definition of n^*. The theorem follows.

The following is trivial.

7. COROLLARY. *Every dendron has property P_n for every nonnegative integer n.*

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