SUMS OF THE DIVISOR FUNCTION

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1. Introduction. Shapiro and Warga **(2)** have proved in an elementary way that all large integers are expressible as the sum of at most 20 primes. In so doing, they proved that

$$(1)\qquad \sum_{\substack{n\leqslant x\\(n,u)=1\\n\,\text{square-free}}} \frac{\tau(n)}{n} = \frac{1}{2}\cdot\prod_{p\mid u}\frac{p}{p+2}\cdot\prod_{p}\frac{(p-1)^2(p+2)}{p^3}\cdot\log^2 x$$

$$+O(\log x\cdot\log\log xu)+O((\log\log 3u)^2),$$

as $x\to\infty$, where u is a positive square-free integer,

$$\tau(n)=\sum_{d\mid n}1,$$

and all the constants implied by O are absolute, here and throughout this paper (except for dependence on ϵ, when it occurs). We shall sum $\tau(n)$ itself over the same range and derive a refined form of (1): for every $\epsilon>0$,

$$(2)\qquad \sum_{\substack{n\leqslant x\\(n,u)=1\\\mu(n)\neq 0}}\frac{\tau(n)}{n}=\frac{1}{2}A\log^2 x+(A+B)\log x+C+O\left(x^{-\frac{1}{2}+\epsilon}\exp\frac{c\sqrt{\log 3u}}{\log\log 3u}\right)$$

for some constant $c>0$, where

$$(2')\qquad \begin{cases} A=\displaystyle\prod_{p\mid u}\frac{p}{p+2}\cdot\prod_{p}\frac{(p-1)^2(p+2)}{p^3},\\ B=A\left\{(2\gamma-1)+2\displaystyle\sum_{p\mid u}\frac{\log p}{p-1}+6\sum_{p\nmid u}\frac{(p-1)\log p}{p^2(p+2)}\right\},\\ C=C(u)=O\left(\exp\dfrac{c\sqrt{\log 3u}}{\log\log 3u}\right),\end{cases}$$

and γ denotes Euler's constant. Note that (2) gives better error terms than (1), for fixed u as $x\to\infty$, because $A=O(1)$ and $B=O(\log\log 3u)$.

2. LEMMA 1. *Let*

$$F_d(x)=F_d^u(x)=\sum_{\substack{n\leqslant x\\(n,u)=1\\d\mid n}}\tau(n).$$

Then for any prime p not dividing du we have, for $\nu=1,2,\ldots,$

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$$F_{dp^\nu}(x) = (\nu+1)F_d\left(\frac{x}{p^\nu}\right) - \nu F_d\left(\frac{x}{p^{\nu+1}}\right). \tag{3}$$

Proof. Since the two sides of (3) are step functions which increase only when $dp^\nu | x$, and since they are equal at $x = 0$, it suffices to verify that they have the same increment at $x = dp^{\nu+\rho}y$, with $\rho \geqslant 0$ and $p \nmid y$. For $\rho = 0$, the requirement is that

$$\tau(p^\nu \cdot dy) = (\nu+1)\tau(dy) = \tau(p^\nu)\cdot\tau(dy),$$

which is true. For $\rho \geqslant 1$, we require

$$\tau(dy)\tau(p^{\nu+\rho}) = (\nu+1)\tau(dy)\tau(p^\rho) - \nu\tau(dy)\tau(p^{\rho-1}),$$

i.e.

$$\nu+\rho+1 = (\nu+1)(\rho+1) - \nu\rho,$$

which is true.

This formula is interesting in itself. It can also be derived by inversion of a functional equation. We need only:

$$F_{p_1{}^2\ldots p_n{}^2}(x) = 3F_{p_1{}^2\ldots p_{n-1}{}^2}\left(\frac{x}{p_n{}^2}\right) - 2F_{p_1{}^2\ldots p_{n-1}{}^2}\left(\frac{x}{p_n{}^3}\right) \qquad (\text{for } p_n \nmid u) \tag{3'}$$

and

$$\sum_{\substack{n\leqslant x\\ n\equiv 0(p_1\ldots p_n)}} \tau(n) = H_{p_1\ldots p_n}(x) = 2H_{p_1\ldots p_{n-1}}\left(\frac{x}{p_n}\right) - H_{p_1\ldots p_{n-1}}\left(\frac{x}{p_n{}^2}\right). \tag{3''}$$

By (3″) we expect to obtain $H_{p_1\ldots p_n}$ in terms of H_1, but all we want is an asymptotic formula. Hence we recall that **(1)**:

$$H_1(x) = \sum_{n\leqslant x}\tau(n) = x\log x + (2\gamma-1)x + O(\sqrt{x}), \tag{4}$$

where γ is Euler's constant.

LEMMA 2. *For square-free* d,

$$H_d(x) = \sum_{\substack{n\leqslant x\\ d|n}}\tau(n) = \prod_{p|d}\frac{2p-1}{p^2}$$
$$\cdot\left\{x\log x + x\left(2\gamma - 1 - 2\sum_{p|d}\frac{p-1}{2p-1}\log p\right)\right\} + O\left(3^{\nu(d)}\sqrt{\frac{x}{d}}\right),$$

where
$$\nu(d) = \sum_{p|d} 1.$$

Proof. It is evident from (3″) and (4) that some formula of the type

$$H_d(x) = a(d)x\log x + b(d)x + R_d(x)$$

must hold, with $R_d(x)$ of order $\sqrt{x}$. Substituting this in (3″) gives: for $p \nmid d$,

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$$a(dp)x\log x + b(dp)x + R_{dp}(x) = 2\left\{a(d)\frac{x}{p}\log\frac{x}{p} + b(d)\frac{x}{p} + R_d\left(\frac{x}{p}\right)\right\}$$
$$-\left\{a(d)\frac{x}{p^2}\log\frac{x}{p^2} + b(d)\frac{x}{p^2} + R_d\left(\frac{x}{p^2}\right)\right\}.$$

Hence we choose $a(d)$ and $b(d)$ so that $a(1) = 1$, $b(1) = 2\gamma - 1$, and

$$a(dp) = a(d)\left(\frac{2}{p} - \frac{1}{p^2}\right),$$

$$b(dp) = a(d)\left\{-2\frac{\log p}{p} + \frac{2\log p}{p^2}\right\} + b(d)\left(\frac{2}{p} - \frac{1}{p^2}\right).$$

Hence,

$$a(d) = \prod_{p|d}\frac{2p-1}{p^2} \quad\text{and}\quad \frac{b(dp)}{a(dp)} = \frac{b(d)}{a(d)} - \frac{2(p-1)}{p^2}\log p\cdot\frac{p^2}{2p-1},$$

and

$$b(d) = \left(2\gamma - 1 - 2\sum_{p|d}\frac{p-1}{2p-1}\log p\right)a(d).$$

When $R_d(x)$ is defined as $H_d(x) - a(d)x\log x - b(d)x$, we get

$$R_{dp}(x) = 2R_d\left(\frac{x}{p}\right) - R_d\left(\frac{x}{p^2}\right).$$

Thus, if

$$|R_d(x)| < K\cdot 3^{\nu(d)}\sqrt{\frac{x}{d}},$$

it follows that

$$|R_{dp}(x)| < 3\cdot K\cdot 3^{\nu(d)}\sqrt{\frac{x}{pd}} = K\cdot 3^{\nu(pd)}\sqrt{\frac{x}{pd}}.$$

Naturally, if the elementary $O(\sqrt{x})$ in the divisor problem is replaced by $O(x^\theta)$ for some $\theta < \frac{1}{2}$, this improves $\sqrt{(x/d)}$ to $(x/d)^\theta$. However, this will not lead to a better result in (2) by our method.

LEMMA 3. *For square-free* u,

$$F_1(x) = \sum_{\substack{n\leqslant x\\(n,u)=1}}\tau(n) = \left(\frac{\phi(u)}{u}\right)^2\left\{x\log x + x\left(2\gamma - 1 + 2\sum_{p|u}\frac{\log p}{p-1}\right)\right\}$$
$$+ O\left(\sum_{d|u}3^{\nu(d)}\sqrt{\frac{x}{d}}\right).$$

Proof.

$$F_1(x) = \sum_{n\leqslant x} \tau(n) \sum_{d\mid(n,u)} \mu(d)$$
$$= \sum_{d\mid u} \mu(d) \sum_{\substack{d\mid n\\ n\leqslant x}} \tau(n)$$
$$= \sum_{d\mid u} \mu(d)\left\{x\log x + x\left(2\gamma - 1 - 2\sum_{p\mid d}\frac{p-1}{2p-1}\log p\right)\right\}\prod_{p\mid d}\frac{2p-1}{p^2} + O\left(\sum_{d\mid u} 3^{\nu(d)}\sqrt{\frac{x}{d}}\right),$$

by use of Lemma 2.

Now

$$\sum_{d\mid u}\mu(d)\prod_{p\mid d}\frac{2p-1}{p^2} = \prod_{p\mid u}\left(1-\frac{2p-1}{p^2}\right) = \prod_{p\mid u}\left(\frac{p-1}{p}\right)^2 = \left(\frac{\phi(u)}{u}\right)^2.$$

Thus, the coefficient of x is the sum of $(2\gamma-1)(\phi(u)/u)^2$ and

$$-2\sum_{d\mid u}\mu(d)a(d)\sum_{p\mid d}\frac{p-1}{2p-1}\log p$$
$$= -2\sum_{p\mid u}\frac{p-1}{2p-1}\log p\sum_{\substack{d\mid u\\ p\mid d}}\mu(d)a(d)$$
$$= 2\sum_{p\mid u}\frac{p-1}{2p-1}\log p\cdot a(p)\sum_{t\mid u/p}\mu(t)a(t)$$
$$= 2\sum_{p\mid u}\frac{p-1}{p^2}\log p\cdot\left(\frac{\phi(u/p)}{u/p}\right)^2$$
$$= 2\left(\frac{\phi(u)}{u}\right)^2\sum_{p\mid u}\frac{\log p}{p-1},$$

as required. As in **(2)**, we note that it is easy to show that

$$\sum_{p\mid u}\frac{\log p}{p-1} \leqslant O(1) + \log p_{\nu(u)} = O(\log\log 3u),$$

since

$$\log u \geqslant \sum_{p\leqslant p_{\nu(u)}}\log p = (1+o(1))p_{\nu(u)},$$

by the prime-number theorem. We could use Tchebychef's inequality instead. As to the coefficient of $\sqrt{x}$ in the O-term, we have

$$S = \sum_{d\mid u}\frac{3^{\nu(d)}}{\sqrt{d}} = \prod_{p\mid u}\left(1+\frac{3}{\sqrt{p}}\right) \leqslant \prod_{p\leqslant p_{\nu(u)}}\left(1+\frac{3}{\sqrt{p}}\right)$$
$$\leqslant \prod_{p\leqslant(1+o(1))\log u}\left(1+\frac{3}{\sqrt{p}}\right).$$

Hence,

$$\log S \leqslant 3\sum_{p\leqslant(1+o(1))\log u}\frac{1}{\sqrt{p}}\cdot(1+o(1)).$$

By elementary means one can show that

$$\sum_{p \leqslant x} \frac{1}{\sqrt{p}} = O\left(\frac{\sqrt{x}}{\log x}\right),$$

while the prime-number theorem gives

$$\sum_{p \leqslant x} \frac{1}{\sqrt{p}} = \frac{\sqrt{x}}{\log x}(1 + o(1)).$$

Hence,

$$S = O\left(\exp\left\{\frac{c\sqrt{\log 3u}}{\log\log 3u}\right\}\right), \tag{5}$$

where c can be taken as $3 + o(1)$ for large u.

LEMMA 4. *For square-free d, we have*

$$F_{d^2}(x) = \alpha(d)x \log x + \beta(d)x + R_d(x),$$

where

$$\alpha(d) = \left(\frac{\phi(u)}{u}\right)^2 \cdot \prod_{p \mid d} \frac{3p - 2}{p^3},$$

$$\beta(d) = \alpha(d)\left\{2\gamma - 1 + 2\sum_{p \mid u} \frac{\log p}{p - 1} - 6\sum_{p \mid d} \frac{p - 1}{3p - 2}\log p\right\},$$

and

$$R_d(x) = O\left(\frac{5^{\nu(d)}}{d}\sqrt{x} \cdot S\right),$$

with S as above.

Proof. This follows the lines of Lemma 2, but (3′) replaces (3″). We find that to get $R_{dp}(x) = 3R_d(x/p^2) - 2R_d(x/p^3)$ we need

$$\alpha(dp) = \alpha(d) \cdot (3p - 2)/p^3,$$

and

$$\frac{\beta(dp)}{\alpha(dp)} = \frac{\beta(d)}{\alpha(d)} - 6\frac{p - 1}{3p - 2}\log p,$$

for $p \nmid d$. These give the desired values, if Lemma 3 is used for evaluating $\alpha(1)$ and $\beta(1)$. The estimation of $R_d(x)$ is now similar to that in Lemma 2.

3. Now the sieve process can be used to compute $\sum \tau(n)$.

THEOREM.

$$\sum_{\substack{n \leqslant x \\ (n,u)=1 \\ n \text{ square-free}}} \tau(n) = Ax \log x + Bx + R(x),$$

where

$$R(x) = O\left(x^{\frac{1}{2}+\epsilon}\cdot \exp \frac{c\sqrt{\log 3u}}{\log\log 3u}\right),$$

for every $\epsilon > 0$, *with* A, B, *and* c *as in* (2′).

Proof.

$$\begin{aligned}\sum_{\substack{n\leqslant x\\(n,u)=1\\ \mu(n)\neq 0}} \tau(n) &= \sum_{\substack{n\leqslant x\\(n,u)=1}} \tau(n) \sum_{s^2\mid n} \mu(s)\\ &= \sum_{\substack{s^2\leqslant x\\(s,u)=1}} \mu(s)\cdot F_{s^2}(x)\\ &= \sum_{\substack{s^2\leqslant x\\(s,u)=1}} \mu(s)\{\alpha(s)x\log x + \beta(s)x + R_s(x)\}\\ &= x\log x\left\{\sum_{(s,u)=1} \mu(s)\alpha(s) - \sum_{\substack{s>\sqrt{x}\\(s,u)=1}} \mu(s)\alpha(s)\right\}\\ &\quad + x\left\{\sum_{(s,u)=1} \mu(s)\beta(s) - \sum_{\substack{s>\sqrt{x}\\(s,u)=1}} \mu(s)\beta(s)\right\}\\ &\quad + O\left\{S\cdot\sqrt{x}\cdot \sum_{\substack{s\leqslant\sqrt{x}\\ \mu(s)\neq 0}} \frac{5^{\nu(s)}}{s}\right\};\end{aligned}$$

$$\begin{aligned}\sum_{(s,u)=1} \mu(s)\alpha(s) &= \alpha(1)\cdot\prod_{p\nmid u}\left(1-\frac{3p-2}{p^3}\right) = \left(\frac{\phi(u)}{u}\right)^2 \prod_{p\nmid u} \frac{(p-1)^2(p+2)}{p^3}\\ &= \prod_{p\mid u} \frac{p}{p+2}\cdot\prod_{p} \frac{(p-1)^2(p+2)}{p^3}\\ &= A;\end{aligned}$$

$$\begin{aligned}&\sum_{(s,u)=1} \mu(s)\beta(s)\\ &= \sum_{(s,u)=1} \mu(s)\alpha(s)\left\{(2\gamma-1) + 2\sum_{p\mid u}\frac{\log p}{p-1} - 6\sum_{p\mid s}\frac{p-1}{3p-2}\log p\right\}\\ &= \left(2\gamma - 1 + 2\sum_{p\mid u}\frac{\log p}{p-1}\right)A - 6\sum_{p\nmid u}\frac{p-1}{3p-2}\log p\cdot \sum_{\substack{(s,u)=1\\ p\mid s}} \mu(s)\alpha(s)\\ &= \left(2\gamma - 1 + 2\sum_{p\mid u}\frac{\log p}{p-1}\right)A + 6\sum_{p\nmid u}\frac{p-1}{3p-2}\cdot\frac{3p-2}{p^3}\log p \sum_{(t,up)=1} \mu(t)\alpha(t)\\ &= A\left[2\gamma - 1 + 2\sum_{p\mid u}\frac{\log p}{p-1} + 6\sum_{p\nmid u}\frac{(p-1)\log p}{p^3}\cdot\frac{p}{p+2}\right]\\ &= B.\end{aligned}$$

For square-free s, we have

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$$\alpha(s) < \prod_{p|s} \frac{3}{p^2} = \frac{3^{\nu(s)}}{s^2} < \frac{s^\epsilon}{s^2}$$

for all large s, because

$$\nu(s) = O\left(\frac{\log s}{\log\log s}\right),$$

by **(1**, Theorem 317**)**. Hence

$$\sum_{s>\sqrt{x}} |\alpha(s)\mu(s)| < \sum_{s>\sqrt{x}} \frac{1}{s^{2-\epsilon}}$$
$$= O(x^{-\frac{1}{2}+\frac{1}{2}\epsilon}).$$

From the formula for $\beta(s)$, we have

$$\sum_{s>\sqrt{x}} |\beta(s)\mu(s)| < O(x^{-\frac{1}{2}+\frac{1}{2}\epsilon} \log\log 2u) + \sum_{s>\sqrt{x}} |\mu(s)\alpha(s) \log s|$$
$$= O(x^{-\frac{1}{2}+\frac{1}{2}\epsilon} \log\log 3u).$$

Finally,

$$\sum_{\substack{s\leqslant\sqrt{x} \\ s \text{ square-free}}} \frac{5^{\nu(s)}}{s} = O\left(\sum_{s\leqslant\sqrt{x}} \frac{s^\epsilon}{s}\right) = O(x^{\frac{1}{2}\epsilon}).$$

Thus,

$$R(x) = O\{x^{\frac{1}{2}(1+\epsilon)}(\log x + \log\log 3u + S)\}$$
$$= O\left(x^{\frac{1}{2}+\epsilon}\cdot\exp\frac{c\sqrt{\log 3u}}{\log\log 3u}\right).$$

To derive (2), write $G(x)$ for the sum in the theorem. Then,

$$\sum_{\substack{n\leqslant x \\ (n,u)=1 \\ \mu(n)\neq 0}} \frac{\tau(n)}{n}$$
$$= \int_{1-}^{x} \frac{dG(t)}{t}$$
$$= G(x)/x + \int_1^x \frac{G(t)}{t^2}\,dt$$
$$= A\log x + B + R(x)/x + \int_1^x \{A \log t\cdot t^{-1} + Bt^{-1} + R(t)t^{-2}\}dt$$
$$= \frac{1}{2}A\log^2 x + (A+B)\log x + \int_1^\infty \frac{R(t)}{t^2}\,dt + \frac{R(x)}{x} - \int_x^\infty \frac{R(t)}{t^2}\,dt + B$$
$$= \frac{1}{2}A\log^2 x + (A+B)\log x + B + \int_1^\infty \frac{R(t)}{t^2}\,dt$$
$$+ O\left(x^{-\frac{1}{2}+\epsilon}\cdot\exp\frac{c\sqrt{\log 3u}}{\log\log 3u}\right),$$

which gives (2).

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4. We conclude by observing that our method would enable $\sum \tau(n)$ over the kth-power-free integers to be similarly treated. As to the effect of replacing (1) by (2) in **(2)**, this does not lead to a reduction of the number 20 in their result.

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