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CUBIC SYMMETRIC GRAPHS OF ORDER TWICE AN ODD PRIME-POWER

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Abstract

An automorphism group of a graph is said to be *s-regular* if it acts regularly on the set of s-arcs in the graph. A graph is *s-regular* if its full automorphism group is s-regular. For a connected cubic symmetric graph X of order $2p^n$ for an odd prime p, we show that if $p \neq 5, 7$ then every Sylow p-subgroup of the full automorphism group $\mathrm{Aut}(X)$ of X is normal, and if $p \neq 3$ then every s-regular subgroup of $\mathrm{Aut}(X)$ having a normal Sylow p-subgroup contains an $(s-1)$-regular subgroup for each $1 \leq s \leq 5$. As an application, we show that every connected cubic symmetric graph of order $2p^n$ is a Cayley graph if $p > 5$ and we classify the s-regular cubic graphs of order $2p^2$ for each $1 \leq s \leq 5$ and each prime p, as a continuation of the authors' classification of 1-regular cubic graphs of order $2p^2$. The same classification of those of order $2p$ is also done.

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1. Introduction

Throughout this paper, we consider a finite graph without loops or multiple edges. Each edge of a graph X gives rise to a pair of opposite arcs and we denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ the vertex set, the edge set and the full automorphism group of X, respectively.

A transitive permutation group P on a set Ω is said to be *regular* if the stabilizer P_α of α in P is trivial for each $\alpha \in \Omega$. Let G be a finite group and S a subset of G such that $1 \notin S$ and $S = S^{-1}$. The *Cayley graph* $X = \mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set $V(X) = G$ and edge set $E(X) = \{(g, sg) \mid g \in G, s \in S\}$.

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By definition, $\mathrm{Cay}(G, S)$ is connected if and only if S generates G. The Cayley graph $\mathrm{Cay}(G, S)$ is vertex-transitive since it admits the *right regular representation* $R(G)$ of G, the acting group of G by right multiplication, as a subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Clearly, $R(G)$ acts regularly on the vertex set. Furthermore, the group $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is also a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Actually, $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A graph X is isomorphic to a Cayley graph on a group G if and only if its automorphism group $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on the vertex set (see [2, Lemma 16.3]).

An *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive or symmetric. A graph X is said to be *s-regular* if for any two s-arcs in X, there is a unique automorphism of X mapping one to the other. In other words, the automorphism group $\mathrm{Aut}(X)$ acts regularly on the set of s-arcs in X. From the definition, one may show that an s-regular graph with regular valency must be connected. Tutte [25, 26] showed that every finite connected cubic symmetric graph is s-regular for some s, and this s can be at most five. A subgroup of the full automorphism group of a graph is said to be s-regular if it acts regularly on the set of s-arcs in the graph. Clearly, a 0-regular subgroup acts regularly on the vertex set of the graph.

The study of s-regular cubic graphs has received considerable attention for more than 50 years and many families of such graphs have been constructed. The first 1-regular cubic graph was constructed by Frucht [15] and later Miller [19] constructed an infinite family of 1-regular cubic graphs of order $2p$, where $p \ge 13$ is a prime congruent to 1 modulo 3. Three infinite families of 1-regular cubic graphs with unsolvable automorphism groups were constructed in [7, 10] and infinitely many 1-regular cubic graphs as regular coverings of small graphs were constructed in [8, 9, 11–14]. Marušič and Xu [23] showed a way to construct a 1-regular cubic graph from a tetravalent half-transitive graph with girth 3, so that one can construct infinitely many 1-regular cubic graphs from the half-transitive graphs constructed by Alspach *et al.* in [1] and by Marušič and Nedela in [21]. Djoković and Miller [6] constructed an infinite family of 2-regular cubic graphs, and Conder and Praeger [5] constructed two infinite families of s-regular cubic graphs for $s = 2$ or 4. Marušič and Pisanski [22] classified the s-regular cubic Cayley graphs on the dihedral groups D_{2n} for each $1 \le s \le 5$. Also, as shown in [21] or [22], one can see the importance of a study for 1-regular cubic graphs in connection with chiral (that is regular and irreflexible) maps on a surface by means of tetravalent half-transitive graphs or in connection with symmetries of hexagonal molecular graphs on the torus.

Feng *et al.* [8, 13] classified the s-regular cubic graphs of order $4p$, $4p^2$, $6p$, $6p^2$, $8p$, and $8p^2$ for each s and each prime p. However, the method of those classifications cannot be applied to classify the s-regular cubic graphs of order $2p$ or $2p^2$. By using Cheng and Oxley's classification of symmetric graphs of order $2p$ [3], one can easily deduce a classification of s-regular cubic graphs of order $2p$ for each $1 \le s \le 5$ and each prime p. In [11], the authors investigated 1-regular cubic graphs of order twice an odd integer, and classified the 1-regular cubic graphs of order $2p^2$. The purpose of this paper is to investigate the automorphism groups of s-regular cubic graphs of order $2p^n$. Let X be a connected cubic symmetric graph of order $2p^n$ for an odd prime p and a positive integer n. It is shown that if $p \ne 5, 7$, every Sylow p-subgroup of the automorphism group $\mathrm{Aut}(X)$ of X is normal, and each s-regular subgroup G $(1 \le s \le 5)$ of $\mathrm{Aut}(X)$ contains an $(s-1)$-regular subgroup if $p \ne 3$ and G has a normal Sylow p-subgroup. This implies that if $p > 7$, every s-regular subgroup $(1 \le s \le 5)$ of $\mathrm{Aut}(X)$ contains an $(s-1)$-regular subgroup. However, this is not true in general as shown in [6]. As an application, we show that every s-regular cubic graph of order $2p^n$ is a Cayley graph if $p > 5$ and we classify the s-regular cubic graphs of order $2p^2$ for each $1 \le s \le 5$ and each prime p.

2. Preliminaries and some notation

Let π be a nonempty set of primes and let π' denote the set of primes that are not in π. A finite group G is called a *π-group* if every prime factor of $|G|$ is in the set π. In this case, we also say that $|G|$ is a *π-number.*

Let G be a finite group. A π-subgroup H of G such that $|G : H|$ is a π'-number is called a *Hall π-subgroup* of G. The following proposition is due to Hall.

PROPOSITION 2.1 ([20, Theorem 9.1.7]). *If G is a finite solvable group, then every π-subgroup is contained in a Hall π-subgroup of G. Moreover, all Hall π-subgroups of G are conjugate.*

The following proposition is known as Burnside's p-q Theorem.

PROPOSITION 2.2 ([20, Theorem 8.5.3]). *Let p and q be primes and let m and n be non-negative integers. Then, any group of order $p^m q^n$ is solvable.*

For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. Then $C_G(H)$ is normal in $N_G(H)$.

PROPOSITION 2.3 ([24, Chapter I, Theorem 6.11]). *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group* $\mathrm{Aut}(H)$ *of H.*

Let G be a finite group and $g \in G$. The function g^τ: $G \to G$ defined by $x^{g^\tau} = g^{-1}xg$ is an automorphism of G. Set $\mathrm{Inn}(G) = \{g^\tau \mid g \in G\}$. We call $\mathrm{Inn}(G)$ the *inner automorphism group of* G. A finite group G is said to be *complete* if its center is trivial and $\mathrm{Inn}(G) = \mathrm{Aut}(G)$. By [24, Chapter III, Theorem 2.19], the symmetry group S_n $(n \geq 3)$ is complete except for $n = 6$.

For any complete normal subgroup N of G, one may easily get the following lemma.

LEMMA 2.4. *If a finite group G has a complete normal subgroup N then $G = C_G(N) \times N$, where $C_G(N)$ is the centralizer of N in G.*

As usual, we denote by $\mathbb{Z}_n$ the cyclic group of order n. The following proposition describes the vertex stabilizer in an s-regular automorphism group of a connected cubic symmetric graph.

PROPOSITION 2.5 ([6, Propositions 2–5]). *Let X be a connected cubic symmetric graph and let G be an s-regular subgroup of* $\mathrm{Aut}(X)$. *Then the stabilizer G_v of $v \in V(X)$ in G is isomorphic to $\mathbb{Z}_3$, S_3, $S_3 \times \mathbb{Z}_2$, S_4, or $S_4 \times \mathbb{Z}_2$ for $s = 1, 2, 3, 4$ or 5, respectively.*

Let $\mathrm{Cay}(G, S)$ be a Cayley graph on a finite group G. Recall that $R(G)$ is the right regular representation of G and $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$. Let $N = N_{\mathrm{Aut}(\mathrm{Cay}(G,S))}(R(G))$ be the normalizer of $R(G)$ in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. By Godsil [16], we have the following.

PROPOSITION 2.6. $N = R(G) \rtimes \mathrm{Aut}(G, S)$.

By Proposition 2.6, $\mathrm{Aut}(\mathrm{Cay}(G, S)) = R(G) \rtimes \mathrm{Aut}(G, S)$ if and only if $R(G) \lhd \mathrm{Aut}(\mathrm{Cay}(G, S))$, that is, $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. In this case, the Cayley graph $\mathrm{Cay}(G, S)$ is called *normal* by Xu [27].

Let n be a positive integer. For the cyclic group $\mathbb{Z}_n$ of order n, we denote by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n. Let p be a prime and let

$$D_{2p^n} = \langle a, b \mid a^2 = b^{p^n} = 1,\ aba = b^{-1} \rangle$$

be the dihedral group of order $2p^n$. If 3 divides $p - 1$ then $\mathbb{Z}_{p^n}^* \cong \mathbb{Z}_{p^n - p^{n-1}}$ and so $\mathbb{Z}_{p^n}^*$ has two elements of order 3, say λ and λ^2. It is easy to show that the map $a \to ab, b \to b^{-1-\lambda^2}$ induces an automorphism of D_{2p^n}, which maps $\{a, ab, ab^{-\lambda}\}$ to $\{a, ab, ab^{-\lambda^2}\}$. It follows that $\mathrm{Cay}(D_{2p^n}, \{a, ab, ab^{-\lambda}\}) \cong \mathrm{Cay}(D_{2p^n}, \{a, ab, ab^{-\lambda^2}\})$. That is, the Cayley graph $\mathrm{Cay}(D_{2p^n}, \{a, ab, ab^{-\lambda}\})$ is independent of the choice of an element λ of order 3 in $\mathbb{Z}_{p^n}^*$. The graph $\mathrm{Cay}(D_{2p^n}, \{a, ab, ab^{-\lambda}\})$ will be used frequently throughout this paper.

PROPOSITION 2.7 ([11, Theorem 3.5]). *Let p be a prime and let X be a connected cubic symmetric graph of order $2p^2$. Then X is 1-regular if and only if $p-1$ is a multiple of 3 and X is isomorphic to* $\mathrm{Cay}(D_{2p^2}, \{a, ab, ab^{-\lambda}\})$. *Furthermore, the Cayley graph* $\mathrm{Cay}(D_{2p^2}, \{a, ab, ab^{-\lambda}\})$ *is normal, that is,* $R(D_{2p^2}) \lhd \mathrm{Aut}(\mathrm{Cay}(D_{2p^2}, \{a, ab, ab^{-\lambda}\}))$.

Combining Theorem 1 of Marušič and Pisanski in [22] and Table 1 of Cheng and Oxley in [3], we have the following classification of s-regular cubic graphs of order $2p$.

PROPOSITION 2.8. *Let p be a prime and let X be a connected cubic symmetric graph of order $2p$. Then X is 1-, 2-, 3- or 4-regular. Furthermore,*

(1) *X is 1-regular if and only if X is isomorphic to the graph* $\mathrm{Cay}(D_{2p}, \{a, ab, ab^{-\lambda}\})$ *for a prime $p \geq 13$ such that $p-1$ is a multiple of 3. In this case, the Cayley graph* $\mathrm{Cay}(D_{2p}, \{a, ab, ab^{-\lambda}\})$ *is normal and it is independent of the choice of an element λ of order 3 in $\mathbb{Z}_p^*$.*
(2) *X is 2-regular if and only if X is isomorphic to the complete graph K_4 of order 4.*
(3) *X is 3-regular if and only if X is isomorphic to the complete bipartite graph $K_{3,3}$ of order 6 or the Petersen graph O_3 of order 10.*
(4) *X is 4-regular if and only if X is isomorphic to the Heawood graph of order 14.*

Let X be a connected cubic symmetric graph and let G be an s-regular subgroup of $\mathrm{Aut}(X)$ for some $s \geq 1$. Let N be a normal subgroup of G and let X/N denote the quotient graph corresponding to the orbits of N. In view of [18, Theorem 9], we have the following result.

PROPOSITION 2.9. *If N has more than two orbits, then N is semiregular and G/N is an s-regular subgroup of* $\mathrm{Aut}(X/N)$.

3. Main results

Let X be a connected cubic symmetric graph of order $2p^n$ with an odd prime p and a positive integer n. First, we show that every Sylow p-subgroup of $\mathrm{Aut}(X)$ is normal if $p \neq 5, 7$. To prove this, we need the following lemma.

LEMMA 3.1. *Let X be a connected cubic symmetric graph of order $2p^n$ with an odd prime p and a positive integer n. If $n \geq 2$, then any minimal normal subgroup of an arc-transitive automorphism subgroup of* $\mathrm{Aut}(X)$ *is an elementary abelian p-group.*

PROOF. Let N be a minimal normal subgroup of an arc-transitive subgroup G of $\mathrm{Aut}(X)$. The stabilizer G_u of $u \in V(X)$ in G is a $\{2, 3\}$-group by Proposition 2.5. Moreover, $|G_u| = 3 \cdot 2^r$ for some $0 \leq r \leq 4$ and $|G| = 2p^n|G_u|$.

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If N is an elementary abelian 2-group, it is semiregular by Proposition 2.9, so that $N \cong \mathbb{Z}_2$. It follows that the quotient graph X/N has an odd number of vertices and odd valency 3, which is impossible.

Let $p = 3$. Then G is a $\{2, 3\}$-group, implying that G is solvable. Thus, N is an elementary abelian 3-group because it cannot be an elementary abelian 2-group.

Now, let $p \geq 5$ be an odd prime. Suppose that N is a product of a non-abelian simple group T, that is $N = T_1 \times T_2 \times \cdots \times T_\ell$, where $\ell \geq 1$ and $T_i \cong T$. Since $|G| = 2p^n|G_u|$, G is a $\{2, 3, p\}$-group and hence T is a $\{2, 3, p\}$-group. By [17, pages 12–14], T is one of the following simple groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, and $U_4(2)$ with $p = 5$, 7, 13 or 17. Most of the groups in this list have order divisible by 9, so they cannot be subgroups of G. Only A_5 and $L_2(7)$ are candidates. In this case, $\ell = 1$ and $N \cong T$ is a non-abelian simple group. Furthermore, since $n \geq 2$ and $p^2 \nmid |T|$, we have $|V(X/N)| \geq p$. By Proposition 2.9, N is semiregular. This is impossible because $|N|$ cannot divide $|V(X)| = 2p^n$. So far, we have proved that N is elementary abelian, but not an elementary abelian 2-group. Also, N cannot be an elementary 3-group because otherwise $N \leq G_u$ and N is not semiregular, which contradicts Proposition 2.9. Thus, N is an elementary abelian p-group. □

THEOREM 3.2. *Let X be a connected cubic symmetric graph of order $2p^n$ with an odd prime p and a positive integer n. If $p \neq 5, 7$, then all Sylow p-subgroups of* Aut(X) *are normal.*

PROOF. We prove this theorem by induction on n. If $n = 1$, one can easily check that all Sylow p-subgroups of Aut(X) are normal from their classification in Proposition 2.8. Assume $n \geq 2$. Let N be a minimal normal subgroup of Aut(X). Then N is an elementary abelian p-group by Lemma 3.1. Denote by X/N the quotient graph of X corresponding to the orbits of N. Since $|V(X)| = 2p^n$, N has at least two orbits. If N has more than two orbits, then Aut$(X)/N$ is a subgroup of Aut(X/N) by Proposition 2.9. By the inductive hypothesis, one can assume that all Sylow p-subgroups of Aut(X/N) are normal, so that all Sylow p-subgroups of Aut$(X)/N$ are normal. Since N is a p-group, all Sylow p-subgroups of Aut(X) are normal. Now, let N have exactly two orbits. If $p \neq 3$, N must be a Sylow p-subgroup of Aut(X) because $|\operatorname{Aut}(X)| = 2p^n \cdot 3 \cdot 2^r$ for some $0 \leq r \leq 4$. However, N is already known to be normal and we are done. Let $p = 3$. If $N_u \neq 1$, then $|N| > 3^n$. Thus, N is a Sylow 3-subgroup of Aut(X) and it is normal, which is what we need to prove. If $N_u = 1$, N acts regularly on each of its orbits. Clearly, X is a bipartite graph with the two orbits of N as its partite sets. By the regularity of N on each partite set of X, one may identify $R(N) = \{R(n) \mid n \in N\}$ and $L(N) = \{L(n) \mid n \in N\}$ with the two partite vertex sets of X. The actions of $n \in N$ on $R(N)$ and on $L(N)$ are just the right multiplication by

n, that is $R(g)^n = R(gn)$ and $L(g)^n = L(gn)$ for any $g \in N$. Let $L(n_1)$, $L(n_2)$, and $L(n_3)$ be the vertices adjacent to $R(1)$. Without loss of generality, one may assume that $n_1 = 1$. By the connectivity of X, we have $N = \langle n_1, n_2, n_3 \rangle$. Thus, $|V(X)| = 18$ because $n \geq 2$. In this case, X is isomorphic to the Pappus graph 9_3 of order 18. By [12], X is a cyclic covering of $K_{3,3}$ and $\text{Aut}(X) \cong \mathbb{Z}_3 : ((S_3 \times S_3) \rtimes \mathbb{Z}_2)$ has normal Sylow 3-subgroups. □

If $p = 5$ or 7, Theorem 3.2 is not true because the automorphism groups of the Petersen graph and the Heawood graph have non-normal Sylow p-subgroups.

Let X be a connected cubic symmetric graph and let G be an s-regular subgroup of $\text{Aut}(X)$. Let $0 \leq t < s$. Does G contain a t-regular subgroup? The answer is negative in general (see [6]), but affirmative if X has order $2p^n$ for a prime $p > 3$ and all Sylow p-subgroups of $\text{Aut}(X)$ are normal. All Sylow p-subgroups of $\text{Aut}(X)$ are normal if $p > 7$, by Theorem 3.2.

THEOREM 3.3. *Let X be a connected cubic symmetric graph of order $2p^n$ for a prime $p > 3$ and let G be an s-regular subgroup of* $\text{Aut}(X)$ *for some* $1 \leq s \leq 5$. *If G has a normal Sylow p-subgroup, then G contains an $(s-1)$-regular subgroup. Furthermore, if $s = 2$, then G contains a normal* 0*-regular subgroup.*

PROOF. Let P be a Sylow p-subgroup of G and let $P \lhd G$. By Proposition 2.5, the stabilizer G_v of $v \in V(X)$ in G is a $\{2, 3\}$-group. Since $|G| = 2p^n|G_v|$, we have that G/P is a $\{2, 3\}$-group. By Burnside's p-q Theorem, a $\{2, 3\}$-group is solvable and hence G/P is solvable. This implies that G is solvable and by Proposition 2.1, there exists a Hall $\{2, 3\}$-subgroup, say H, of G such that $G_v \leq H$. Since $P \lhd G$, X is bipartite with two orbits of P as its partite sets. Clearly, $G = PH$. Since $|G| = 2p^n|G_v|$, we have $|H : G_v| = 2$, which forces $G_v \lhd H$. Also, G_v fixes the partite sets of X setwise, but H does not because $G = PH$ is transitive on $V(X)$. Thus for each $h \in H \backslash G_v$, h interchanges the two partite sets of X. This fact will be used repeatedly in the remainder of the proof.

Without any loss of generality, by Proposition 2.5 one may assume that $G_v = \mathbb{Z}_3$, S_3, $S_3 \times \mathbb{Z}_2$, S_4, or $S_4 \times \mathbb{Z}_2$ for $s = 1, 2, 3, 4$ or 5 respectively. We now consider five cases for s.

Case I: $s = 1$. In this case $G_v = \mathbb{Z}_3$ and $|H| = 6$. Let c be an involution in H. Then $c \notin G_v$ and hence c interchanges the two partite sets of X. Since $P \lhd G$, $P\langle c \rangle$ is a 0-regular subgroup of G.

Case II: $s = 2$. In this case, $G_v = S_3$ and $|H| = 12$. Since $G_v \lhd H$ and S_3 is complete, we have $H = S_3 \times \mathbb{Z}_2$ by Lemma 2.4. Clearly, the non-trivial element of $\mathbb{Z}_2$ interchanges the two partite sets of X. Let $\mathbb{Z}_3$ be the unique Sylow 3-subgroup of H. Then, $P(\mathbb{Z}_3 \times \mathbb{Z}_2)$ is 1-regular.

We now prove that G contains a normal 0-regular subgroup. Clearly, $P\mathbb{Z}_2$ acts regularly on $V(X)$, that is, $P\mathbb{Z}_2$ is a 0-regular subgroup of G. We claim that $P\mathbb{Z}_2 \lhd G$. Since $\mathbb{Z}_2 \lhd H$ and $P \lhd G$, we have that $P\mathbb{Z}_2$ is normalized by any $h \in H$. Also, $P\mathbb{Z}_2$ is normalized by any $r \in P$ since $P \leq P\mathbb{Z}_2$. So, $P\mathbb{Z}_2$ is normalized by $PH = G$.

Case III: $s = 3$. In this case, $G_v = S_3 \times \mathbb{Z}_2$ and $|H| = 24$. Since $\mathbb{Z}_3$ is characteristic in G_v and $G_v \lhd H$, $\mathbb{Z}_3 \lhd H$. Let $C = C_H(\mathbb{Z}_3)$ be the centralizer of $\mathbb{Z}_3$ in H. Then, $C \lhd H$. By Proposition 2.3, H/C is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{Z}_3)$. It follows that $|H| = |C|$ or $|H| = 2|C|$. Suppose $|H| = |C|$. Then C is a Hall $\{2, 3\}$-subgroup of G. This implies that $C = H$ and $G_v \leq C$, which is impossible because $S_3 \not\leq C$. Thus, $|H| = 2|C|$, forcing $|C| = 12$. Since $G_v \neq C$, $H = CG_v$. Thus, there exists a $c \in C$ such that $c \notin G_v$. Recalling that c interchanges the two partite parts of X, PC is vertex-transitive on $V(X)$. Since $|PC| = 12p^n$, PC is 2-regular.

Case IV: $s = 4$. In this case, $G_v = S_4$ and $|H| = 48$. Since S_4 is complete and normal in H, Lemma 2.4 implies that $H = G_v \times \mathbb{Z}_2 = S_4 \times \mathbb{Z}_2$. Let A_4 be the alternating subgroup of S_4. Then, $P(A_4 \times \mathbb{Z}_2)$ is 3-regular.

Case V: $s = 5$. In this case, $G_v = S_4 \times \mathbb{Z}_2$ and $|H| = 96$. One may easily show that $A_4 \leq S_4$ is characteristic in G_v, so it is normal in H. Let $\mathbb{Z}_2 \times \mathbb{Z}_2$ be the Sylow 2-subgroup of A_4. Then, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is characteristic in A_4 and hence $\mathbb{Z}_2 \times \mathbb{Z}_2 \lhd H$. Set $C = C_H(\mathbb{Z}_2 \times \mathbb{Z}_2)$. By Proposition 2.3, H/C is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$, which has order 6. Let P_3 be a Sylow 3-subgroup of A_4 and let P_2 be a Sylow 2-subgroup of S_4. Clearly, $P_3 \not\leq C$ and $P_2 \not\leq C$. Thus $|H/C| = 6$ and $|C| = 96/6 = 16$. If $C \leq G_v$, then C is a normal Sylow 2-subgroup of G_v because $C \lhd H$. It follows that $P_2 \leq C$, a contradiction. Thus, $C \not\leq G_v$ and $H = G_vC$, implying that there is a $c \in C$ such that $c \notin G_v$. Noting that c interchanges the two partite sets of X, we have that PC is vertex-transitive on $V(X)$. Since $PC \lhd PH = G$, PCP_3 is a subgroup of G and since $|PCP_3| = 48p^n$, PCP_3 is 4-regular. □

The following example shows that Theorem 3.3 is not true when $p = 3$. Consider the complete bipartite graph $K_{3,3}$ of order 6. It is 3-regular and $\mathrm{Aut}(K_{3,3}) \cong (S_3 \times S_3) \rtimes \mathbb{Z}_2$ is of order $3^2 2^3$, so that $\mathrm{Aut}(K_{3,3})$ has a normal Sylow 3-subgroup, say P. Clearly, there is an element a of order 4 in $\mathrm{Aut}(K_{3,3})$ which interchanges the two partite sets of $K_{3,3}$. It follows that $\mathrm{Aut}(K_{3,3})$ has a 2-regular subgroup $P\langle a\rangle$ which contains neither a 1-regular subgroup nor a 0-regular subgroup.

Let X be a connected cubic symmetric graph of order $2p^n$ with a prime p. By Theorems 3.2 and 3.3, if $p > 7$, then each s-regular ($s \geq 2$) subgroup of $\mathrm{Aut}(X)$ contains a 2-regular subgroup which has a normal 0-regular subgroup. However, a 1-regular subgroup of $\mathrm{Aut}(X)$ may not have a normal 0-regular subgroup. In fact, by a straightforward analysis of Case II in the proof of Theorem 3.3, each 2-regular subgroup of $\mathrm{Aut}(X)$ contains a 1-regular subgroup, which has no normal 0-regular subgroup. This implies that, in some sense, it is easier to classify the s-regular ($s \geq 2$)

cubic graphs of order $2p^n$ than to classify the 1-regular ones.

COROLLARY 3.4. *Let $p > 5$ be a prime and let n be a positive integer. Then all connected cubic symmetric graphs of order $2p^n$ are Cayley graphs.*

PROOF. Let X be a connected symmetric cubic graph of order $2p^n$. Set $A = \mathrm{Aut}(X)$. If $p \geq 11$, by Theorems 3.2, all Sylow p-subgroups of A are normal and by Theorems 3.3, A has a 0-regular subgroup, that is, X is a Cayley graph. Thus, we only need to prove the corollary for $p = 7$. To complete the proof, it suffices to show that each s-regular subgroup of A for $1 \leq s \leq 5$, contains a 0-regular subgroup. We show this by induction on n.

If $n = 1$, then X is the Heawood graph of order 14 and its automorphism group is $\mathrm{PSL}(2, 7) \cdot 2$. Since $\mathrm{PSL}(2, 7)$ is simple, $\mathrm{PSL}(2, 7) \cdot 2$ has neither a 2- nor 3-regular subgroup because of the bipartiteness of the Heawood graph. Let B be a 1-regular subgroup of $\mathrm{PSL}(2, 7) \cdot 2$. Then $|B| = 2 \times 7 \times 3$ and B is solvable. By [11, Lemma 3.2], a solvable 1-regular automorphism group of a connected cubic graph contains a regular subgroup, so B contains a 0-regular subgroup. Thus, the claim is true for $n = 1$.

Assume that $n \geq 2$. Let G be an s-regular subgroup of A for some $1 \leq s \leq 5$ and let N be a minimal normal subgroup of G. By Lemma 3.1, N is an elementary abelian 7-group. If N is a Sylow 7-subgroup then, the claim is true by Theorem 3.3. If N is not a Sylow 7-subgroup, by Proposition 2.9, N is semiregular and G/N is an s-regular automorphism group of the quotient graph X/N corresponding to the orbits of N. By the inductive hypothesis, one can assume that G/N contains a 0-regular subgroup, say H/N, on $V(X/N)$. Clearly, H is transitive on $V(X)$. By the semiregularity of N on $V(X)$ and the 0-regularity of H/N on $V(X/N)$, we have that $|H| = |V(X)|$. It follows that H is a 0-regular subgroup of G on $V(X)$. □

A connected cubic symmetric graph of order 6, 18 or 54 is a Cayley graph (see [8]). We conjecture that the corollary is true for $p = 3$, but the proof is still elusive. The corollary is not true for $p = 5$ because the Petersen graph is not Cayley. In fact, one may construct infinitely many non-Cayley graphs of order $2 \cdot 5^n$ by considering the regular coverings of the Petersen graph [2, Chapter 19].

From elementary group theory we know that there are three non-abelian groups of order $2p^2$ up to isomorphism:

$$G_1(p) = \langle a, b \mid a^2 = b^{p^2} = 1,\ aba = b^{-1}\rangle;$$
$$G_2(p) = \langle a, b, c \mid a^p = b^p = c^2 = [a, b] = 1,\ c^{-1}ac = a^{-1},\ c^{-1}bc = b^{-1}\rangle;$$
$$G_3(p) = \langle a, b, c \mid a^p = b^p = c^2 = 1,\ [a, b] = [a, c] = 1,\ c^{-1}bc = b^{-1}\rangle.$$

Now, we are ready to classify the s-regular cubic graphs of order $2p^2$ for $1 \leq s \leq 5$, where p is any prime.

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THEOREM 3.5. *Let X be a connected cubic symmetric graph of order $2p^2$ with a prime p. Then X is 1-, 2- or 3-regular. Furthermore,*

(1) *X is 1-regular if and only if X is isomorphic to the graph*

$$\mathrm{Cay}(G_1(p), \{a, ab, ab^{-\lambda}\}) \tag{3.1}$$

for a prime p such that $p-1$ is a multiple of 3, where λ is an element of order 3 in the multiplicative group $\mathbb{Z}^*_{p^2}$. Here, the Cayley graph (3.1) is independent of the choice of λ in $\mathbb{Z}^*_{p^2}$.

(2) *X is 2-regular if and only if X is isomorphic to either the three dimensional hypercube Q_3 of order 8 or one of the Cayley graphs $\mathrm{Cay}(G_2(p), \{c, ca, cb\})$ for a prime $p \neq 2, 3$.*

(3) *X is 3-regular if and only if X is isomorphic to the Pappus graph 9_3 of order 18.*

PROOF. If $p = 2$, then $|V(X)| = 8$. There is only one connected cubic symmetric graph of order 8, and that is the 2-regular three dimensional hypercube Q_3. Thus, we may assume that $p \geq 3$ from now on.

Clearly, (1) follows from Proposition 2.7. Thus, we assume that X is s-regular for some $s \geq 2$. It is straightforward to show that $\mathrm{Aut}(G_2(p), \{c, ca, cb\})$ is 2-transitive on $\{c, ca, cb\}$. Consequently, the Cayley graphs $\mathrm{Cay}(G_2(p), \{c, ca, cb\})$, $p \geq 3$, are 2-arc-transitive. By Conder and Dobcsányi [4]'s lists of cubic symmetric graphs of order up to 768, there is only one connected cubic symmetric graph of order $2p^2$ for each prime $p = 3, 5$ or 7, that are 2-regular for $p = 5$ or 7 and 3-regular for $p = 3$. It follows that all connected cubic symmetric graphs of order $2p^2$ with $p = 3, 5$ or 7 are the 2-regular graphs $\mathrm{Cay}(G_2(p), \{c, ca, cb\})$ for $p = 5$ or 7 and the 3-regular graph $\mathrm{Cay}(G_2(p), \{c, ca, cb\})$ for $p = 3$, of which the last one is the Pappus graph 9_3 of order 18. Thus, we assume $p \geq 11$. To complete the proof, it suffices to show that X is isomorphic to $\mathrm{Cay}(G_2(p), \{c, ca, cb\})$ and that it is 2-regular.

Since X is s-regular for some $s \geq 2$, by Theorems 3.2 and 3.3, $\mathrm{Aut}(X)$ contains a 2-regular subgroup, which contains a normal 0-regular subgroup. Thus, X is a Cayley graph on a finite group G of order $2p^2$, say $X = \mathrm{Cay}(G, S)$. Moreover, there is a 2-regular subgroup M of $\mathrm{Aut}(X)$ such that the right regular representation $R(G)$ is normal in M. By Proposition 2.6, $M \leq R(G) \rtimes \mathrm{Aut}(G, S)$ and the 2-regularity of M implies that $\mathrm{Aut}(G, S)$ is 2-transitive on S. Since X has valency 3, S contains at least one involution and the transitivity of $\mathrm{Aut}(G, S)$ on S implies that S consists of three involutions. If G is abelian, then $\langle S \rangle = G$ implies that $|G| \leq 8$, contrary to the hypothesis $p \geq 11$. Thus, G is non-abelian and $G = G_1(p)$, $G_2(p)$, or $G_3(p)$.

Suppose $G = G_1(p)$. Since $p \geq 11$, by Theorem 1 of Marušič and Pisanski in [22], X is 1-regular, contrary to the hypothesis that $s \geq 2$. Since $G_3(p)$ cannot be generated by involutions, we have $G = G_2(p)$. One may check that $\mathrm{Aut}(G)$ is 2-transitive on the set of involutions of G. Thus, we assume that $S = \{c, ca, ca^ib^j\}$. Since $\langle S \rangle = G$,

$j \neq 0$. Then, the map $c \to c$, $a \to a$, and $b \to a^i b^j$ induces an automorphism of $G_2(p)$ that maps $\{c, ca, cb\}$ to S. This implies that $X \cong \mathrm{Cay}(G_2(p), \{c, ca, cb\})$.

Now we shall prove that X is 2-regular. Since X is 2-arc-transitive, it is at least 2-regular. Then $\mathrm{Cay}(G_2(p), \{c, ca, cb\})$ has girth 6 and there are exactly two girth cycles passing through any given edge. Thus, X is not 3-arc-transitive because otherwise there is at least four girth cycles passing through any given edge. It follows that $\mathrm{Cay}(G_2(p), \{c, ca, cb\})$ is 2-regular, as required. □

COROLLARY 3.6. *All connected cubic symmetric graphs of order $2p^2$ are Cayley graphs.*

This corollary is not true for connected cubic symmetric graphs of order $2p$ because the Petersen graph of order 10 is not Cayley.

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