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Free Locally Convex Spaces and the k-space Property

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Abstract. Let $L(X)$ be the free locally convex space over a Tychonoff space X. Then $L(X)$ is a k-space if and only if X is a countable discrete space. We prove also that $L(D)$ has uncountable tightness for every uncountable discrete space D.

1 Introduction

The free (resp., abelian) topological group $F(X)$ (resp., $A(X)$) and the free locally convex space $L(X)$ over a Tychonoff space X were introduced by Markov [10] and intensively studied over the last half-century (see [7, 9, 15, 18, 19]).

Recall that the *free locally convex space* $L(X)$ over a Tychonoff space X is a pair consisting of a locally convex space $L(X)$ and a continuous mapping $i\colon X \to L(X)$ such that every continuous mapping f from X to a locally convex space E gives rise to a unique continuous linear operator $\bar{f}\colon L(X) \to E$ with $f = \bar{f} \circ i$. The free locally convex space $L(X)$ always exists and is unique. The set X forms a Hamel basis for $L(X)$, and the mapping i is a topological embedding [4, 5, 15, 19]. It is known that the identity map $\mathrm{id}_X\colon X \to X$ extends to a canonical homomorphism $\mathrm{id}_{A(X)}\colon A(X) \to L(X)$, which is an embedding of topological groups [18, 20].

One of the most important topological properties is the property to be a k-space. Recall that a Hausdorff space X is called a *k-space* if its topology is defined by compact subsets of X; *i.e.,* for each $A \subseteq X$, the set A is closed in X provided that the intersection of A with any compact subset K of X is closed in K. In the partial case when the topology of a k-space X is defined by an increasing sequence of its compact subsets, the space X is called a *k_ω-space*. It is known ([7, 9]) that for any k_ω-space X, the groups $F(X)$ and $A(X)$ are also k_ω-spaces. Arhangel'skii, Okunev, and Pestov (see [1]) described all metrizable spaces X for which the groups $F(X)$ and $A(X)$ are k-spaces.

Theorem 1.1 ([1]) *Let X be a metrizable space.*

(i) *$F(X)$ is a k-space if and only if X is locally compact separable or discrete.*
(ii) *$A(X)$ is a k-space if and only if X is locally compact and the set X' of all non-isolated points in X is separable.*

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It is natural to ask for which Tychonoff spaces X is the free locally convex space $L(X)$ a k-space. Consider two simple cases. If X is a finite space of cardinality n, then $L(X) \cong \mathbb{R}^n$. If X is a countably infinite discrete space, then $L(X) \cong \phi$, where ϕ is the countable inductive limit of the increasing sequence $(\mathbb{R}^k)_{k\in\mathbb{N}}$. It is well known that ϕ is even a sequential k_ω-space. It turns out that except for these two simplest cases the space $L(X)$ is never a k-space. The following theorem is the main result of the article.

Theorem 1.2 *For a Tychonoff space X, $L(X)$ is a k-space if and only if X is a countable discrete space.*

The paper is organized as follows. In Section 2 we study the box, the maximal, and maximal locally convex vector topologies on direct sums of the reals $\mathbb{R}$. The main theorem of this section (see Theorem 2.1) generalizes some of the main results of [13], and it is essentially used to prove Theorem 1.2. In Section 3 we prove Theorem 1.2.

2 Maximal Vector Topologies on Direct Sums of the Reals $\mathbb{R}$

Let κ be an infinite cardinal, let $\mathbb{V}_\kappa = \bigoplus_{i<\kappa} \mathbb{R}_i$ be a vector space of dimension κ over $\mathbb{R}$, let $\boldsymbol{\tau}_\kappa$ be the box topology on $\mathbb{V}_\kappa$, and let $\boldsymbol{\mu}_\kappa$ and $\boldsymbol{\nu}_\kappa$ be the maximal and maximal locally convex vector topologies on $\mathbb{V}_\kappa$, respectively. Clearly, $\boldsymbol{\tau}_\kappa \subseteq \boldsymbol{\nu}_\kappa \subseteq \boldsymbol{\mu}_\kappa$ and $L(D) \cong (\mathbb{V}_\kappa, \boldsymbol{\nu}_\kappa)$, where D is a discrete space of cardinality κ. It is well known that $\boldsymbol{\tau}_\omega = \boldsymbol{\nu}_\omega = \boldsymbol{\mu}_\omega$ (see [8, Proposition 4.1.4]). However, if κ is uncountable the situation changes [13] (see also Theorem 2.1).

We denote by $\overline{A}^\tau$ the closure of a subset A of a topological space (X, τ).

For a topological group (G, τ) denote by $k(\tau)$ the finest group topology for G coinciding on compact sets with τ. In particular, τ and $k(\tau)$ have the same family of compact subsets. Clearly, $\tau \le k(\tau)$. If $\tau = k(\tau)$, the group (G, τ) is called a *k-group* [12]. The group $\mathbf{k}(G, \tau) := (G, k(\tau))$ is called the *k-modification* of X. The class of all k-groups contains all topological groups whose underlying space is a k-space.

The next theorem generalizes [13, Theorems 1(i), 3(i), and 5] and simplifies their proofs (we give an independent proof of item (i)).

Theorem 2.1 *Let κ be an uncountable cardinal and let τ be a vector topology on $\mathbb{V}_\kappa$ such that $\boldsymbol{\tau}_\kappa \subseteq \tau \subseteq \boldsymbol{\nu}_\kappa$.*

(i) $\boldsymbol{\tau}_\kappa \subsetneq \boldsymbol{\nu}_\kappa \subsetneq \boldsymbol{\mu}_\kappa$ *(see [13]).*
(ii) $(\mathbb{V}_\kappa, \tau)$ *has uncountable tightness.*
(iii) $(\mathbb{V}_\kappa, \tau)$ *is not a k-group and hence not a k-space.*

Proof We shall use the following simple description of the topology $\boldsymbol{\mu}_\kappa$ given in the proof of [13, Theorem 1]. For each $i \in \kappa$, choose some $\lambda_i \in \mathbb{R}_i^+$, $\lambda_i > 0$, and denote by $\mathcal{S}_\kappa$ the family of all subsets of $\mathbb{V}_\kappa$ of the form

$$\bigcup_{i<\kappa} \Big([-\lambda_i, \lambda_i] \times \prod_{j<\kappa,\ j\neq i} \{0\} \Big).$$

For every sequence $\{S_k\}_{k\in\omega}$ in $\mathcal{S}_\kappa$, we put

$$\sum_{k\in\omega} S_k := \bigcup_{k\in\omega} (S_0 + S_1 + \cdots + S_k)$$

and denote by $\mathcal{N}_\kappa$ the family of all subsets of $\mathbb{V}_\kappa$ of the form $\sum_{k\in\omega} S_k$. It is easy to check that $\mathcal{N}_\kappa$ is a base at $\mathbf{0}$ for $\boldsymbol{\mu}_\kappa$ and the family $\widehat{\mathcal{N}}_\kappa := \{\mathrm{conv}(V) : V \in \mathcal{N}_\kappa\}$ is a base at $\mathbf{0}$ for $\boldsymbol{\nu}_\kappa$ (see [13]). For $(x_i) \in \mathbb{V}_\kappa$, we denote $\mathrm{supp}(x_i) := \{i \in \kappa : x_i \neq 0\}$.

We prove the theorem in four steps.

Step 1. For every natural number n, set

$$E_n := \Big\{ (x_i) \in \mathbb{V}_\kappa : |\mathrm{supp}(x_i)| \geq n, \text{ and } x_i \geq \frac{1}{n^2} \text{ for every } i \in \mathrm{supp}(x_i) \Big\}.$$

Set $E := \bigcup_{n\in\mathbb{N}} E_n$. Clearly, $\mathbf{0} \notin E$. We show that

(a) $\mathbf{0} \in \overline{E}^{\boldsymbol{\nu}_\kappa}$;
(b) $\mathbf{0} \notin \overline{B}^{\boldsymbol{\tau}_\kappa}$ for any countable subset B of E;
(c) $\mathbf{0} \notin \overline{E}^{\boldsymbol{\mu}_\kappa}$.

Take arbitrarily an open convex neighborhood W of $\mathbf{0}$ in $\boldsymbol{\nu}_\kappa$. Choose a neighborhood $\sum_{k\in\omega} S_k$ of $\mathbf{0}$ in $\boldsymbol{\mu}_\kappa$ such that $\sum_{k\in\omega} S_k \subseteq W$. Since κ is uncountable, there is a positive number $c > 0$ and an uncountable set J of indices such that $\lambda_j^0 > c$ for all $j \in J$, where the positive numbers λ_j^0 define S_0. Take $n \in \mathbb{N}$ with $1/n < c$ and a finite subset $J_0 = \{j_1, \ldots, j_n\}$ of J. For every $1 \leq l \leq n$ we set $\mathbf{x}_l = (x_i^l)_{i<\kappa}$, where $x_i^l = \frac{1}{n}$ if $i = j_l$, and $x_i^l = 0$. So $\mathbf{x}_l \in S_0 \subset \sum_{n\in\omega} S_n \subseteq W$ for every $1 \leq l \leq n$. Since W is convex, the element

$$\mathbf{x} := \frac{1}{n}(\mathbf{x}_1 + \cdots + \mathbf{x}_n)$$

belongs to W. By construction, $\mathbf{x} \in E_n$. Thus $\mathbf{0} \in \overline{E}^{\boldsymbol{\nu}_\kappa}$ and (a) holds.

To prove (b) let $B = \{(x_i^n)_{i<\kappa}\}_{n\in\mathbb{N}}$ be a countable subset of E. Denote by I the set of all indices $i, i < \kappa$, such that $x_i^n \neq 0$ for some $n \in \mathbb{N}$. We can assume that I is countably infinite and hence $I = \{i_k\}_{k\in\mathbb{N}}$. Set

$$U := \Big\{ (x_i)_{i<\kappa} \in \mathbb{V}_\kappa : x_{i_k} \in \Big(-\frac{1}{2^{2k}}, \frac{1}{2^{2k}}\Big), \forall k \in \mathbb{N} \Big\}.$$

Clearly, U is an open neighborhood of $\mathbf{0}$ in $\boldsymbol{\tau}_\kappa$. For each $(x_i)_{i<\kappa} \in U$ and every $n \in \mathbb{N}$, if $x_{i_k} \geq \frac{1}{n^2}$, then $\frac{1}{2^{2k}} \geq \frac{1}{n^2}$ and hence $k \leq \log_2 n < n$. This means that the size of the set of indices i_k for which $x_{i_k} \geq \frac{1}{n^2}$ is strictly less than n. So $(x_i)_{i<\kappa} \notin E_n \cap B$ for each $(x_i)_{i<\kappa} \in U$ and every $n \in \mathbb{N}$. Thus $U \cap B = \varnothing$ and (b) is proved.

Now we prove (c). For every $k \in \omega$ and each $i < \kappa$, set

$$\lambda_i^k = \frac{1}{(k+4)^3 4^{k+1}} \quad \text{and} \quad S_k := \bigcup_{i<\kappa} \Big([-\lambda_i^k, \lambda_i^k] \times \prod_{j<\kappa,\ j\neq i} \{0\} \Big).$$

We show that $E \cap \sum_{k\in\omega} S_k = \varnothing$. Clearly, $S_0 \cap E = \varnothing$. Taking into account that for $n > 0$,

$$(2.1) \qquad \sum_{k=n-1}^{\infty} \frac{1}{(k+4)^3 4^{k+1}} < \frac{1}{(n+3)^3} \sum_{k=n-1}^{\infty} \frac{1}{4^{k+1}} < \frac{1}{(n+3)^3} < \frac{1}{n^2},$$

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we obtain that any element $\mathbf{x} \in \sum_{k\in\omega} S_k$ has at most $n-1$ coordinates that are greater than or equal to $\frac{1}{n^2}$. So $E_n \cap \sum_{k\in\omega} S_k = \varnothing$ for every $n \in \mathbb{N}$. Thus $E \cap \sum_{k\in\omega} S_k = \varnothing$ and (c) holds.

Now (a) and (b) prove (ii), and (a) and (c) show that $\boldsymbol{\nu}_\kappa \subsetneq \boldsymbol{\mu}_\kappa$.

Step 2. We claim that $\boldsymbol{\tau}_\kappa \subsetneq \boldsymbol{\nu}_\kappa$. Indeed, set

$$A := \Big\{ (x_i)_{i<\kappa} \in \mathbb{V}_\kappa : x_i \geq 0, \forall i < \kappa, \text{ and } \sum_{i<\kappa} x_i > 1 \Big\}.$$

Clearly, $\mathbf{0} \notin A$. To prove the claim it is enough to show the following:

(d) $\mathbf{0} \in \overline{A}^{\boldsymbol{\tau}_\kappa}$;
(e) $\mathbf{0} \notin \overline{A}^{\boldsymbol{\nu}_\kappa}$.

We first show (d). Take an arbitrary neighborhood U of $\mathbf{0}$ in the box topology $\boldsymbol{\tau}_\kappa$ of the form $\mathbb{V}_\kappa \cap \prod_{i<\kappa}(-\lambda_i, \lambda_i)$. Since κ is uncountable, there is a positive number $c > 0$ and an uncountable set J of indices such that $\lambda_j > c$ for all $j \in J$. Pick a finite subset F of J such that $c|F| > 1$ and set $y_i = c$ if $i \in F$, and $y_i = 0$ otherwise. Clearly, $(y_i)_{i<\kappa} \in A \cap U$. Thus $\mathbf{0} \in \overline{A}^{\boldsymbol{\tau}_\kappa}$.

Now we prove (v). Take λ_i^k as in the proof of (iii), and note that each element $\mathbf{x}$ of $\operatorname{conv}(\sum_{k\in\omega} S_k)$ has the form

$$\mathbf{x} = c_1(x_i^1) + \cdots + c_m(x_i^m),$$

where $c_1, \ldots, c_m > 0$, $c_1 + \cdots + c_m \leq 1$ and $(x_i^1), \ldots, (x_i^m) \in \sum_{k\in\omega} S_k$. The inequality (2.1) for $n = 1$ implies

$$\sum_{i<\kappa}(c_1 x_i^1 + \cdots + c_m x_i^m) < c_1 \frac{1}{4^3} + \cdots + c_m \frac{1}{4^3} \leq \frac{1}{4^3} < 1.$$

So $\mathbf{x} \notin A$. Thus $\operatorname{conv}(\sum_{k\in\omega} S_k) \cap A = \varnothing$ and (v) is proved.

Step 3. For the convenience of the reader we prove the following well-known fact: compact subsets of $(\mathbb{V}_\kappa, \boldsymbol{\tau}_\kappa)$, and hence compact subsets of $(\mathbb{V}_\kappa, \boldsymbol{\mu}_\kappa)$ and $(\mathbb{V}_\kappa, \tau)$, are finite-dimensional. Indeed, suppose for a contradiction that $(\mathbb{V}_\kappa, \boldsymbol{\tau}_\kappa)$ has an infinite-dimensional compact subset K. Then the intersection of K with some countably infinite-dimensional subspace is also infinite-dimensional. So the space $(\mathbb{V}_\omega, \boldsymbol{\tau}_\omega) = (\mathbb{V}_\omega, \boldsymbol{\mu}_\omega)$ has an infinite-dimensional compact subset. But this contradicts [17, Lemma 9.3].

Step 4. Now we prove (iii). By Step 3 the k-modifications $\mathbf{k}(\tau)$ and $\mathbf{k}(\boldsymbol{\mu}_\kappa)$ of τ and $\boldsymbol{\mu}_\kappa$ respectively coincide. Since $\boldsymbol{\mu}_\kappa \subseteq \mathbf{k}(\boldsymbol{\mu}_\kappa)$, (i) implies that

$$\tau \subsetneq \boldsymbol{\mu}_\kappa \subseteq \mathbf{k}(\tau) = \mathbf{k}(\boldsymbol{\tau}_\kappa).$$

Thus $(\mathbb{V}_\kappa, \tau)$ is not a k-group and hence not a k-space. ■

Remark 2.2 Theorem 5 of [13] states that $(\mathbb{V}_\omega, \boldsymbol{\tau}_\omega) = \phi$ is not sequential, because the zero vector $\mathbf{0}$ belongs to the closure of the set X defined in the proof of this theorem. However, $\mathbf{0} \notin \overline{X}$, since $X \cap V = \varnothing$ for $V = \prod_{n\in\mathbb{N}}\left(-\frac{1}{2n}, \frac{1}{2n}\right)$. So the proof of [13, Theorem 5] is wrong.

We end this section with the following question in which $t(X)$ denotes the tightness of a space X.

Question 2.3 For a cardinal $\kappa > \aleph_0$, is $t(\mathbb{V}_\kappa, \boldsymbol{\mu}_\kappa) = t(\mathbb{V}_\kappa, \boldsymbol{\nu}_\kappa) = t(\mathbb{V}_\kappa, \boldsymbol{\tau}_\kappa) = \kappa$?

3 Proof of Theorem 1.2

Let $\mathbf{s} = \{\frac{1}{n}\}_{n\in\mathbb{N}} \cup \{0\}$ be the convergent sequence with the usual topology induced from $\mathbb{R}$. It is well known that $A(\mathbf{s})$ is a sequential non-Fréchet–Urysohn space.

Recall (see [2]) that a topological space Y has *countable* cs^**-character* if for each $y \in Y$, there exists a countable family $\mathcal{D}$ of subsets of Y such that for each nontrivial sequence in Y converging to y and each neighbourhood U of y, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and D contains infinitely many elements of that sequence. Note that the free locally convex space $L(\mathbf{s})$ has countable cs^*-character by [6, Proposition 5]. Recall also that a topological group G is an $\mathcal{MK}_\omega$*-group* if its topology is defined by an increasing sequence of compact metrizable subsets.

In the next proposition we consider an important partial case of Theorem 1.2.

Proposition 3.1 *The space $L(\mathbf{s})$ is not a k-space.*

Proof We note first that $L(\mathbf{s})$ is not Fréchet–Urysohn, because it contains $A(\mathbf{s})$ as a closed subgroup. Further, $L(\mathbf{s})$ has countable cs^*-character [6].

Suppose for a contradiction that $L(\mathbf{s})$ is a k-space. Define the following embedding p of $\mathbf{s}$ into the classical Banach space c_0:

$$p(0) = \mathbf{0} \quad \text{and} \quad p\Big(\frac{1}{n}\Big) := \Big(0, \dots, 0, \frac{1}{n}, 0, \dots\Big),$$

where $1/n$ is placed in position n. So there is a continuous linear monomorphism $\widetilde{p}\colon L(\mathbf{s}) \to c_0$ such that $\widetilde{p}(x) = p(x)$ on $\mathbf{s}$. Hence any compact subset of the k-space $L(\mathbf{s})$ is metrizable. Thus $L(\mathbf{s})$ is a sequential space.

Since $L(\mathbf{s})$ is a sequential non-metrizable space with countable cs^*-character, [2, Theorem 1] implies that $L(\mathbf{s})$ has an open $\mathcal{MK}_\omega$-subgroup. So $L(\mathbf{s})$ is an $\mathcal{MK}_\omega$-group as it is (arcwise) connected. Thus $L(\mathbf{s})$ is complete [14, 4.1.6]. However $L(\mathbf{s})$ is not complete by a corollary of [19, Theorem 5]. This contradiction shows that $L(\mathbf{s})$ is not a k-space. ∎

The next two lemmas help us to reduce the proof of Theorem 1.2 for simpler cases.

Lemma 3.2 *For every infinite compact space K there is a quotient mapping f of K onto an infinite metrizable compact space C.*

Proof We show first that there is a continuous function $f\colon K \to [0,1]^{\mathbb{N}}$ such that $f(K)$ is infinite. Indeed, the compact space K can be considered as an infinite subspace of a Tychonoff cube [3, 3.2.5]. Now take for f the projection to an appropriate countable face.

Now we set $C := f(K)$. By construction, C is an infinite metrizable compact space. Since K is compact, f is a quotient map by [3, 2.4.8 and 3.1.12]. ∎

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Lemma 3.3 *If Y is a compact subspace of a Tychonoff space X, then $L(Y)$ can be identified with the closed subspace $L(Y,X)$ of $L(X)$ generated by Y.*

Proof Note that the topology of the free lcs $L(Z)$ over a Tychonoff space Z is determined by continuous seminorms arising from pseudometrics on Z (see [18, 20]). As Y is compact, each continuous pseudometric on Y can be extended to a continuous pseudometric on X (see [3, 8.5.6]). These two facts imply that $L(Y)$ can be identified with $L(Y,X)$. Since Y is closed in X we can repeat word for word the proof of [16, Proposition 3.8] to show that $L(Y,X)$ is closed in $L(X)$. ∎

We need also the following standard fact.

Lemma 3.4 *Let X and Y be Tychonoff spaces, let $f\colon X \to Y$ be a quotient mapping, and let $\Phi\colon L(X) \to L(Y)$ be a continuous linear operator such that $\Phi(x) = f(x)$ for each $x \in X$. Then Φ is a quotient map.*

Proof Let H be the kernel of Φ, let $q\colon L(X) \to L(X)/H$ be the quotient map and $i\colon L(X)/H \to L(Y)$ be the induced linear operator. So $\Phi = i \circ q$, and i is a continuous linear isomorphism. Since $f = i \circ q|_X$, the restriction $j := i|_{q(X)}$ is a quotient map by [3, 2.4.5]. As $j\colon q(X) \to Y$ is a continuous isomorphism, we obtain that j is a topological isomorphism of $q(X)$ onto Y (see [3, 2.4.7]). Consider the locally convex topology τ_i on the underlying linear space $L_a(Y)$ of $L(Y)$ induced by i from the quotient space $L(X)/H$. Then τ_i is finer than the topology τ of $L(Y)$ and $\tau_i|_Y = \tau|_Y$. By the definition of τ we obtain that $\tau_i = \tau$. Thus $L(Y)$ is a quotient space of $L(X)$. ∎

Proposition 3.5 *For each infinite compact space K, the space $L(K)$ is not a k-space.*

Proof By Lemma 3.2, there is a quotient mapping f of K onto an infinite metrizable compact space C. Since C contains a subspace that is homeomorphic to $\mathbf{s}$, Proposition 3.1 and Lemma 3.3 imply that $L(C)$ is not a k-space. Also taking into account that a quotient space of a k-space is a k-space, Lemma 3.4 implies that the space $L(K)$ is not a k-space. ∎

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2 Assume that $L(X)$ is a k-space. We show first that each compact subset K of X is finite. Indeed, suppose for a contradiction that X has an infinite compact subset K. Then, by Lemma 3.3 and Proposition 3.5, the k-space $L(X)$ has a closed subspace $L(K)$ that is not a k-space, a contradiction.

As $L(X)$ is a k-space we obtain that X is a k-space as well. Since each compact subset of X is finite we deduce that X is a discrete space. Now Theorem 2.1(iii) implies that X is countable.

The converse assertion is clear. ∎

Recall that a topological space X is called a k_R-*space* if it is Tychonoff and every $f\colon X \to \mathbb{R}$, whose restriction to each compact subset $K \subset X$ is continuous, is continuous on X. Clearly, all Tychonoff k-spaces are k_R-spaces; the converse is false.

Question 3.6 Let X be a Tychonoff space that is not a discrete countable space. Is $L(X)$ a non-k_R-space? What about $L(\mathbf{s})$ and $L(\kappa)$ for uncountable κ?

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