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Genus 2 Curves with Quaternionic Multiplication

Srinath Baba and Håkan Granath

Abstract. We explicitly construct the canonical rational models of Shimura curves, both analytically in terms of modular forms and algebraically in terms of coefficients of genus 2 curves, in the cases of quaternion algebras of discriminant 6 and 10. This emulates the classical construction in the elliptic curve case. We also give families of genus 2 QM curves, whose Jacobians are the corresponding abelian surfaces on the Shimura curve, and with coefficients that are modular forms of weight 12. We apply these results to show that our j-functions are supported exactly at those primes where the genus 2 curve does not admit potentially good reduction, and construct fields where this potentially good reduction is attained. Finally, using j, we construct the fields of moduli and definition for some moduli problems associated to the Atkin–Lehner group actions.

1 Introduction

An abelian surface is said to have quaternionic multiplication, or QM for short, if its endomorphism ring admits an embedding by an order $\mathcal{O}$ in an indefinite non-split quaternion algebra over $\mathbb{Q}$. In this paper we will only consider cases where $\mathcal{O}$ is a maximal order. A genus 2 curve is said to have QM if its Jacobian does. Given $\mathcal{O}$, let $\mathcal{O}^1$ denote the elements in $\mathcal{O}$ with norm 1. This group acts naturally, via an embedding of $\mathcal{O}$ into $M_2(\mathbb{R})$, on the upper half-plane $\mathcal{H}$. The quotient $V = \mathcal{H}/\mathcal{O}^1$, which is a compact Riemann surface called a Shimura curve, is the moduli space of the natural moduli problem of classifying abelian surfaces with QM by $\mathcal{O}$. Shimura proved that this curve admits a model over $\mathbb{Q}$. Our main goal in this paper is to construct explicitly this canonical model of the Shimura curves of discriminants 6 and 10.

The complex multiplication (or CM) points on V will play a big role for us. A QM abelian surface has complex multiplication if the center of the ring $\mathrm{End}(A)$ is a complex quadratic order $\mathfrak{O}$. Associated to every optimal embedding ι of a quadratic order $\mathfrak{O}$ into $\mathcal{O}$ is a fixed point $z_\iota \in \mathcal{H}$ such that the corresponding abelian surface has CM by $\mathcal{O}$. If $\mathrm{disc}(\mathfrak{O}) = D$, we say that the resulting point on the Shimura curve has CM by D.

The Shimura curve V comes equipped with a natural group of involutions which is called the Atkin–Lehner group. This group has 2^{2n} elements, where $2n$ is the number of primes dividing the discriminant of the quaternion algebra. The natural forgetful maps from V to other moduli spaces, which are induced by forgetting parts of the information given by the embedding of $\mathcal{O}$ into $\mathrm{End}(A)$, factor through quotients

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of V by subgroups of the Atkin–Lehner group. In particular, the natural map from V to the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces factors as a projection $V \to V/G$, where G has 2 or 4 elements, and a generically injective map. The case where G has 4 elements is called the twisting case.

The problem that we consider has two sides to it: on one side there are the abelian surfaces with QM considered as complex tori. There is a classical construction associating to a point z on the upper half-plane $\mathcal{H}$ the period matrix of a QM abelian surface. On the other side, there are the genus 2 curves whose Jacobians have QM. Families of such curves have been constructed by Hashimoto and Murabayashi in the cases of discriminant 6 and 10. What is missing is an explicit way to link a genus 2 curve in such a family to the point z in the upper half-plane giving its complex Jacobian. In addition, equations for Shimura curves have been computed by Kurihara and others. For example, it is known that in the case of $\mathcal{O}$ having discriminant 6, the Shimura curve has equation $x^2 + y^2 + 3 = 0$ defined over $\mathbb{Q}$. However, there is no explicit correspondence between points on this conic and points $z \in \mathcal{H}$. In this paper, we construct such correspondences in the discriminant 6 and 10 cases, and derive a complex uniformized family of genus 2 curves whose Jacobians are exactly those given by the classical construction.

The strategy in the paper is the following. On the one hand, there is a rational map of the Shimura curve into the moduli space $\mathcal{M}_2$ of genus 2 curves factoring through the action of the Atkin–Lehner group (the cases we consider are twisting cases in which the Atkin–Lehner groups have exactly 4 elements). We compute equations for the image in $\mathcal{M}_2$ in terms of Igusa's invariants $J_2, \dots, J_{10}$ using the work of Hashimoto and Murabayashi on families of genus 2 curves with QM [10]. Then we solve these equations, and get a parametrization of the image curve. This is now given in terms of the coefficients of the sextics defining the genus 2 curves. On the other hand, we determine the ring of modular forms with respect to $\mathcal{O}^1$, and use this to construct a generator of the rational function field in terms of modular forms. Finally, we are able to find some CM points on the Shimura curve where we know both sides, *i.e.,* both the genus 2 curve and the values of the modular forms. We then use this information to show that our two functions are equal. We denote the function we have constructed by j. This function is really defined on the quotient curve of the Shimura curve by the Atkin–Lehner involutions, so finally we construct functions on V which give the canonical structure of V as a curve over $\mathbb{Q}$.

By using a construction of Mestre, we recover the genus 2 curve from its j-invariant. Using the analytic description of j, we are thus able to construct a sextic whose coefficients are modular forms of weight 12, such that at any point z in $\mathcal{H}$, except for a finite set of orbits which will also be determined in the course of our investigations, the genus 2 curve C_z described by the sextic has Jacobian A_z. An important difference between our families and the ones given in [10] is that the former are families over V, while the latter factor via a cover of V.

The j-function that we define has many of the properties that its namesake in the elliptic curve case has. First, it follows immediately from our construction that $j(C)$ generates the field of moduli of the genus 2 curve C. Contrary to the elliptic curve case, the genus 2 curve can in general not be defined over its field of moduli. We express the Mestre obstruction for this curve in terms of j, and hence we know

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in principle every possible field of definition. Furthermore, we show that for any algebraic point on V, the prime places where j has non-zero valuation are exactly those where the genus 2 curve does not have potentially good reduction. This shows that our j-function has some arithmetic significance. Finally, using j, we construct the fields of moduli for the various moduli problems described by the Atkin–Lehner involutions.

One point in involving modular forms is, of course, to be able to numerically compute values of our j-function. To calculate values of modular forms on a compact Shimura curve, one can embed the curve in a Hilbert modular surface. By restricting Hilbert modular forms on $\mathcal{H} \times \mathcal{H}$ to the embedded curve and putting in natural correction factors, one gets modular forms on the Shimura curve. We describe this in Appendix A. In our cases, the curves can be embedded into the Hilbert modular surface corresponding to $\mathbb{Q}(\sqrt{5})$, and one can use the corresponding Eisenstein series on $\mathcal{H} \times \mathcal{H}$ to efficiently compute values of the modular forms occurring in this paper to arbitrary precision.

The obvious directions that should emerge from this work are extensions to higher discriminant and higher level. In the higher discriminant case, the difficulty lies in the absence of the Hashimoto–Murabayashi family. On the other hand, understanding the ring of modular forms may yield some success. In the cases that are considered here, better understanding of the behaviour of the j-invariant at algebraic points (in particular singular moduli) should prove useful in arithmetic applications.

The computations required in this paper were done using the computer algebra systems Macaulay 2 [7] and Pari/GP [20].

2 Preliminaries

2.1 Shimura curves

Let $B = B_\Delta$ denote the indefinite quaternion algebra of discriminant Δ over $\mathbb{Q}$, where $\Delta = p_1 \cdots p_{2n}$. Let $x \mapsto x^*$ be the canonical involution on B, so $\mathrm{nr}(x) = x^*x$, and let $\mathcal{O}$ be a maximal order in B_Δ. Following Rotger [23], we fix an element $\mu \in \mathcal{O}$ with $\mu^2 = -\Delta$ and we call the pair $(\mathcal{O}, \mu)$ a principally polarized maximal order in B. Define the positive anti-involution $a \mapsto a'$ on B by $a' = \mu^{-1}a^*\mu$, *i.e.,* the quadratic form $a \mapsto \mathrm{tr}(a'a)$ is positive definite. Let R be a maximally embedded subring of $\mathcal{O}$ invariant under $'$, *i.e.,* $R = \mathbb{Q}R \cap \mathcal{O}$ and $R' = R$. We call such rings *stable*.

Let $S_{(\mathcal{O},\mu)}$ be the set of triples $[A, \rho, \iota]$, where A is an abelian surface, ρ is a principal polarization on A, and $\iota\colon \mathcal{O} \to \mathrm{End}(A)$ is an embedding such that the Rosati involution defined by ρ on $\iota(\mathcal{O})$ is $'$. We recall how such triples are constructed. Fix an embedding $\theta\colon B \to M_2(\mathbb{R})$. For any point $z \in \mathcal{H}$, consider the lattice $\Lambda_z = \theta(\mathcal{O})v_z$ in $\mathbb{C}^2$, where $v_z = \begin{pmatrix} z & 1 \end{pmatrix}^t$. Define $A_z = \mathbb{C}^2/\Lambda_z$. On Λ_z, define the Riemann form

$$E_z\colon \Lambda_z \times \Lambda_z \to \mathbb{Z}$$
$$E_z(\theta(\lambda_1)v_z, \theta(\lambda_2)v_z) = \mathrm{tr}(\lambda_1^*\mu\lambda_2).$$

Then E_z defines a principal polarization ρ_z on A_z such that the Rosati involution on $\mathrm{End}_{\mathbb{Q}}(A_z)$ corresponds to the positive anti-involution $a \mapsto a'$ on B. This defines a

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triple $[A_z, \rho_z, \iota_z]$, where A_z is an abelian surface, ρ_z a principal polarization on A_z and $\iota_z\colon \mathcal{O} \to \mathrm{End}(A_z)$ an injection.

Let R be a stable subring of $\mathcal{O}$. Define an equivalence relation $\sim_R$ on $S_{(\mathcal{O},\mu)}$ as follows: $[A_1, \rho_1, \iota_1] \sim_R [A_2, \rho_2, \iota_2]$ if and only if there exists an isomorphism $\phi\colon A_1 \to A_2$ such that $\phi^*(\rho_2) = \rho_1$, and for every $r \in R$, the following diagram commutes.

$$\begin{array}{ccc} A_1 & \xrightarrow{\phi} & A_2 \\ {\scriptstyle \iota_1(r)}\downarrow & & \downarrow{\scriptstyle \iota_2(r)} \\ A_1 & \xrightarrow{\phi} & A_2 \end{array}$$

It turns out that there exists a coarse moduli space, which we denote by V_R, that classifies triples $[A, \rho, \iota]$ up to equivalence by the relation $\sim_R$. This situation was studied by Jordan [14] and Rotger [23], we describe this moduli space below.

The normalizer group $N_{B^+}(\mathcal{O})$ acts on $\mathcal{H}$ as fractional linear transformations through θ. This generates a subgroup of $\mathrm{Aut}(\mathcal{H})$ which we denote by $\tilde{\Gamma}$. The subgroup of $\tilde{\Gamma}$ generated by elements of $\mathcal{O}$ of norm 1 is denoted by Γ. Let $V = \mathcal{H}/\Gamma$, which is a compact Riemann surface. The group $\tilde{\Gamma}/\Gamma \cong (\mathbb{Z}/2)^{2n}$, the Atkin–Lehner group, acts on V. Any coset $w \in \tilde{\Gamma}/\Gamma$ is of the form $w = \gamma\Gamma$, where $\gamma \in \mathcal{O}$ with $\mathrm{nr}(\gamma) = d > 0$ and $d \mid \Delta$. We write $w = w_d$. For any subgroup G of $\tilde{\Gamma}/\Gamma$, we denote $V_G = V/G$. We also let $V_d = V/\langle w_d \rangle$ for any positive divisor d of Δ.

In [23], an element $\chi \in \mathcal{O} \cap N_B(\mathcal{O})$ is called *twisting* if $\mathrm{tr}(\chi) = \mathrm{tr}(\chi\mu) = 0$. Note that this implies that $\mathrm{nr}(\chi) < 0$, so if, additionally, $\chi \in \mathcal{O}$ is assumed to be primitive, then we have $B = \mathbb{Q}(\mu, \chi)$ where $\mu^2 = -\Delta$, $\chi^2 = m$, and $\mu\chi = -\chi\mu$, for some positive integer m dividing Δ. A quadratic stable ring R is called *twisting* if it contains a twisting element. Not all pairs $(\mathcal{O}, \mu)$ have twisting subrings, and if they exist there are exactly 2 of them. From now on, we assume that this is the case (it holds for $\Delta = 6, 10$). The twisting rings naturally correspond to two elements w_m, $w_{m'}$ of the Atkin–Lehner group with $mm' = \Delta$. We denote the twisting rings by R_m and $R_{m'}$ respectively and let $R_\Delta = \mathbb{Z}[\mu]$. The subgroup of the Atkin–Lehner group generated by w_m and $w_{m'}$ has 4 elements and is denoted by W.

The following result shows how we can identify the moduli spaces above with quotients of V by appropriate subgroups of the Atkin–Lehner groups.

Theorem 2.1 *There are natural identifications $V_{\mathcal{O}} = V$, $V_{R_d} = V_d$ for $d \in \{m, m', \Delta\}$ and $V_{\mathbb{Z}} = V_W$. For any other stable quadratic ring R, we have $V_R = V$.*

This theorem is a reformulation of more general results proved in [23]. In fact, the following holds:

> $[A_z, \rho_z, \iota_z] \sim_R [A_{z'}, \rho_{z'}, \iota_{z'}]$ if and only if the points z and z' are related by $z' = \gamma z$, where $\gamma = w\varepsilon \in N_{B^+}(\mathcal{O})$, $w \in Z(R)$, $\varepsilon \in \mathcal{O}^*$ and $w\mu = \mathrm{nr}(\varepsilon)\mu w$.

Here $Z(R)$ denotes the centralizer of R in $\mathcal{O}$. It is straightforward to verify that this claim is equivalent to Theorem 2.1.

We recall some fundamental facts that will be used in this paper. First, the curve V has a canonical model defined over $\mathbb{Q}$ (see [26]). The Atkin–Lehner involutions are also defined over $\mathbb{Q}$, so all quotient curves V_G are defined over $\mathbb{Q}$. Second, for any imaginary quadratic order $\mathfrak{O}_D$ that embeds into $\mathcal{O}$, the coordinates of a point with complex multiplication by $\mathfrak{O}_D$ on any rational model of V are in the ring class field $H(\mathfrak{O}_D)$ of $\mathfrak{O}_D$ (see [27]). The number of points on V with given CM can be computed by Eichler's theory of optimal embeddings, see [28].

Define the field of moduli $k_{\mathbb{Z}}$ of $[A, \rho]/\bar{\mathbb{Q}}$ as the smallest extension of $\mathbb{Q}$ such that for any $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}}/k_{\mathbb{Z}})$, there is an isomorphism $\phi_\sigma \colon A \to A^\sigma$ such that $\phi_\sigma^*(\rho^\sigma) = \rho$. For any subring R of $\operatorname{End}(A)$, we define the field of moduli k_R as the minimal number field containing $k_{\mathbb{Z}}$ with the property that for any $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}}/k_R)$, there is an isomorphism $\phi_\sigma \colon A \to A^\sigma, \phi_\sigma^*(\rho^\sigma) = \rho$ such that the following diagram commutes for each $r \in R$

$$\begin{array}{ccc} A & \longrightarrow & A^\sigma \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r^\sigma} \\ A & \longrightarrow & A^\sigma \end{array}$$

Let $\mathcal{M}_2$ denote the moduli space of genus 2 curves, and $\mathcal{A}_2$ denote the moduli space of principally polarized abelian surfaces. The open Torelli map assigns to a point $C \in \mathcal{M}_2$ the pair $(\operatorname{Pic}^0(C), \Theta)$, where Θ is the theta divisor that embeds C into $\operatorname{Pic}^0(C)$. Thus, $\mathcal{M}_2$ maps onto a Zariski open subset of $\mathcal{A}_2$, and so the general principally polarized abelian surface is a Jacobian. Let $\tilde{E} = \tilde{E}_\Delta$ denote the image of the natural forgetful map $V \to \mathcal{A}_2$, and let $E = E_\Delta$ denote the intersection with $\mathcal{M}_2$. We have the following picture:

$$\begin{array}{ccccc} & & \mathcal{A}_2 & \hookleftarrow & \mathcal{M}_2 \\ & & \uparrow & & \uparrow \\ V & \xrightarrow{4:1} & \tilde{E} & \hookleftarrow & E \end{array}$$

2.2 Igusa Invariants and Mestre's Construction

Let C be a non-singular genus 2 curve, with a hyperelliptic model

$$C : z^2 = a_6x^6 + a_5x^5y + a_4x^4y^2 + a_3x^3y^3 + a_2x^2y^4 + a_1xy^5 + a_0y^6.$$

Igusa [13] defined invariants $J_i = J_i(C)$, for $i = 2, 4, 6, 10$, which are homogenous polynomials of degree i in the coefficients a_k having rational coefficients. In particular, $2^{10}J_{10}$ is the discriminant of the sextic. Two curves are isomorphic if and only if they define the same point $p = [J_2, J_4, J_6, J_{10}]$ in the weighted projective space $\mathbb{P}(2, 4, 6, 10)$. The minimal field k over which the point p is defined is called the field of moduli of C.

Mestre [18] showed how to solve the inverse problem of constructing the genus 2 curve C from its Igusa invariants. He constructed a conic

$$L = \sum_{1 \le i, j \le 3} A_{ij}x_ix_j, \tag{2.1}$$

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and a cubic

$$M = \sum_{1 \le i,j,k \le 3} a_{ijk} x_i x_j x_k, \tag{2.2}$$

where the coefficients A_{ij} and a_{ijk} can be expressed in terms of the J_i's. The conic L is degenerate if and only if the curve C has more automorphisms than just the hyperelliptic involution. In this case, we say that C has non-trivial involutions. If so, then it follows from [3] that C is defined over k. If not, then the curve C can in general not be defined over k. In this case, C can be defined over a field extension K/k if and only if L is isotropic over K. It is therefore natural to consider the even Clifford algebra over k associated with L, and we denote it H_C. The quaternion algebra H_C is called the Mestre obstruction of C, and it has the property that C can be defined over K if and only if K splits H_C. One recovers a hyperelliptic model $z^2 = f(x, y)$ from the Igusa invariants by finding a parametrization $x_i = x_i(x, y)$ of the solutions to the equation $L = 0$ and setting

$$f(x, y) = \sum_{1 \le i,j,k \le 3} a_{ijk} x_i(x, y) x_j(x, y) x_k(x, y).$$

3 Discriminant 6 Case

If k is a field and $a, b \in k^*$, then we use the notation $(a, b)_k$ for the quaternion algebra over k generated by elements i and j satisfying $i^2 = a$, $j^2 = b$ and $ij + ji = 0$.

Let $B = (2, -3)_{\mathbb{Q}}$, so $\Delta = \operatorname{disc}(B) = 6$. We choose a maximal order $\mathcal{O} = \mathbb{Z}[i, (j + 1)/2]$ in B, and an element $\mu = 2j + ij \in \mathcal{O}$ with $\mu^2 = -6$. We let $R_6 = \mathbb{Z}[\mu] \cong \mathbb{Z}[\sqrt{-6}]$. There are two twisting rings in this case, namely $R_2 = \mathbb{Z}[i] \cong \mathbb{Z}[\sqrt{2}]$ and $R_3 = \mathbb{Z}[j + ij] \cong \mathbb{Z}[\sqrt{3}]$.

From [25] it follows that the curve V has genus 0 and that there are 2 classes of elliptic fixed points of order 2 and 3 respectively for the action of Γ on $\mathcal{H}$. The involutions w_d on V, for $d = 2, 3, 6$, each have 2 fixed points. In the cases of $d = 2$ and $d = 3$, these are exactly the elliptic points of order d. Let z_d, z'_d denote the fixed points of w_d on V. Let x_d and x'_d be a choice of points on the upper half plane that descend to the points z_d, z'_d, for $d = 2, 3, 6$. By abuse of notation, we denote also by z_d, z'_d the images of these points on quotient surfaces V_G, for any subgroup G of the Atkin–Lehner group.

Lemma 3.1 *The curve V_W is isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$ and $z_2, z_3, z_6 \in V_W(\mathbb{Q})$.*

Proof The points z_2, z'_2 are points with complex multiplication by $\mathbb{Q}(\sqrt{-1})$, so $z_2, z'_2 \in V(H(\mathbb{Q}(\sqrt{-1}))) = V(\mathbb{Q}(\sqrt{-1}))$. These two points are transposed by w_3 and also by complex conjugation. We conclude that $z_2, z'_2 \in V_3(\mathbb{Q})$. In particular $z_2 \in V_W(\mathbb{Q})$ and we get that $V_W \cong \mathbb{P}^1_{\mathbb{Q}}$.

The same argument, using $z_3, z'_3 \in V(H(\mathbb{Q}(\sqrt{-3}))) = V(\mathbb{Q}(\sqrt{-3}))$, gives that $z_3 \in V_W(\mathbb{Q})$.

The fixed points of w_2 on V_3 are the two points z_2 and z_6. Since $V_3 \cong \mathbb{P}^1_{\mathbb{Q}}$, we conclude that z_2 and z_6 are defined over a common minimal field F at most quadratic

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over $\mathbb{Q}$. If $F \neq \mathbb{Q}$, then we would have that z_2 and z_6 are conjugates under the nontrivial element $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$. Hence the images on V_W are also conjugates, but since $z_3 \in V_W(\mathbb{Q})$ we would get $z_3 = z_6$ on V_W, a contradiction. ■

3.1 The Ring of Modular Forms

For any Fuchsian group G acting on $\mathcal{H}$, we denote by $v_i(G)$ the number of elliptic fixed points of order i for G, and by $s_k(G)$ the dimension of the space of holomorphic weight k modular forms for G. Let $\Gamma_d = \Gamma \cup w_d\Gamma$ and $\Gamma_W = \bigcup_{w\in W} w\Gamma$. Using the formulae in [25], it follows that the degree of the divisor of a modular form of weight k is $k/6$ and that $s_k = 1 - k + 2\lfloor k/4\rfloor + 2\lfloor k/3\rfloor$ for k even and $k > 2$, and 0 otherwise. We compute the following table:

G	v_2	v_3	v_4	v_6	s_4	s_6	s_{12}
Γ	2	2	0	0	1	1	3
Γ_2	0	1	2	0	0	1	1
Γ_3	1	0	0	2	0	0	2
Γ_6	3	1	0	0	1	0	2
Γ_W	1	0	1	1	0	0	1

We conclude that $S_4(\Gamma)$ is generated by a form $h_4(z)$ which vanishes necessarily at x_3 and x_3'. Similarly, $S_6(\Gamma)$ is generated by a form $h_6(z)$ which vanishes necessarily at x_2 and x_2'. We assume that h_6 is normalized so that $h_4^3/h_6^2(x_6) = \sqrt{-3}$. We fix a basis of $S_{12}(\Gamma)$ by the forms h_4^3, h_6^2, h_{12}, where h_{12} is chosen so that it vanishes at x_6 and x_6'. We normalize h_{12} so that $h_{12}^2 + 3h_6^4 + h_4^6 = 0$. Finally, the action of the Atkin–Lehner group on the modular forms is given by

$$w_6(h_4) = h_4 = -w_2(h_4) = -w_3(h_4),$$
$$w_2(h_6) = h_6 = -w_3(h_6) = -w_6(h_6),$$
$$w_3(h_{12}) = h_{12} = -w_2(h_{12}) = -w_6(h_{12}).$$

Proposition 3.2 *The modular forms h_4 and h_6 are algebraically independent, and*

$$\bigoplus_{k=0}^{\infty} S_{2k}(\Gamma) = \mathbb{C}[h_4(z), h_6(z), h_{12}(z)] \cong \mathbb{C}[h_4, h_6, h_{12}]/(h_{12}^2 + 3h_6^4 + h_4^6)$$

as graded rings.

Proof To see that $h_4(z)$ and $h_6(z)$ are algebraically independent, first note that any linear combination of modular forms that is identically 0 must have terms of the same weight, and so any polynomial in $h_4(z)$ and $h_6(z)$ that vanishes identically must be a linear combination of monomials of the same weight. Also, any polynomial in $h_4(z)$ and $h_6(z)$ that is identically zero is divisible by h_6, for if not, we could write it as $ch_4(z)^m + h_6(z)P(h_4(z), h_6(z))$ for some polynomial P and non-zero constant c, and this would vanish at x_2 but not at x_3. Thus, dividing by $h_6(z)$ would give an algebraic relation of lower degree, and so algebraic independence follows by induction on the degree of the polynomial.

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To see that the whole ring of modular forms is generated by h_4, h_6 and h_{12}, we observe that for weights $k \leq 12$ it follows from the dimension formula that all forms are combinations of h_4, h_6 and h_{12}. It also follows that $\dim S_{k+12} = \dim S_k + 2$ for all k. The same recursion formula holds for the graded polynomial ring

$$\mathbb{C}[h_4, h_6, h_{12}]/(h_{12}^2 + 3h_6^4 + h_4^6),$$

and we are done. ■

Now, we define the weight 0 modular form j_m, invariant under the action of w_6, as $j_m = 4h_6^2/3h_4^3$. Since the degree of the divisor of a modular form of weight 12 is 2, j_m defines a double cover $V \to \mathbb{P}^1$, and we get the following.

Proposition 3.3 *The holomorphic map $j_m \colon V_6 \to \mathbb{P}^1$ is an isomorphism. Furthermore $j_m(w_2(z)) = j_m(w_3(z)) = -j_m(z)$.*

3.2 Equations for E_6 and the Arithmetic j-Function

Our goal is to give equations for E_6 and construct an embedding of E_6 into $\mathbb{P}^1$ in terms of the Igusa invariants J_n. Our starting point is the following important result from [10].

Theorem 3.4 *The following equations give a family of QM-curves with respect to $\mathcal{O}$:*

$$y^2 = x(x^4 - Px^3 + Qx^2 - Rx + 1)$$

with

$$4s^2t^2 - s^2 + t^2 + 2 = 0,$$

$$P = -2(s+t), \quad R = -2(s-t), \quad Q = \frac{(1+2t^2)(11 - 28t^2 + 8t^4)}{3(1-t^2)(1-4t^2)}.$$

We rewrite these equations in the coordinates on $\mathcal{M}_2$ given by the Igusa invariants as follows.

Proposition 3.5 *The equations for E_6 are*

$$J_2^6 - 68J_2^4J_4 + 1296J_2^2J_4^2 + 216J_2^3J_6 - 4608J_4^3 - 6912J_2J_4J_6 + 15552J_6^2 = 0, \tag{3.1a}$$

$$J_2^5 - 60J_2^3J_4 + 864J_2J_4^2 + 216J_2^2J_6 - 5184J_4J_6 + 248832J_{10} = 0. \tag{3.1b}$$

The map given by $\varphi(p) = [J_2(24J_4 - J_2^2), 432J_6 - 96J_4J_2 + 3J_2^3] \in \mathbb{P}^1$ for $p \in E_6$ gives an embedding $\varphi \colon E_6 \to \mathbb{P}^1$.

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Proof A calculation gives that (3.1a) and (3.1b) are satisfied by the family of genus 2 curves given in Theorem 3.4. Let

$$A = J_2, \quad B = J_2^2 - 24J_4, \quad C = 432J_6 - 96J_4J_2 + 3J_2^3, \quad D = 12^5 J_{10}.$$

In these variables, equations (3.1) become

$$(AB + C)(AB - C) = 4B^3, \quad 2D = B(AB - C). \tag{3.2}$$

These equations define an irreducible reduced rational curve $E_{6,0}$ in $\mathbb{P}(2, 4, 6, 10)$ with a single simple node at $[A, B, C, D] = [1, 0, 0, 0]$. Hence (3.1) generate all relations. It is easy to see that the map given by $p \mapsto [AB, C]$ induces an isomorphism from the non-singular resolution of this curve to $\mathbb{P}^1$. ■

With notations as in the proof of Proposition 3.5, we define the following function on E_6

$$j = j_6 = \frac{AB - C}{AB + C} = \frac{D^2}{B^5}.$$

where the second equality follows from (3.2). Since the image of E_6 in $\mathbb{P}(2, 4, 6, 10)$ is the non-singular locus of the curve given by (3.2), we get the following.

Proposition 3.6 *The map j defines an isomorphism $E_6 \to \mathbb{P}^1 \setminus \{0, \infty\}$.*

The inverse map is given by $[A, B, C, D] = [j + 1, j, j(1 - j), j^3]$, or, equivalently, the Igusa invariants are given in terms of j by

$$[J_2, J_4, J_6, J_{10}] = [12(j + 1), 6(j^2 + j + 1), 4(j^3 - 2j^2 + 1), j^3].$$

Since $\tilde{E}_6$ is the resolution of the nodal curve $E_{6,0}$, there are two points on $\tilde{E}_6$ which are not Jacobians of genus 2 curves. By [11], we know that these points correspond to the points on $\tilde{E}_6$ with CM by $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. What is happening in these cases, is that the abelian surfaces are products of elliptic curves with the product principal polarization.

Remark. The map from the family in Theorem 3.4 to the curve E_6 is of degree 24 and is given by

$$j = \frac{16(2t^2 + 1)^4(t^2 - 1)^2}{27(4t^2 - 1)^3}.$$

3.3 Some Explicit Points on E_6

To compare the algebraically and analytically defined functions, we will need to compute a few explicit points on E_6. In this section, we compute the curves with discriminant 6 QM and D CM for $D = -24$ and $D = -19$.

We consider first the case $D = -24$. Let $K = \mathbb{Q}(\sqrt{-6})$. The class number $h(K) = 2$, and a non-trivial ideal class in $\mathfrak{O} = \mathbb{Z}[\sqrt{-6}]$ is given by the ideal

$\mathfrak{a} = (2, \sqrt{-6})$. Hence, there are two isomorphism classes of elliptic curves with $\mathbb{Z}[\sqrt{-6}]$ CM, namely the curves $\mathcal{E}_1 = \mathbb{C}/\mathfrak{O}$ and $\mathcal{E}_2 = \mathbb{C}/\mathfrak{a}$ with j-invariants

$$j(\mathcal{E}_1) = j(\sqrt{-6}) = 1728(1399 + 988\sqrt{2})$$

and

$$j(\mathcal{E}_2) = j(\sqrt{-6}/2) = 1728(1399 - 988\sqrt{2}).$$

Consider the cubic

$$f(t) = (1 + \sqrt{2})t^3 - 3(7 - 3\sqrt{2})t^2 - 3(7 + 3\sqrt{2})t + (1 - \sqrt{2}),$$

and define a genus 2 curve $C_{(-24)}$ by $z^2 = f(x^2)$. By a direct computation of Igusa invariants one gets that $C_{(-24)}$ corresponds to a point on E_6, and has invariant $j = -16/27$.

Proposition 3.7 *The Jacobian J of $C_{(-24)}$ is isomorphic to $\mathcal{E}_1 \times \mathcal{E}_2$ and has endomorphism ring $\left(\begin{smallmatrix} \mathfrak{O} & \mathfrak{a} \\ \mathfrak{a}^{-1} & \mathfrak{O} \end{smallmatrix}\right)$. Furthermore, this curve defines a point on E_6 with $j(C_{(-24)}) = -16/27$.*

Proof Consider the non-hyperelliptic involutions u and v on $C_{(-24)}$ given by $u(x, z) = (-x, z)$ and $v(x, z) = (-x, -z)$. The quotient $C_{(-24)}/u$ has equation $s^2 = f(t)$ (let $s = z$ and $t = x^2$), which is an elliptic curve with j-invariant $1728(1399 - 988\sqrt{2})$. Hence $C_{(-24)}/u \cong \mathcal{E}_2$.

Similarly the quotient $C_{(-24)}/v$ has equation $s^2 = tf(t)$ (where $s = xz$, $t = x^2$), which has j-invariant $1728(1399 + 988\sqrt{2})$, so $C_{(-24)}/v \cong \mathcal{E}_1$.

Let $f_i \colon C_{(-24)} \to \mathcal{E}_i$, $i = 1, 2$, be the corresponding quotient maps. They induce a natural surjective homomorphism of abelian varieties $\varphi\colon J \to \mathcal{E}_1 \times \mathcal{E}_2$, given by

$$\varphi(\mathcal{O}(a - b)) = (f_1(a) - f_1(b), f_2(a) - f_2(b)),$$

where $a, b \in C_{(-24)}$ (the minus sign on the left-hand side is given by the group laws of the elliptic curves). By a direct computation, one gets that $f_i(a) = f_i(b)$ for $i = 1, 2$ if and only if $a = b$. Hence φ is an isomorphism. The claim about the endomorphism ring follows immediately.

The final claim follows from a direct computation. ■

Remark. By Proposition 3.7, the endomorphism ring is a maximal order in $M_2(K)$ which is not isomorphic to $M_2(\mathfrak{O})$ (since the class of $\mathfrak{a}$ is not a square in the class group of K). Hence one gets in particular that the Jacobian of $C_{(-24)}$ cannot be isomorphic to the square $\mathcal{E} \times \mathcal{E}$ of an elliptic curve $\mathcal{E}$.

We now consider the case $D = -19$. Let $K = \mathbb{Q}(\sqrt{-19})$ and $\mathfrak{O} = \mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$. The class number of K is one, so there is only one elliptic curve $\mathcal{E}_3 = \mathbb{C}/\mathfrak{O}$ with CM by $\mathfrak{O}$. It has j-invariant $j(\mathcal{E}_3) = -96^3$. Consider the cubic

$$f(t) = 2t^3 - 3(1 + 9\sqrt{-19})t^2 - 3(1 - 9\sqrt{-19})t + 2,$$

and define a genus 2 curve $C_{(-19)}$ by $z^2 = f(x^2)$. This curve corresponds to a point on E_6 with $j = 81/64$, and by an argument similar to the proof of Proposition 3.7 one gets the following.

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Proposition 3.8 *The Jacobian J of $C_{(-19)}$ is isomorphic to $\mathcal{E}_3 \times \mathcal{E}_3$ and has endomorphism ring $M_2(\mathfrak{O})$. Furthermore, this curve defines a point on E with $j(C_{(-19)}) =$ $81/64$.*

3.4 The Arithmetic Function and the Analytic Function

The main results of this paper relate the arithmetically and the analytically defined functions and give an explicit map from V to its rational model in terms of modular forms. The first step in this direction is the following.

Proposition 3.9 *We have $j = j_m^2$, considered as functions on $\tilde{E}_6$.*

Proof It is clear, from Propositions 3.3 and 3.6, that j and j_m^2 are related by a linear fractional transformation. We noted in Section 3.2 that the points on V_W corresponding to $J_{10} = 0$ on E_6 are the points z_2 and z_3. These points are therefore the zeroes and poles of j. Furthermore, $j(z_6) = -16/27$ by Proposition 3.7. Now, we have $j_m^2(z_2) = 0$, $j_m^2(z_3) = \infty$, and $j_m^2(z_6) = -16/27$ by our choices of normalization. Hence we conclude that $j = j_m^2$ or $j = \frac{16^2}{27^2 j_m^2}$.

To determine which possibility is the correct one, we use the -19 CM point. By Proposition 3.8, we get that $j = 81/64$ at this point. Now we can numerically compute j_m^2 at the point $z_{(-19)}$, which is the fixed point attached to the order $\mathbb{Z}[i + (1 + 3j)/2]$. This calculation can be done with full control of the size of the error terms, and doing this shows that the first possibility is the correct one. ■

By Proposition 3.9 and the identity $-27j - 16 = (4h_{12}/h_4^3)^2$, we see that $\sqrt{j}$ and $\sqrt{-27j - 16}$ lift to meromorphic functions on V.

Theorem 3.10 *The holomorphic map $f\colon V \to X = \{x^2 + 3y^2 + z^2 = 0\}$ given by*

$$f(z) = [h_4^3(z), h_6^2(z), h_{12}(z)] = [4, 3\sqrt{j}, \sqrt{-27j - 16}]$$

descends to an isomorphism from the canonical $\mathbb{Q}$-model of V to X viewed as a curve over $\mathbb{Q}$.

Proof By [16], we know that there exists an isomorphism $g\colon V \to X$ of curves over $\mathbb{Q}$, so we only need to show that the automorphism $f \circ g^{-1}$ of X is defined over $\mathbb{Q}$.

Let $K_1 = \mathbb{Q}(\sqrt{-1}, \sqrt{-3})$. By [25], we have that the two elliptic points z_2, z_2' of order 2 on V belong to $V(H(\mathbb{Q}(\sqrt{-1}))) \subset V(K_1)$. Similarly, the two elliptic points z_3, z_3' of order 3 belong to $V(K_1)$. Furthermore $j(z) = 0$, so $f(z) = [1, 0, \pm\sqrt{-1}]$ for $z = z_2, z_2'$, and similarly $f(z) = [0, 1, \pm\sqrt{-3}]$ for $z = z_3, z_3'$. Hence, f maps 4 points on $V(K_1)$ to 4 points on $X(K_1)$. We conclude that $f \circ g^{-1}$ maps 4 points on $X(K_1)$ into $X(K_1)$, and since X is isomorphic to $\mathbb{P}^1$ over K_1, we get that $f \circ g^{-1}$ is defined over K_1.

Let $K_2 = \mathbb{Q}(\sqrt{-19})$ and consider the 4 points $z \in V(K_2)$, with -19 CM. We have, by Proposition 3.8 that $j(z) = 81/64$, so $f(z) = [32, \pm 27, \pm 13\sqrt{-19}] \in X(K_2)$ for these points z. Hence as above $f \circ g^{-1}$ is defined over K_2.

We conclude that $f \circ g^{-1}$ is defined over $K_1 \cap K_2 = \mathbb{Q}$. ■

Remark. The equation for V given in Theorem 3.10 is, of course, known, see [16]. What is new with this result is the explicitly given isomorphism.

Corollary 3.11 *The Atkin–Lehner action on the curve X is given by*

$$w_2(p) = [x, -y, z], \quad w_3(p) = [-x, y, z],$$

where $p = [x, y, z] \in X$. In particular, we get $V_d \cong \mathbb{P}^1_{\mathbb{Q}}$ for $d = 2, 3, 6$, and the projection maps $\pi_d \colon V \to V_d$, $\pi_W \colon V \to V_W$ are given by $\pi_2(p) = [x, z]$, $\pi_3(p) = [y, z]$, $\pi_6(p) = [x, y]$, $\pi_W(p) = [x^2, y^2]$.

Proof This follows immediately from Theorem 3.10, since we know the action of W on the modular forms. ■

3.5 Mestre's Obstruction and a Family over E_6

Let $L(j)$ be the matrix of the quadratic form (2.1) where we have made the substitutions $J_2 = 12(j + 1)$, $J_4 = 6(j^2 + j + 1)$, $J_6 = 4(j^3 - 2j^2 + 1)$, and $J_{10} = j^3$.

Proposition 3.12 *The only curves on E_6 which have non-trivial automorphisms are the two curves with -24 and -19 CM respectively. In both of these cases, the automorphism group is isomorphic to $C_2 \times C_2$.*

Proof The curve C with $j(C) = j$ has non-trivial automorphisms if and only if the matrix $L(j)$ has vanishing determinant. Now we get

$$\det(L(j)) = -2^{33}3^{-18}5^{-20}j^7(64j - 81)^2(27j + 16),$$

so the first claim follows immediately from our computations in Section 3.3.

It is well known, see [1], that any genus 2 curve with automorphism group other than C_2 or $C_2 \times C_2$ has a model of the form $y^2 = x^5 + tx^3 + x$ or $y^2 = x^6 + tx^3 + 1$ for some $t \in \mathbb{C}$, or is the exceptional curve $y^2 = x^5 - x$. By comparing Igusa invariants, one checks that neither of the curves $C_{(-24)}$ and $C_{(-19)}$ can be written in this form. ■

Consider now the matrix $N(j) =$

$$3^2\begin{pmatrix} -32(96j^3 - 76j^2 + 75j - 108) & 48(296j^3 - 13j^2 - 564j) & 16(2856j^3 - 2389j^2 + 684j + 864) \\ 1800(8j^2 - 6j - 9) & 38700(2j^2 - 3j) & 900(242j^2 - 65j - 72) \\ 16875(8j - 9) & 50625j & 16875(j - 36) \end{pmatrix}.$$

We have $\det(N(j)) = 2^7 3^{11} 5^{10} j^3(64j - 81)^2$, and get

$$N(j)^t L(j) N(j) = -2^{15}3j^4(64j - 81)^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6j & 0 \\ 0 & 0 & 2(27j + 16) \end{pmatrix}. \tag{3.3}$$

Hence we have the following (*cf.* [2, Theorem 4.5]).

Proposition 3.13 *If C is on E_6 and has trivial automorphisms, then the Mestre obstruction H_C is the quaternion algebra $(-6j, -2(27j + 16))_{k_Z}$.*

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Now we can get explicit equations for the sextic defining C when C has trivial automorphism group.

Theorem 3.14 *The curve C is defined over, for example, the field $K = \mathbb{Q}(\sqrt{-6j})$. An explicit equation is*

$$f(x) = (-4+3s)x^6 + 6tx^5 + 3t(28+9s)x^4 - 4t^2x^3 + 3t^2(28-9s)x^2 + 6t^3x - t^3(4+3s),$$

where $t = -2(27j+16)$, $s = \sqrt{-6j}$, i.e., *the corresponding genus 2 curve $C : y^2 = f(x)$ lies on E_6 and $j(C) = j$.*

Proof We see immediately from (3.3) that the Mestre conic (2.1) is parametrised by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = N(j) \begin{pmatrix} x^2 + ty^2 \\ x^2 - ty^2 \\ -2sxy \end{pmatrix}.$$

Plugging this into the Mestre cubic (2.2), we get the result after dehomogenisation and a slight simplification. ∎

Corollary 3.15 *For any point $z \in \mathcal{H}$, the curve $C_z : y^2 = g_z(x)$, where*

$$g_z(x) = h_4^3(z)(x^6 - 21x^4 - 21x^2 + 1) + \sqrt{-6}h_6^2(z)(x^6 + 9x^4 - 9x^2 - 1) + 2\sqrt{2}h_{12}(z)x(3x^4 - 2x^2 + 3)$$

is a point on E_6 with $j(C_z) = j(z)$ and Jacobian isomorphic to (A_z, ρ_z).

Proof Using Theorem 3.10, we get that

$$s = \frac{4\sqrt{-6}h_6^2(z)}{3h_4^3} \quad \text{and} \quad t = \frac{32h_{12}^2(z)}{h_4^6(z)},$$

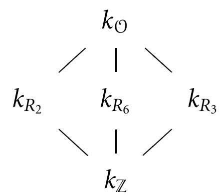
which gives the result in the case of trivial automorphism group. One can also verify that the model works also for the two points with -19 and -24 CM respectively. ∎

3.6 Arithmetic Properties

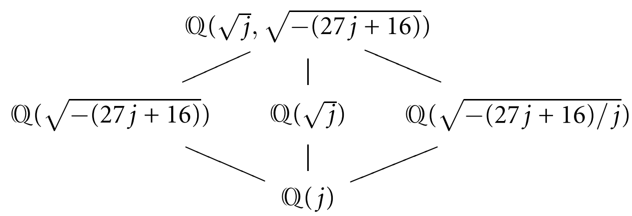
Proposition 3.16 *The field of moduli of the curve C is $k_{\mathbb{Z}} = \mathbb{Q}[j(C)]$.*

Proof If σ is an automorphism of $\mathbb{C}$, then $C^\sigma \cong C$ if and only if $j(C^\sigma) = j(C)$ if and only if $j(C)^\sigma = j(C)$. The claim follows. ∎

Theorem 3.17 *The diagram*



is given by



Proof That $k_{\mathcal{O}} = \mathbb{Q}(\sqrt{j}, \sqrt{-(27j+16)})$ follows from Theorem 3.10. That $k_{\mathbb{Z}}$ is $\mathbb{Q}[j(C)]$ is Proposition 3.16. By the moduli property of V_d and by the explicit descriptions of the maps given in Theorem 3.10, we see that the two diagrams are the same. ■

Proposition 3.18 *If K is a field of definition of C, then $L = K \cdot k_{\mathcal{O}}$ is a field of definition of the endomorphisms,* i.e., $\mathrm{End}_L(A) \cong \mathcal{O}$.

Proof By [15, Theorem 1.1], it is enough to show that the field L splits B. Now we get

$$B \otimes_{\mathbb{Q}} k_{\mathcal{O}} \cong H_C \otimes_{k_{\mathbb{Z}}} k_{\mathcal{O}},$$

since $H_C \otimes_{k_{\mathbb{Z}}} k_{\mathcal{O}} \cong (-6j, -2(27j+16))_{k_{\mathcal{O}}} \cong (-6, 2)_{k_{\mathcal{O}}} \cong B \otimes_{\mathbb{Q}} k_{\mathcal{O}}$. But

$$H_C \otimes_{k_{\mathbb{Z}}} K \cong M_2(K)$$

since K is a field of definition for C, so we get $B \otimes_{k_{\mathbb{Z}}} L \cong M_2(L)$. ■

Remark. It follows that the curve C is always defined over the field $k_{\mathcal{O}}[\sqrt{-6}]$.

The following results show that our choice of j-function is reasonable from an arithmetic point of view.

Proposition 3.19 *Let C on E_6 be such that $k_{\mathbb{Z}}$ is a number field. Then C has potentially good reduction at a prime p not dividing* 6 *if and only if*

$$v_p(j) = 0. \tag{3.4}$$

In fact, the curve attains good reduction over the field

$$K = k_{\mathbb{Z}}[\sqrt{-6j}, \sqrt{-2(27j+16)}] \subseteq k_{\mathcal{O}}[\sqrt{2}, \sqrt{-3}].$$

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Proof First we prove that (3.4) is necessary. Recall that, with notations as in the proof of Proposition 3.5, we have $j = D^2/B^5$. Also, equations (3.2) show that $p \mid D$ if and only if $p \mid B$. If the curve has potentially good reduction at p, then there exists an integral model over some extension field such that D is a unit at p. Hence B is also a unit at p, so (3.4) follows.

To prove that (3.4) is sufficient, we use our model in Theorem 3.14. If we let $u = \sqrt{t}$, the model simplifies to

$$y^2 = (-4 + 3s)x^6 + 6ux^5 + 3(28 + 9s)x^4 - 4ux^3 + 3(28 - 9s)x^2 + 6ux - (4 + 3s).$$

In this model, we have $j = -s^2/6$ and $J_{10} = 2^{27}3^{12}s^6$. Hence, if j is a unit in $\mathfrak{O}_p$, then the discriminant of the sextic, which equals $2^{12}J_{10}$, is also a unit and consequently the curve has good reduction at p. ■

4 Discriminant 10 Case

Now we want to do exactly the same thing in the discriminant 10 case as we have done for discriminant 6. The presentation will, however, be briefer this time, and we omit proofs in those cases where they are completely analogous to what we have already done.

Let $B = \mathbb{Q}(i, j)$, where $i^2 = 2$, $j^2 = 5$ and $ij + ji = 0$, so $\Delta = \operatorname{disc}(B) = 10$. We choose the maximal order $\mathcal{O} = \mathbb{Z}[i, (1 + j)/2]$ in B, and the element $\mu = ij \in \mathcal{O}$ with $\mu^2 = -10$. We let $R_{10} = \mathbb{Z}[\mu] \cong \mathbb{Z}[\sqrt{-10}]$. There are two twisting rings in this case, namely $R_2 = \mathbb{Z}[i] \cong \mathbb{Z}[\sqrt{2}]$ and $R_5 = \mathbb{Z}[(1 + j)/2] \cong \mathbb{Z}[(1 + \sqrt{5})/2]$.

The ring of modular forms with respect to Γ is generated by one element $g(z)$ of weight 4 and three forms $a_2(z)$, $a_5(z)$ and $a_{10}(z)$ of weight 6. The ring of modular forms with respect to Γ_d is generated by $g(z)$ and $a_d(z)$, for $d = 2, 5$ and 10. The ring with respect to $\tilde{\Gamma}$ is generated by $g(z)$, a_2^2 and $a_2a_5a_{10}$. We can normalize the forms such that

$$a_2^2 + 2a_5^2 + a_{10}^2 = 0, \quad 4g^3 + 27a_5^2 + a_{10}^2 = 0.$$

Furthermore, we get a holomorphic isomorphism $j_m \colon V_W(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ by

$$j_m(z) = \frac{g^3(z)}{a_2^2(z)}.$$

It is clear from our choice of normalization, that

$$(4.1) \qquad \begin{aligned} j_m(z_{(-3)}) &= 0, & j_m(z_{(-8)}) &= \infty, \\ j_m(z_{(-20)}) &= 1/4, & j_m(z_{(-40)}) &= 27/8. \end{aligned}$$

where $z_{(D)}$ denotes a point on $V_W(\mathbb{C})$ such that the corresponding abelian surface has D CM (which is unique for the above values of D).

Theorem 4.1 **([10])** *The following equations give a family of QM-curves with respect to* $\mathcal{O}_{10}$*:* $y^2 = x(P^2x^4 + P^2(1+R)x^3 + PQx^2 + P(1-R)x + 1)$ *with*

$$P = \frac{4(2t+1)(t^2-t-1)}{(t-1)^2}, \quad R = \frac{(t-1)s}{t(t+1)(2t+1)},$$
$$Q = \frac{(t^2+1)(t^4+8t^3-10t^2-8t+1)}{t(t-1)^2(t+1)^2},$$

where $s^2 - t(t-2)(2t+1) = 0$.

The equations for E_{10} in terms of $J_2, \dots, J_{10}$ are rather complicated, so we do not write them here. Let $E_{10,0}$ be the closure of the image of E_{10} in $\mathbb{P}(2,4,6,10)$. It turns out that $E_{10,0}$ has one singular point $[12,6,4,0]$, where it has two cusps meeting with different tangent directions. This point is the only intersection point with $E_{10,0}$ and the curve $J_{10} = 0$. It also turns out that $E_{10,0}$ is not a complete intersection, which makes it more difficult to find nice equations as we had in the discriminant 6 case. We can, however, find a parametrization of E_{10}.

Proposition 4.2 *We have an isomorphism* $j = j_{10} \colon E_{10} \to \mathbb{P}^1 \setminus \{0, \infty\}$ *which is given by*

$$j = \frac{23751C^2 - 501060ABC + 2641541A^2B^2 - 37046420B^3}{2169C^2 - 34404ABC - 16709A^2B^2 + 37046420B^3}$$

where $A = 5J_2$, $B = J_2^2 - 24J_4$, $C = 5(33J_2^3 - 992J_2J_4 + 3600J_6)$*. The inverse map is given by* $[J_2, J_4, J_6, J_{10}] = [J_2(j), J_4(j), J_6(j), J_{10}(j)]$*, where*

$$J_2(j) = 12j^2 - 16j + 12, \quad J_4(j) = 6j^4 - 16j^3 + 6j^2 - 16j + 6,$$
$$J_6(j) = 4j^6 - 16j^5 + 32j^3 - 8j^2 - 16j + 4, \quad J_{10}(j) = j^4.$$

For future reference, we also note the formula

$$j^2 = \frac{(J_2^2 - 24J_4)^5}{20^{10}J_{10}^2}. \tag{4.2}$$

Remark. The map from the family in Theorem 4.1 to the curve E_{10} is of degree 12 and is given by

$$j = \frac{(t^2-1)^3}{4t(t^2-2t-1)^2}.$$

Proposition 4.3 *The following curves* $C_{(D)}$ *have D CM:*

$$C_{(-20)} : y^2 = x^5 - \sqrt{5}x^3 + x,$$
$$C_{(-40)} : y^2 = (2-\sqrt{5})x^6 + (30+51\sqrt{5})x^4 + (30-51\sqrt{5})x^4 + (2+\sqrt{5}),$$
$$C_{(-27)} : y^2 = x^6 - (189+64\sqrt{-3})x^4 - (189-64\sqrt{-3})x^2 + 1,$$
$$C_{(-35)} : y^2 = (2-\sqrt{5})x^6 + (30+19\sqrt{5})x^4 + (30-19\sqrt{5})x^4 + (2+\sqrt{5}).$$

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Proof The proof of the last three cases is analogous to the proof of Proposition 3.7. One gets the first case by observing that the Jacobian is isomorphic to $\mathcal{E} \times \mathcal{E}$, where $\mathcal{E} = \mathbb{C}/\mathbb{Z}[\sqrt{-5}]$. A splitting map $C_{(-20)} \to \{Y^2 = (X+2)(X^2 - 2 - \sqrt{5})\} \cong \mathcal{E}$ is given by

$$X = \frac{x^2+1}{x}, \quad Y = \frac{y(x^3+1)}{x^2(x^2-x+1)}. \qquad \blacksquare$$

It is easy to verify that the curves $C_{(D)}$ from Proposition 4.3 lie on E_{10}, and we have

$$j(C_{(-20)}) = 1/4, \qquad j(C_{(-40)}) = 27/8,$$
$$j(C_{(-27)}) = -24/25, \quad j(C_{(-35)}) = 8/7.$$

Now we consider the matrix $L(j)$ corresponding to (2.1). We get

$$\det(L(j)) = -2^{33}3^{-18}5^{-20}j^{10}(4j-1)^2(25j+24)^2(7j-8)^2(8j-27).$$

In this case, one can find a matrix $N(j)$ with coefficients in $\mathbb{Z}[j]$ such that

$$N(j)^t L(j) N(j) =$$
$$-2^{15}5j^6(7j-8)^2(25j+24)^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10(1-4j) & 0 \\ 0 & 0 & -2(8j-27)(1-4j) \end{pmatrix}.$$

Hence we get the following.

Proposition 4.4 *If C is on E_{10} and has trivial automorphisms, then the Mestre obstruction H_C is the quaternion algebra $(-10(1-4j), 5(8j-27))_{k_{\mathbb{Z}}}$.*

Lemma 4.5 *The -3 and -8 CM surfaces on $\tilde{E}_{10}$ have the product polarizations, hence neither of them is a Jacobian of a genus 2 curve.*

Proof In the -3 CM case, there is only the product polarization (see [11]), so there is nothing to prove.

In the -8 CM case, one can use the formulas for period matrices given in [10, p. 290] and find a period matrix $Z = \Omega(z)$ of the abelian surface. It is then straightforward to find a matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{Z})$ such that

$$Z' = (AZ+B)(CZ+D)^{-1} = \begin{pmatrix} \sqrt{-2} & 0 \\ 0 & \sqrt{-2} \end{pmatrix}.$$

Now, Z' is clearly the period matrix of a split abelian surface with product principal polarization. ■

Proposition 4.6 *We have $j = j_m$ on $\tilde{E}_{10}$.*

Proof It is clear that j and j_m are related by a linear fractional transformation. We know the values of j_m at the 4 points given in (4.1). Now it follows from Lemma 4.5 that the set $\{j(z_{(-3)}), j(z_{(-8)})\}$ equals $\{0, \infty\}$, and we have $j(z_{(-20)}) = 1/4$ and $j(z_{(-40)}) = 27/8$ by Proposition 4.3. The claim follows. ■

Theorem 4.7 *The expressions $\sqrt{1-4j}$ and $\sqrt{8j-27}$ lift to meromorphic functions on V. Furthermore, the holomorphic map $f\colon V \to X = \{x^2+2y^2+z^2=0\}$ given by*

$$f(z) = [a_2(z), a_5(z), a_{10}(z)] = [5, \sqrt{1-4j}, \sqrt{8j-27}]$$

descends to an isomorphism from the canonical $\mathbb{Q}$-model of V to X viewed as a curve over $\mathbb{Q}$.

Proof The first claim follows from the identities $1-4j = (5a_5/a_2)^2$ and $8j-27 = (5a_{10}/a_2)^2$.

Let $K_1 = \mathbb{Q}(\sqrt{-2}, \sqrt{-3})$. The elliptic points z_3, z_3' and the fixed points $z_{(-8)}, z'_{(-8)}$ of w_2 belong to $V(K_1)$, and map to $[5, 1, \pm\sqrt{-3}]$ and $[0, 1, \pm\sqrt{-2}]$ respectively. Hence the map is defined over K_1.

Let $K_2 = H(\mathbb{Q}(\sqrt{-35})) = \mathbb{Q}(\sqrt{-7}, \sqrt{5})$. The 4 points with -35 CM are defined over K_2, and by Proposition 4.3, we have that $j = 8/7$ for these points, so they map to $[\pm\sqrt{-7}, 1, \pm\sqrt{5}]$. Hence the map is also defined over K_2 and we are done. ∎

Theorem 4.8 *The curve C is defined over the field $K = \mathbb{Q}(\sqrt{-10(1-4j)})$. An explicit equation is*

$$\begin{aligned} f(x) = t^3(s^2+2s-10)x^6 - 4t^3(3s+4)x^5 + 15t^2(3s^2+2s+2)x^4 - 40t^2sx^3 \\ + 15t(-3s^2+2s-2)x^2 - 4t(3s-4)x - (s^2-2s-10), \end{aligned}$$

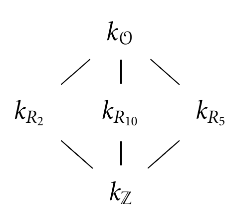
where $s = \sqrt{-10(1-4j)}/5$, $t = (8j-27)/5$.

Corollary 4.9 *For any point $z \in \mathcal{H}$, the curve $C_z : y^2 = g_z(x)$, where*

$$\begin{aligned} g_z(x) = {} & 5a_2^2(z)(x^6 - 3x^4 + 3x^2 - 1) + a_5^2(z)(x^6 + 45x^4 - 45x^2 - 1) \\ & - \sqrt{-2}a_2(z)a_5(z)(x^6 + 15x^4 + 15x^2 + 1) + 8\sqrt{5}a_2(z)a_{10}(z)(x^5 - x) \\ & + 2\sqrt{-10}a_5(z)a_{10}(z)(3x^5 + 10x^3 + 3x) \end{aligned}$$

is a point on E_{10} with $j(C_z) = j(z)$ and has Jacobian isomorphic to (A_z, ρ_z).

Theorem 4.10 *The diagram*



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is given by

$$\begin{array}{ccccc} & & \mathbb{Q}(\sqrt{1-4j}, \sqrt{8j-27}) & & \\ & \diagup & | & \diagdown & \\ \mathbb{Q}(\sqrt{(1-4j)/(8j-27)}) & & \mathbb{Q}(\sqrt{1-4j}) & & \mathbb{Q}(\sqrt{8j-27}) \\ & \diagdown & | & \diagup & \\ & & \mathbb{Q}(j) & & \end{array}$$

Proposition 3.18 holds also in this case, with an analogous proof. We conclude that the genus 2 curve is defined over $k_{\mathcal{O}}[\sqrt{-10}]$, exactly corresponding to the result for the $D = 6$ case.

Proposition 4.11 *The only curves on E_{10} which have non-trivial automorphism groups are the -20 CM curve, which has group D_4, and the curves with -40, -27, and -35 CM, where the group is $C_2 \times C_2$.*

Proposition 4.12 *Let C on E_{10} be such that $k_{\mathbb{Z}}$ is a number field. Then C has potentially good reduction at a prime p not dividing 10 if and only if*

$$v_p(j) = 0. \tag{4.3}$$

In fact, the curve attains good reduction over the field

$$K = k_{\mathbb{Z}}[\sqrt{-10(1-4j)}, \sqrt{5(8j-27)}] \subseteq k_{\mathcal{O}}[\sqrt{-2}, \sqrt{5}].$$

Proof First we prove that (4.3) is necessary. Assume that C has good reduction over a field K at a prime p, so there is an integral model over K whose discriminant $2^{12}J_{10}$ is not divisible by p. We introduce the variables

$$A = 5J_2, \quad B = J_2^2 - 24J_4, \quad C = 5(33J_2^3 - 992J_2J_4 + 3600J_6), \quad D = 20^5 J_{10}.$$

One can verify that A, B, C, and D are integral at any prime not dividing 10, and D is a unit at p. Now we have, by (4.2), that $j^2 = B^5/D^2$, so it is sufficient to show that B also is a unit. Now, there is a relation $31D^2 = B(DC - 49B^4)$, so we are done in case $p \nmid 31$. However, modulo 31 there is a relation

$$D^2 = B(11ABD - 7A^2B^3 + 13CD - 8AB^2C + 11BC^2),$$

so B is must be a unit in this case too.

To prove that (4.3) is sufficient, we use the model in Theorem 4.8 and argue exactly as in the proof of Proposition 3.19. ∎

Proposition 4.13 *The intersection of E_6 and E_{10} in $\mathcal{M}_2$ is the single point corresponding to the curve with -43 CM.*

Proof Plugging the parametrization of E_{10} from Proposition 4.2 into equations (3.1), we get the only solution $j = 216/1225$. Now, in Table 2 we see that this j-value corresponds to the -43 CM curve. ∎

5 Examples and Tables

Elkies [6] found that there are 27 rational CM points on E_6 and 22 on E_{10}. We list them in Tables 1 and 2 respectively, together with the additional information that we now can give about these points. The points are ordered in increasing height of j. The relations between our uniformization j and Elkies' t is $j = 16(t-1)/27$ in the discriminant 6 case, and $j = t/8$ in the discriminant 10 case. Unfortunately, the caveat in [6] that not all of these curves are proved to have the correct CM still applies, for example the -163 curve on E_6. Probably it should be easier to prove these cases now that we can give the explicit sextics, but it seems to the authors that these are still computationally difficult problems.

D	j	k_{R_2}	k_{R_6}	k_{R_3}	disc(H_C)
-3	∞	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3})$	no curve
-4	0	$\mathbb{Q}(\sqrt{-1})$	$\mathbb{Q}$	$\mathbb{Q}$	no curve
-24	$-2^4/3^3$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}$	–
-19	$3^4/2^6$	$\mathbb{Q}(\sqrt{-19})$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-19})$	–
-40	$2^4 3^4/5^3$	$\mathbb{Q}(\sqrt{-10})$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{-2})$	$2\cdot\infty$
-51	$-7^4/(2^6 3^3)$	$\mathbb{Q}(\sqrt{17})$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-51})$	1
-84	$-2^6 7^2/3^6$	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\sqrt{-1})$	$\mathbb{Q}(\sqrt{-3})$	$2\cdot 3$
-52	$-2^6 3^4/5^6$	$\mathbb{Q}(\sqrt{-13})$	$\mathbb{Q}(\sqrt{-1})$	$\mathbb{Q}(\sqrt{13})$	$2\cdot 13$
-120	$2^4 7^4/(3^6 5^3)$	$\mathbb{Q}(\sqrt{-15})$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{-3})$	$3\cdot\infty$
-75	$11^4/(2^6 3^6 5^1)$	$\mathbb{Q}(\sqrt{-15})$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{-3})$	$3\cdot\infty$
-132	$-2^8 11^2/(3^3 5^6)$	$\mathbb{Q}(\sqrt{-1})$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{3})$	1
-43	$3^4 7^4/(2^6 5^6)$	$\mathbb{Q}(\sqrt{-43})$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-43})$	$43\cdot\infty$
-168	$-2^4 7^2 11^4/(3^3 5^6)$	$\mathbb{Q}(\sqrt{6})$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-2})$	$2\cdot 3$
-88	$-2^4 3^4 7^4/(5^6 11^3)$	$\mathbb{Q}(\sqrt{-22})$	$\mathbb{Q}(\sqrt{-11})$	$\mathbb{Q}(\sqrt{2})$	$2\cdot 11$
-100	$2^8 3^4 5^1 7^4/11^6$	$\mathbb{Q}(\sqrt{-1})$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{-5})$	$5\cdot\infty$
-123	$-7^4 19^4/(2^6 3^3 5^6)$	$\mathbb{Q}(\sqrt{41})$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-123})$	1
-228	$2^{10} 7^4 19^2/(3^9 5^6)$	$\mathbb{Q}(\sqrt{-1})$	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\sqrt{-3})$	$2\cdot\infty$
-67	$3^4 7^4 11^4/(2^{12} 5^6)$	$\mathbb{Q}(\sqrt{-67})$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-67})$	$67\cdot\infty$
-147	$-11^4 23^4/(2^6 3^6 5^6 7^1)$	$\mathbb{Q}(\sqrt{21})$	$\mathbb{Q}(\sqrt{-7})$	$\mathbb{Q}(\sqrt{-3})$	$3\cdot 7$
-148	$-2^6 3^4 7^4 11^4/(5^6 17^6)$	$\mathbb{Q}(\sqrt{-37})$	$\mathbb{Q}(\sqrt{-1})$	$\mathbb{Q}(\sqrt{37})$	$2\cdot 37$
-312	$-2^4 7^4 23^4/(3^3 5^6 11^6)$	$\mathbb{Q}(\sqrt{-39})$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{13})$	$3\cdot 13$
-372	$-2^6 7^4 19^4 31^2/(3^6 5^6 11^6)$	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\sqrt{-1})$	$\mathbb{Q}(\sqrt{-3})$	$2\cdot 3$
-408	$-2^4 7^4 11^4 31^4/(3^9 5^6 17^3)$	$\mathbb{Q}(\sqrt{17})$	$\mathbb{Q}(\sqrt{-51})$	$\mathbb{Q}(\sqrt{-3})$	1
-267	$-7^4 31^4 43^4/(2^{12} 3^3 5^6 11^6)$	$\mathbb{Q}(\sqrt{89})$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-267})$	1
-232	$2^4 3^4 7^4 11^4 19^4/(5^6 23^6 29^3)$	$\mathbb{Q}(\sqrt{-58})$	$\mathbb{Q}(\sqrt{29})$	$\mathbb{Q}(\sqrt{-2})$	$2\cdot\infty$
-163	$3^8 7^4 19^4 23^4/(2^6 5^6 11^6 17^6)$	$\mathbb{Q}(\sqrt{-163})$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-163})$	$163\cdot\infty$
-708	$-2^{12} 7^4 11^4 47^4 59^2/(3^3 5^6 17^6 29^6)$	$\mathbb{Q}(\sqrt{-1})$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{3})$	1

Table 1: Rational CM points on E_6

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D	j	k_{R_2}	$k_{R_{10}}$	k_{R_5}	disc(H_C)
-3	0	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3})$	no curve
-8	∞	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}$	$\mathbb{Q}$	no curve
-20	$1/2^2$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-1})$	–
-35	$2^3/7$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{-7})$	$\mathbb{Q}(\sqrt{-35})$	–
-27	$-2^3 3/5^2$	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-3})$	–
-40	$3^3/2^3$	$\mathbb{Q}$	$\mathbb{Q}\sqrt{-2}$	$\mathbb{Q}$	–
-52	$-3^3/(2^2 5^2)$	$\mathbb{Q}(\sqrt{-13})$	$\mathbb{Q}(\sqrt{13})$	$\mathbb{Q}(\sqrt{-1})$	$13 \cdot \infty$
-120	$-3^3/(2^3 7^2)$	$\mathbb{Q}(\sqrt{-15})$	$\mathbb{Q}(\sqrt{10})$	$\mathbb{Q}(\sqrt{-6})$	$3 \cdot \infty$
-72	$5^3/(2^3 3^1 7^2)$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{6})$	$\mathbb{Q}(\sqrt{-3})$	$3 \cdot \infty$
-43	$2^3 3^3/(5^2 7^2)$	$\mathbb{Q}(\sqrt{-43})$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-43})$	$43 \cdot \infty$
-180	$-11^3/(2^2 13^2)$	$\mathbb{Q}(\sqrt{-15})$	$\mathbb{Q}(\sqrt{15})$	$\mathbb{Q}(\sqrt{-1})$	$2 \cdot \infty$
-88	$3^3 5^3/(2^4 7^2)$	$\mathbb{Q}(\sqrt{-22})$	$\mathbb{Q}(\sqrt{-11})$	$\mathbb{Q}(\sqrt{2})$	$2 \cdot 11$
-115	$2^6 3^3/(13^2 23)$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{-23})$	$\mathbb{Q}(\sqrt{-115})$	$5 \cdot 23$
-67	$-2^3 3^3 5^3/(7^2 13^2)$	$\mathbb{Q}(\sqrt{-67})$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-67})$	$67 \cdot \infty$
-280	$3^3 11^3/(2^4 7^1 23^2)$	$\mathbb{Q}(\sqrt{10})$	$\mathbb{Q}(\sqrt{-35})$	$\mathbb{Q}(\sqrt{-14})$	$2 \cdot 7$
-340	$3^3 23^3/(2^2 7^2 29^2)$	$\mathbb{Q}(\sqrt{17})$	$\mathbb{Q}(\sqrt{-17})$	$\mathbb{Q}(\sqrt{-1})$	$2 \cdot 17$
-148	$3^3 11^3/(2^2 5^2 7^2 13^2)$	$\mathbb{Q}(\sqrt{-37})$	$\mathbb{Q}(\sqrt{37})$	$\mathbb{Q}(\sqrt{-1})$	$37 \cdot \infty$
-235	$2^3 3^3 17^3/(7^2 37^2 47)$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{-47})$	$\mathbb{Q}(\sqrt{-235})$	$5 \cdot 47$
-520	$3^3 29^3/(2^6 7^2 13^1 47^2)$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{13})$	$\mathbb{Q}(\sqrt{-26})$	$2 \cdot \infty$
-232	$3^3 11^3 17^3/(2^5 5^2 7^2 23^2)$	$\mathbb{Q}(\sqrt{-58})$	$\mathbb{Q}(\sqrt{-2})$	$\mathbb{Q}(\sqrt{29})$	1
-163	$-2^6 3^3 5^3 11^3/(7^2 13^2 29^2 31^2)$	$\mathbb{Q}(\sqrt{-163})$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{-163})$	$163 \cdot \infty$
-760	$3^3 17^3 47^3/(2^3 7^2 31^2 71^2)$	$\mathbb{Q}(\sqrt{-95})$	$\mathbb{Q}(\sqrt{-38})$	$\mathbb{Q}(\sqrt{10})$	$5 \cdot 19$

Table 2: Rational CM points on E_{10}

Example 1 Consider the curve

$$y^2 = (x^2+5)((-1/6+\sqrt{2})x^4 + 20x^3 - 490/6x^2 + 100x + 25(-1/6-\sqrt{2}))$$

studied in [5]. This has discriminant 6 QM with $j = -9/128$. We get $k_{R_6} = \mathbb{Q}(\sqrt{-2})$, $k_{R_2} = \mathbb{Q}(\sqrt{-10})$, $k_{R_3} = \mathbb{Q}(\sqrt{5})$, and $k_{\mathcal{O}} = \mathbb{Q}(\sqrt{5}, \sqrt{-2})$, which is consistent with and somewhat more precise than the results in [5].

Example 2 The curve

$$y^2 = 1/48x(9075x^4 + 3025(3+2\sqrt{-3})x^3 - 6875x^2 + 220(-3+2\sqrt{-3}x + 48)),$$

which is studied in [4, 24], has discriminant 10 QM. We have $j = -32/147$, and hence we get

$$k_{R_{10}} = \mathbb{Q}(\sqrt{33}), \quad k_{R_2} = \mathbb{Q}(\sqrt{-11}), \quad k_{R_5} = \mathbb{Q}(\sqrt{-3}), \quad k_{\mathcal{O}} = \mathbb{Q}(\sqrt{-3}, \sqrt{-11}).$$

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Example 3 From Table 1, we see, for example, that the curve with -132 CM on E_6 is defined over $\mathbb{Q}$. An explicit equation is given by

$$y^2 = 73x^6 - 750x^5 + 966x^4 + 2000x^3 - 876x^2 - 3000x - 1288.$$

To find this explicit model from the invariants $J_2, \dots, J_{10}$, we used Paul B. van Wamelen's PARI/GP package for computations of genus 2 curves.

Example 4 We consider an example of CM points with non-rational j-value. There are two points with -91 CM points on E_6, which we denote $z_{(-91)}$ and $z'_{(-91)}$. The j-values of these points belong to $H(\mathbb{Q}(\sqrt{-91})) = \mathbb{Q}(\sqrt{-7}, \sqrt{13})$. Numerically (up to several hundred decimals) we get

$$j(z_{(-91)}) = \overline{j(z'_{(-91)})} = \frac{3^4 p_7^4 p_{11}^4}{p_2^6 \overline{p}_2^{12} \overline{p}_{11}^6},$$

where $p_2 = (1 + \sqrt{-7})/2$, $p_7 = \sqrt{-7}$ and $p_{11} = 2 + \sqrt{-7}$.

A Restriction of Hilbert Modular Forms

In this appendix, we describe how to construct modular forms with respect to a co-compact quaternionic group over $\mathbb{Q}$ in a way that is suitable for numerical computations. We start by embedding the curve into a suitable Hilbert modular surface. Note that our description of this embedding is just a reformulation of the classical construction of Hirzebruch–Zagier cycles [12]. The key to constructing modular forms with respect to the quaternionic group is to introduce a certain factor such that restriction of modular forms on $\mathcal{H} \times \mathcal{H}$ times this factor gives modular forms with respect to the quaternionic group. A special case of this occurs in [9].

Let $k = \mathbb{Q}(\sqrt{d})$, where $d > 1$ is a square free integer. Let D be the discriminant of the field k, $\mathfrak{O}$ the ring of integers in k and the nontrivial automorphism of k is denoted by $x \mapsto \overline{x}$. Consider the algebra $A = M_2(k)$. The canonical involution on A is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Let $\Lambda = M_2(\mathfrak{O})$. The Hilbert modular group $\Gamma = \mathrm{SL}_2(\mathfrak{O})$ acts on $\mathcal{H} \times \mathcal{H}$ by $\gamma(z_1, z_2) = (\gamma z_1, \overline{\gamma} z_2)$. Let $\beta \in \Lambda$ with $\overline{\beta}^* = \beta$, $\beta \in \mathbb{Z} + \sqrt{D}\Lambda$ and $\det(\beta) > 0$. Furthermore, we assume that β is primitive, *i.e.,* if $\beta = n\beta_0$ where β_0 has the same properties and $n \in \mathbb{Z}$, then $n = \pm 1$. Consider

$$\Lambda_\beta = \{\lambda \in \Lambda \mid \beta\lambda = \overline{\lambda}\beta\},$$

which is an order in an indefinite quaternion algebra over $\mathbb{Q}$. The discriminant of the order Λ_β is $\det(\beta)$ (*cf.* [8]). Let $C_\beta = \{(z, \beta z) \mid z \in \mathcal{H}\} \subset \mathcal{H} \times \mathcal{H}$. The group $\Gamma_\beta = \{\gamma \in \Lambda^1 \mid \beta\gamma = \pm\overline{\gamma}\beta\} \supseteq \Lambda^1_\beta$, which is an extension of Λ^1_β of degree at most 2, acts on $C_\beta \cong \mathcal{H}$.

Our aim is to construct modular forms with respect to the group Γ_β. Let F be a Hilbert modular form with respect to Γ of weight (k_1, k_2), *i.e.,* $F\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is holomorphic and satisfies

$$F(\gamma z_1, \overline{\gamma} z_2) = j(\gamma, z_1)^{k_1} j(\overline{\gamma}, z_2)^{k_2} F(z_1, z_2),$$

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for all $\gamma \in \Gamma$. Here $j(\alpha, z) = cz + d$ for any real matrix $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We define a function $f(z) = j(\beta, z)^{-k_2} F(z, \beta z)$, $z \in \mathcal{H}$. Now, for $\gamma \in \Gamma_\beta$, we get $j(\beta, \gamma z) j(\gamma, z) = j(\beta\gamma, z) = \pm j(\overline{\gamma}\beta, z) = \pm j(\overline{\gamma}, \beta z) j(\beta, z)$, so

$$\frac{j(\overline{\gamma}, \beta z)}{j(\beta, \gamma z)} = \pm \frac{j(\gamma, z)}{j(\beta, z)}, \tag{A.1}$$

where the sign is the same as the sign in the equation $\beta\gamma = \pm\overline{\gamma}\beta$.

Proposition A.1 *The function f is a modular form with respect to Λ^1_β of weight $k_1 + k_2$. If k_2 is even, then it is also a modular form with respect to Γ_β.*

Proof The function f is obviously holomorphic. By (A.1), we get

$$\begin{aligned} f(\gamma z) &= j(\beta, \gamma z)^{-k_2} F(\gamma z, \beta\gamma z) = j(\beta, \gamma z)^{-k_2} F(\gamma z, \overline{\gamma}\beta z) \\ &= j(\beta, \gamma z)^{-k_2} j(\gamma, z)^{k_1} j(\overline{\gamma}, \beta z)^{k_2} F(z, \beta z) \\ &= j(\beta, z)^{-k_2} j(\gamma, z)^{k_1} j(\gamma, z)^{k_2} F(z, \beta z) \\ &= j(\gamma, z)^{k_1+k_2} f(z), \end{aligned}$$

for any $z \in \mathcal{H}$ and $\gamma \in \Lambda^1_\beta$. The last statement is now clear. ∎

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Department of Mathematics and Statistics, Concordia University, Montréal, QC, H3G 1M8
e-mail: sbaba@mathstat.concordia.ca

Department of Mathematics, Karlstad University, 65188 Karlstad, Sweden
e-mail: hakan.granath@kau.se

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