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MAXIMAL AREAS OF REULEAUX POLYGONS(1)

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1. In this paper we provide new proofs of some interesting results of Firey [2] on isoperimetric ratios of Reuleaux polygons. Recall that a Reuleaux polygon is a plane convex set of constant width whose boundary consists of a finite (odd) number of circular arcs. Equivalently, it is the intersection of a finite number of suitably chosen congruent discs. For more details, see [1, p. 128].

If a Reuleaux polygon has n sides (arcs) of positive length (where n is odd and ≥ 3), we will refer to it as a Reuleaux n-gon, or sometimes just as an n-gon. If all of the sides are equal, it is termed a *regular* n-gon.

Firey [2] proved the following theorems.

THEOREM 1. *The isoperimetric ratio (ratio of area to squared perimeter) of regular Reuleaux polygons strictly increases with the number of sides.*

THEOREM 2. *Among all Reuleaux polygons having the same number of sides, the regular Reuleaux polygons (and only these) attain the greatest isoperimetric ratio.*

THEOREM 3. *For any odd integer $n>3$ and any $\epsilon>0$, there is an n-sided Reuleaux polygon whose isoperimetric ratio exceeds that of the Reuleaux triangle by an amount less than ϵ.*

In the next section of the paper we will describe a construction for modifying Reuleaux polygons and prove a key lemma. In the concluding sections we will apply the construction to prove the theorems.

2. Since, by Barbier's Theorem [4], all sets of the same constant width in the plane have the same perimeter, it suffices to compare areas of the figures under consideration. Moreover, without loss of generality, we may assume that all figures have width l.

Let P be a Reuleaux polygon, let q be a point near P but exterior to it, and let r, s be the points on the boundary of P which are at a distance l from q. Denoting the vertices of P which are no more than distance l from q by $v_1, \ldots, v_t$, let Q be the intersection of the discs of radius l centered at $q, r, s, v_1, \ldots, v_t$. It is easily verified that if the arc between r and s furthest from q contains no opposite points (that is, points which are distance l apart), then Q is a Reuleaux polygon. We will call Q the (r, s)-*variant* of P. (See also [3].)

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LEMMA 1. *Let P be a Reuleaux n-gon which is not regular. Then there exists a variant of P of greater area which is also a Reuleaux n-gon.*

Proof. Let the vertices of P in order be $v_0, \ldots, v_{2n}$ and assume that $\alpha = \angle v_{n+1}, v_0, v_n < \beta = \angle v_{2n}, v_n, v_0$. Since P is not regular such unequal angles exist. Let Q be the (w_n, v_{n+1})-variant of P, where w_n is a point on the boundary arc of P between v_{n-1} and v_n. Note that if w_n is sufficiently near v_n, then Q is not only a $(2n+1)$-gon, but has exactly the same vertices as P except for w_n and w_0, a vertex near v_0 which lies on an extension of the arc (v_0, v_1).

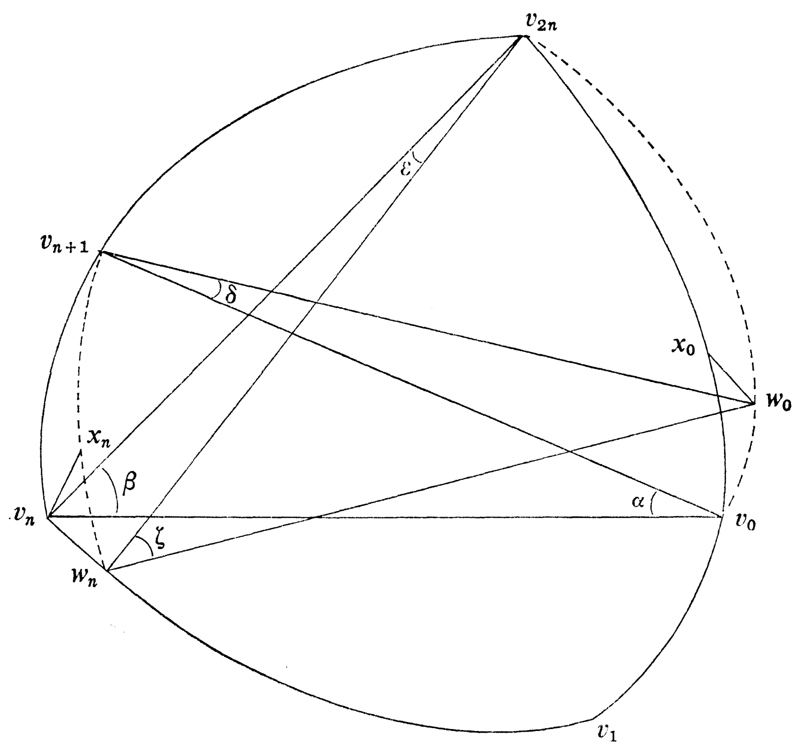

Figure 1.

Denote the area of S by $A(S)$. We wish to show that the area $A(X)$ of the (curvilinear) triangle X, with vertices (v_0, w_0, v_{2n}), is greater than $A(Y)$, the area of the triangle Y, with vertices (v_n, w_n, v_{n+1}), for w_n sufficiently near v_n. This, of course, will prove the lemma. See Figure 1 for an illustration in the case of the Reuleaux pentagon.

We break up X into two pieces by a line through v_n parallel to $[w_0, v_0]$. Call x_n the intersection of this line with (w_n, v_{n+1}), let X_1 be the part of X containing v_{n+1} and let X_2 be the other part. Note that $A(X_1) = O(\epsilon)$ while $A(X_2) = O(\epsilon^2)$ (where $\epsilon = \angle v_n, v_{2n}, w_n$) and so X_2 will be the dominant section of X as $\epsilon \to 0$. By similar reasoning, we break up Y into subsets Y_1 and Y_2 by a line segment $[x_0, w_0]$ parallel to $[v_n, w_n]$. Y_1 denotes the section containing v_{2n}. Observe that $A(Y_1)=O(\delta)$ while $A(Y_2)=O(\delta^2)$, where $\delta=\angle w_0, v_{n+1}, v_0$. Thus it suffices to compute $\lim_{\varepsilon\to 0} A(X_1)/A(Y_1)$, since $\delta\to 0$ as $\epsilon\to 0$.

Except near v_{n+1}, X_1 intersects every line parallel to $[w_0, v_0]$ in a segment of length $|w_0-v_0|$. Thus

(1)
$$A(X_1) = |w_0-v_0|\cdot|\text{proj }[\mathbf{u}, (v_n, v_{n+1})]|+O(\delta^3)$$

where $\mathbf{u}=[\mathrm{v}_0, \mathrm{w}_0]$ and $|\text{proj }[\mathbf{u}, S]|$ denotes the length of the projection of S in direction $\mathbf{u}$. Replacing $\mathbf{u}$ by $\mathbf{u}_0$, the vector perpendicular to $[v_0, v_{n+1}]$, changes only a tiny triangle with $[x_n, v_n]$ as one edge, so we may also write

(2)
$$A(X_1) = |w_0-v_0|\cdot|\text{proj }[\mathbf{u}_0, (v_n, v_{n+1})]|+O(\delta^2).$$

Clearly

(3)
$$|\text{proj }[\mathbf{u}_0, (v_n, v_{n+1})]| = 1-\cos\alpha.$$

It remains to evaluate $|w_0-v_0|$.

To do this, we introduce cartesian coordinates so that $v_n=(0, 0)$, $v_0=(1, 0)$ and w_0 lies in the upper half-plane. It is easily computed that

(4)
$$\begin{aligned} v_{2n} &= (\cos\beta, \sin\beta)\\ v_{n+1} &= (1-\cos\alpha, \sin\alpha)\\ w_n &= (\cos\beta-\cos(\beta+\epsilon), \sin\beta-\sin(\beta+\epsilon))\\ w_0 &= (1-\cos\alpha+\cos(\alpha-\delta), \sin\alpha-\sin(\alpha-\delta)).\end{aligned}$$

When ϵ and δ are small, we may simplify the expressions for w_n and w_0 by using first-order approximations.

(4′)
$$\begin{aligned} w_n &= (\epsilon\sin\beta+O(\epsilon^2), -\epsilon\cos\beta+O(\epsilon^2))\\ w_0 &= (1+\delta\sin\alpha+O(\delta^2), \delta\cos\alpha+O(\delta^2)).\end{aligned}$$

We now apply (4′) to compute

$$|w_0-v_0|^2 = \delta^2+O(\delta^3).$$

So

$$|w_0-v_0| = \delta+O(\delta^2).$$

Thus

(5)
$$A(X_1) = \delta(1-\cos\alpha)+O(\delta^2).$$

Reasoning in precisely the same manner, we find

(6)
$$A(Y_1) = \epsilon(1-\cos\beta)+O(\epsilon^2).$$

Thus, to compare $A(X_1)$ and $A(Y_1)$, it is only necessary to determine a relation

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between δ and ϵ. We may do this by using w_n to compute another set of coordinates for w_0, namely:

(7) $w_0 = (\cos\beta - \cos(\beta+\epsilon) + \cos(\beta+\epsilon-\zeta),\ \sin\beta - \sin(\beta+\epsilon) + \sin(\beta+\epsilon-\zeta))$

If we set $\omega = \beta-\zeta$ and observe that $\omega \to 0$ as $\epsilon \to 0$, we may approximate the x-coordinate of w_0 as given in (7) by

$$1+\epsilon\sin\beta + O(\epsilon^2) + O(\omega^2). \tag{8}$$

Setting this equal to the x-coordinate of w_0 from (4′), we may infer that δ, ϵ, and ω are of the same order and that

$$\lim_{\varepsilon\to 0} \delta/\epsilon = \sin\beta/\sin\alpha. \tag{9}$$

From (5), (6), and (9), we then find

$$\begin{aligned}\lim_{\varepsilon\to 0} A(X_1)/A(Y_1) &= \lim_{\varepsilon\to 0} [\delta(1-\cos\alpha)/\epsilon(1-\cos\beta)] \\ &= \tan(\alpha/2)/\tan(\beta/2).\end{aligned} \tag{10}$$

This limit is positive and less than 1 because $\alpha < \beta < \pi$, and $\tan(x/2)$ is positive and increasing in x over $[0, \pi]$.

Thus

$$A(Q) > A(P) \text{ for } w_n \text{ sufficiently near } v_n.$$

3. **Proofs of Theorems 1 and 2.** By the Blaschke Selection Theorem a set S of constant width l and maximal area exists which can be approximated arbitrarily closely by Reuleaux r-gons. Since each of the approximating figures has r vertices, S has at most r vertices [1, p. 128]. Suppose then that S is an m-gon. From Lemma 1 we infer that S must be a regular m-gon. It only remains to show that $m=r$.

Let R_p denote the regular Reuleaux p-gon. Note that we may consider R_p as a degenerate $(p+2)$-gon where two (opposite) pairs of vertices coincide. Thus, we may construct a $(p+2)$-gon of greater area than R_p by letting Q be the (w_{n+1}, w_n)-variant of R_p where w_{n+1} lies on the arc (v_n, v_{n+1}) and w_n on (v_{n-1}, v_n). The same estimates as before are valid as we only used the fact that v_{n+1} was a vertex in order to preserve the number of sides.

Thus, the set $S \neq R_m$ for $m<r$. Since Lemma 1 implies S is regular, $S=R_r$. Moreover, the arguments above show that $A(R_p) < A(R_{p+2})$. This concludes the proofs of Theorems 1 and 2.

4. **Proof of Theorem 3.** We will prove a stronger statement which will yield Theorem 3 as an immediate corollary. Recall that the symmetric difference of X and Y, $X \triangle Y = (X \sim Y) \cup (Y \sim X)$.

LEMMA 2. *Let T be any Reuleaux n-gon and let $\epsilon>0$ be given. Then for any (odd) $p>n$, there exists a Reuleaux p-gon, U, of the same width as T such that $A(T \triangle U)<\epsilon$.*

Proof. First, suppose $p=n+2$. Number the vertices of T in order, $v_1, \ldots, v_n$. Let w_i denote a point on the arc (v_{i-1}, v_i). Then if w_j and w_{j+1} are sufficiently near v_j and v_{j+1}, respectively, and U denotes the (w_j, w_{j+1})-variant of T, we have $A(T \triangle U) < \epsilon$.

Now if $p=n+2k$, construct $U_1, \ldots, U_k$ inductively so that $|A(U_i \triangle U_{i-1})| < \epsilon/2^i$ for $i=1, \ldots, k$, where $U_0=T$ and $U_k=U$. By the triangle inequality $|A(T \triangle U)| < \epsilon$. This proves the lemma.

The proof of Theorem 3 is immediate.

5. Remarks. The notion of a set of constant width can be extended to any Minkowski space $\mathscr{M}$ with unit ball B. In such a space K is of relative constant width ω if and only if $K-K=2\omega B$. A relative Reuleaux polygon in $\mathscr{M}$ is a set of relative constant width which is the intersection of a finite number of translates of B.

It is possible to duplicate the analysis given above to give necessary conditions on the angles of a relative Reuleaux n-gon to guarantee that it has maximum area among all such n-gons. The relations are messier, of course.

It might even be possible to use this technique to find the three-dimensional set of constant width having minimal volume. The estimates here, however, look very difficult.

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