ON SOME AXIOMS IN FOUNDATIONS OF CARTESIAN SPACES

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The present paper seeks to put into a clearer focus the roles of certain axioms in axiomatic studies of finite dimensional Cartesian spaces over various classes of ordered fields. Following Tarski [3], we define a Cartesian space as follows. Let $\mathfrak{F} = (F, +, \cdot, \leq)$ be an ordered field. By an n-dimensional Cartesian space over $\mathfrak{F}$, we understand a relational structure $\mathcal{C}_n(\mathfrak{F}) = (A_{\mathfrak{F}}, B_{\mathfrak{F}}, D_{\mathfrak{F}})$, where $A_{\mathfrak{F}}$ is the set of all n-tuples $\overline{x} = (x_1, x_2, \ldots, x_n)$ of elements of F, and $B_{\mathfrak{F}}$, $D_{\mathfrak{F}}$ are respectively the 3-place and 4-place relations over A defined by the following stipulations: for $\overline{x}, \overline{y}, \overline{z}, \overline{u} \in A$,

$B_{\mathfrak{F}}(\overline{x}\ \overline{y}\ \overline{z})$ if and only if there exists an element $k \in F$, $0 \leq k \leq 1$, such that

$$y_i = (1-k)\, x_i + kz_i \quad (1 \leq i \leq n);$$

$$D_{\mathfrak{F}}(\overline{x}\ \overline{y}\ \overline{z}\ \overline{u}) \text{ if and only if } \sum_{i=1}^{n} (x_i - y_i)^2 = \sum_{i=1}^{n} (z_i - u_i)^2 .$$

(The symbols 1, $x - y$, x^2, $\sum_{i=1}^{n}$ are understood as defined in the usual way.)

Given an ordered field $\mathfrak{F}$, the language appropriate for the study of $\mathcal{C}_n(\mathfrak{F})$ will contain two non-logical constants, viz., a 3-place predicate symbol β and a 4-place predicate symbol δ to stand respectively for the relations $B_{\mathfrak{F}}$ and $D_{\mathfrak{F}}$. We shall refer to this language by $L_{\beta\delta}$. The set of all first-order sentences in $L_{\beta\delta}$, which are true in $\mathcal{C}_n(\mathfrak{F})$ will be called the elementary theory of $\mathcal{C}_n(\{\mathfrak{F}\})$ and be denoted by $\mathcal{E}_n(\{\mathfrak{F}\})$. In the same vein, given a class K of ordered fields we can define the elementary theory of n-dimensional Cartesian spaces over

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fields in K to be the set of first-order sentences in $L_{\beta\delta}$ holding in all $\mathcal{C}_n(\mathfrak{F})$, $\mathfrak{F} \in K$. This theory will be denoted by $\mathcal{E}_n(K)$. Going one step further, the common part of theories $\mathcal{E}_n(K)$ for all dimension values n, $n \geq 2$, will be denoted by $\mathcal{E}(K)$. Given a class **K** of ordered fields, an $n \geq 2$, and a sentence σ_n in $L_{\beta\delta}$, we say that σ_n is a <u>dimension axiom of index</u> n for $\mathcal{E}_n(K)$ if the following conditions hold: (i) σ_n is in $\mathcal{E}_n(K)$ and (ii) $\sim \sigma_n$ (the denial of σ_n) is in $\mathcal{E}_m(K)$ for all $m \neq n$. The importance of $\mathcal{E}(K)$ and dimension axioms is brought out by the following theorem.

THEOREM. <u>For any class</u> K <u>of ordered fields and an integer</u> n, $n \geq 2$, <u>if</u> σ_n <u>is a dimension axiom of index</u> n <u>for</u> $\mathcal{E}(K)$, <u>then</u> $\mathcal{E}_n(K)$ <u>coincides with the set of sentences which are logical consequences of the set</u> $\mathcal{E}(K) \cup \{\sigma_n\}$.

In view of the above theorem, it appears that for an axiomatic foundation of $\mathcal{E}_n(K)$ for a given class K of ordered fields, it would be ideal first to have ready a set of axioms for the dimension-free part $\mathcal{E}(K)$ and then a dimension axiom of appropriate index. It will be seen from our definition of dimension axioms that if a sentence σ is a dimension axiom of index n for $\mathcal{E}(K)$ then σ is also a dimension axiom of index n for $\mathcal{E}(L)$, where L is a subclass of K. Thus a dimension axiom of index n for the largest class OF of <u>all</u> ordered fields will serve simultaneously for all classes of ordered fields. Such dimension axioms are easy to find. For example, the sentence σ_n which guarantees existence of n mutually orthogonal lines at every point and excludes existence of $n+1$ mutually orthogonal lines at all points is such an axiom. (The notion of orthogonality though not belonging in $L_{\beta\delta}$ can be easily defined in terms of β and δ.) Although a dimension axiom is readily available, it may by no means be easy to obtain an axiom system for $\mathcal{E}(K)$. To illustrate this we mention the class OF of all ordered fields. It is not known whether $\mathcal{E}(OF)$ is axiomatizable, i.e. whether there is a recursive set of sentences the set of all logical consequences of which coincides with $\mathcal{E}(OF)$. Nevertheless, it is now known that for each n, $n \geq 2$, the theory $\mathcal{E}_n(OF)$ is axiomatizable, in fact, finitely axiomatizable. In [1], the author gave a system of axioms for $\mathcal{E}_n(OF)$, for each n. For this axiom system the following representation theorem is established: <u>A relational structure</u> $\mathfrak{A} = (A, B, D)$ <u>is a model of the system if and only if</u> $\mathfrak{A}$ <u>is isomorphic to a Cartesian space</u> $\mathcal{C}_n(\mathfrak{F})$ <u>for some ordered field</u> $\mathfrak{F}$. The system consists of a set $\overline{\mathcal{E}}$ of axioms and a dimension axiom $\overline{\sigma}_n$ which differs from σ_n referred to earlier in that $\overline{\sigma}_n$ guarantees existence of n mutually orthogonal <u>and</u> segment-comparable lines, and excludes as before existence of $n+1$ mutually

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orthogonal lines. $\overline{\sigma}_n$ is thus stronger than σ_n. Its strength is brought home by the fact that while $\overline{\mathcal{E}}$ together with $\overline{\sigma}_n$ provides an axiom system for $\mathcal{E}_n(OF)$, $\overline{\mathcal{E}}$ together with σ_n fails. This last fact is shown by constructing a model for $\overline{\mathcal{E}} \cup \{\sigma_n\}$ in which $\overline{\sigma}_n$ fails. This leads us to conclude that $\overline{\mathcal{E}} \cup \{\sigma_n\}$ does not constitute an axiom system for $\mathcal{E}_n(OF)$, and, therefore, the theory of $\overline{\mathcal{E}}$ is actually weaker than $\mathcal{E}(OF)$. Just to what extent should one have to strengthen $\overline{\mathcal{E}}$ in order to obtain an axiom system for $\mathcal{E}(OF)$? The question is open. As remarked already, we do not even know whether $\mathcal{E}(OF)$ admits of an axiomatization at all. The author is inclined to make the negative conjecture. One can, however, show that the theory $\mathcal{E}(OF)$, as well as the theories $\mathcal{E}_n(OF)$ for all $n \geq 2$ is undecidable.

The situation improves when one considers the narrower class PF of all Pythagorean ordered fields. (An ordered field is called Pythagorean if for any two elements $a, b \in F$ there exists an element $c \in F$ such that $c^2 = a^2 + b^2$.) It is known that in $\mathcal{E}_n(PF)$ the axiom of segment-construction plays an important role. For ready reference we take the following formulation of Tarski [3, A 10]

$$(SC) \quad : \bigwedge x\,y\,u\,v \bigvee z \; (\beta(x\,y\,z) \wedge \delta(y\,z\,u\,v)).$$

The importance of this axiom becomes clearer by the fact that the addition of this axiom to an axiom system for $\mathcal{E}_n(OF)$ leads at once to an axiom system for $\mathcal{E}_n(PF)$. But is $\mathcal{E}(PF)$ axiomatizable? The answer is affirmative. Utilizing ideas contained in Scott [2] the author established that the axioms A 1 - A 11 in Tarski [3] constitute an axiomatization of $\mathcal{E}(PF)$. Now the axioms in $\overline{\mathcal{E}}$ are obtained from those in [3] by omitting (SC) and adding some of its special cases. Thus $\overline{\mathcal{E}}$ together with (SC) is logically equivalent to the system A 1 - A 11 of Tarski. Thus although $\overline{\mathcal{E}}$ is inadequate for $\mathcal{E}(OF)$, it serves admirably as a base for constructing axiom systems for $\mathcal{E}_n(OF)$, $\mathcal{E}(PF)$, and $\mathcal{E}_n(PF)$. We have thus a choice, namely, either to add (SC) to $\mathcal{E}_n(OF)$ or to add a dimension axiom to $\mathcal{E}(PF)$, both $\mathcal{E}_n(OF)$ and $\mathcal{E}(PF)$ being axiomatizable. Speaking of an axiom of segment-construction, we might replace (SC) above containing five variables by the following axiom containing four variables:

$$(SC_1) \quad : \bigwedge x\,y\,u \bigvee z \; (\beta(x\,y\,z) \wedge \delta(y\,z\,y\,u)).$$

In $\overline{\mathcal{E}}$ the two axioms (SC) and (SC_1) are equivalent. Moreover, this equivalence is established without the use of the axiom of parallels. We shall have occasion to speak of yet another equivalent axiom later. In view of the foregoing observations, the author would plead for a due recognition of the axiom of segment construction in axiomatic studies by reserving its use only for later stages when its use becomes really necessary.

Another class of ordered fields to be discussed in this paper is that of the so-called Euclidean ordered fields, EF. (An ordered field is said to be Euclidean if for every element a in F, $a \geq 0$, there exists an element c such that $c^2 = a$.) One sees that the class EF is a subclass of PF. It is known that in $\mathcal{E}_n(EF)$ the circle axiom plays an important role. Again for ready reference, we take it in Tarski's formulation, viz. [3; A 13']. We refer to it as (CA). In this formulation, as in other known ones, the notions of points lying inside and lying outside a circle are used. In the absence of segment-comparability of all lines these notions are inapplicable.

$\mathcal{E}_n(EF)$ is axiomatizable by adding the axioms (SC) and (CA) both to an axiom system for $\mathcal{E}_n(OF)$. Similarly, by adding these two axioms to $\overline{\mathcal{E}}$ one obtains an axiom system for $\mathcal{E}(EF)$. This approach, though sound, is defective on aesthetic grounds. We should have a single intuitively simple axiom which when added to an axiom system for $\mathcal{E}_n(OF)$ will give an axiom system for $\mathcal{E}_n(EF)$. It should have two other desirable features, namely (i) by adding this axiom to $\overline{\mathcal{E}}$ one should arrive at an axiom system for $\mathcal{E}(EF)$ and (ii) the axiom of segment construction should be an easy consequence of this axiom in $\overline{\mathcal{E}}$. (Just as every Euclidean field is necessarily Pythagorean.) In other words, this single axiom should by itself be equivalent in $\overline{\mathcal{E}}$ to the conjunction of (SC) and (CA). The following is such an axiom:

$$(E) : \bigwedge xyzu \ \{\beta(xyz) \rightarrow \bigvee v \bigvee w \ [\beta(uyv) \wedge \beta(zvw) \wedge \delta(zvvw) \wedge \delta(xzxw)]\}.$$

(Intuitively, this axiom expresses the fact that if y is between x and z and u is an arbitrary point, then the segment uy can be extended to a point v such that the points xvz form a right angle at v.) One can show that this axiom holds in all $\mathcal{C}_n(\mathfrak{F})$, where $\mathfrak{F}$ is Euclidean. Conversely, if this axiom holds in a $\mathcal{C}_n(\mathfrak{F})$, $\mathfrak{F}$ any ordered field, then $\mathfrak{F}$ is necessarily Euclidean. It will therefore follow that this axiom when added to an axiom system for $\mathcal{E}_n(OF)$ leads to an axiomatization of $\mathcal{E}_n(EF)$. To see that (E) implies (SC) in $\overline{\mathcal{E}}$ we observe first that (E) logically implies the following sentence:

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(SC_2) : $\bigwedge xyzu \{\beta(xyz) \wedge \delta(xyyz) \rightarrow \bigvee v \bigvee w[\beta(uyv) \wedge \beta(zvw)$

$$\wedge\ \delta(zvvw) \wedge \delta(xzxw)]\} .$$

(Intuitively, (SC_2) states that if y is the midpoint of the segment xz, and u is an arbitrary point then the segment uy can be extended to a point v such that the points xvz form a right angle.) At a first glance, (SC_2) does not look like an axiom of segment-construction. But we can show that (SC_1) is equivalent to (SC_2) in $\overline{\mathcal{E}}$, and hence that (S C), (SC_1) and (SC_2) are all equivalent in $\overline{\mathcal{E}}$. However, for showing the equivalence of (SC_1) and (SC_2) in $\overline{\mathcal{E}}$ we seem to need the parallel axiom. Next, one shows that (CA) is also a consequence of (E). The demonstration of this last fact is slightly involved. The proof that (CA) and (S C) together imply (E) in $\overline{\mathcal{E}}$ is straightforward. Another interesting candidate in connection with the class EF is the following:

$$(B) \quad \beta(xyz) \rightarrow \bigvee v(\tau(vyx) \wedge \tau(xvz)),$$

where $\tau(xyz)$ is intended to express the fact that x, y, z form a right angle at y. It is essentially known that by adding (B) to an axiom system for $\mathcal{E}_n(PF)$ one obtains an axiom system for $\mathcal{E}_n(EF)$. It follows that in $\mathcal{E}_n(PF)$ the three axioms (CA), (E) and (B) are all equivalent. They are also equivalent in $\mathcal{E}(PF)$. Axiom (B) is preferable to (CA) in the axiomatic studies of $\mathcal{E}(EF)$ in that it is meaningful even in the absence of an axiom of segment construction. The implication (E) $\rightarrow$ (B) holds in $\overline{\mathcal{E}}$. The author has not succeeded in showing whether the opposite inclusion also holds in $\mathcal{E}_n(OF)$, in general, far less in $\overline{\mathcal{E}}$. (This implication holds in $\mathcal{E}_2(OF)$. Analytically expressed, this leads to the following problem:

<u>Problem.</u> Let $\mathfrak{F}$ be any ordered field such that for some fixed positive integer $n \geq 2$, and all $a_1, \ldots a_n \in F$, $1 \geq 0$ the system of equations

$$a_1 x_1 + \ldots + a_n x_n = 0$$

$$x_1^2 + \ldots + x_n^2 = 1 \quad (a_1^2 + \ldots + a_n^2)$$

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has always a solution for $x_1 \ldots x_n$ in F. Is the field necessarily Euclidean? (In the case when $n = 2$, and, in the case when $\mathfrak{F}$ is Pythagorean, the answer is affirmative.)

We have left the class RCF of all real closed fields out of the present discussion. The author intends to discuss problems related to $\mathcal{E}_n(RCF)$ and $\mathcal{E}(RCF)$ in a future paper.

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