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ON THE CHERN CLASSES OF THE REGULAR REPRESENTATIONS OF SOME FINITE GROUPS

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In studying the cohomology of the symmetric groups and its applications in topology one is led to certain questions concerning the representation rings of special subgroups of $\mathscr{S}_n$. In this note we calculate the Chern classes of the regular representation of $(Z/p)^n$ where p is a fixed odd prime in terms of certain modular invariants first described by L. E. Dickson in 1911. In a later paper [**9**] we apply these results to study the odd primary torsion in the PL cobordism ring. Some indications of this application are given in Sections 10–12 where we apply the result above to obtain information about the cohomology of $\mathscr{S}_{p^n}$. After circulation of this note in preprint form we learned that H. Mui [**10**], has also proved Theorem 6.2.

1.

Recall that the regular representation $R(G)$ of a finite group G is obtained via its natural action as a group of automorphisms of the group ring $C(G)(g\sum z_i g_i = \sum z_i g g_i)$. In particular, from the theory of representations of finite groups [3] we have the decomposition

$$R(G) = \sum_{S(G)} (\dim S(G))S(G) \qquad 1.1$$

where $S(G)$ runs over all irreducible complex representations of G.

In the case $G = (Z/p)^n$ all the irreducible representations are one dimensional and are obtained explicitly as follows. Let

$$\lambda: Z/p \to U(1) \qquad (1.2)$$

be the homomorphism defined by

$$\lambda(T) = e^{(2\pi i/p)}$$

where T is a specified generator of Z/p. Then to each n-tuple $I = (i_1 \dots i_n) \in (Z/p)^n$ define

$$V_I: (Z/p)^n \to Z/p$$

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as the homomorphism given on the jth generator T_j by

$$V_I(T_j) = T^{(i_j)}.$$

The composites

$$\lambda \circ V_I = S_I : (Z/p)^n \to U(1)$$

are irreducible, and indeed give all the irreducible representations of $(Z/p)^n$.

2.

The cohomology of $(Z/p)^n$ is given by

$$H^*((Z/p)^n, Z/p) = E(e_1 \dots e_n) \otimes P(b_1, \dots, b_n)$$

where $\dim(e_i) = 1$, $\dim(b_i) = 2$. Let I_i denote $(0 \dots 010 \dots 0)$ then

$$e_i = V^*_{I_i} e$$

and

$$b_i = V^*_{I_i} b \tag{2.1}$$

with e, b, the canonical generators of $H^*(Z/p, Z/p) = E(e) \otimes P(b)$.

It is well known that the homomorphism $\lambda_i : Z/p \to Z/p$ defined by $\lambda_i(T) = T^i$ induces the cohomology map $(\lambda_i)^*(e) = ie$, $(\lambda_i)^*(b) = ib$.

From this and the Künneth theorem it is easy to see that

$$\begin{aligned} V^*_{(1,1,\dots,1)}(e) &= \Sigma(e_i) \\ V^*_{(1,1,\dots,1)}(b) &= \Sigma(b_i) \end{aligned} \tag{2.2}$$

(Indeed b is the Bochstein of e so it suffices to check on e, and here use the maps

$$\varphi_{1,j} : (Z/p) \to (Z/p)^n \quad \text{defined by}$$

$$\varphi_{1j}(T) = T_1 \circ T_j^{-1}$$

together with the observation $V_{(1,1,\dots,1)} \varphi_{1j}(T) = 1$.)

Lemma 2.3. *Let $I = (i_1, \dots, i_n)$ as in Section 1 then*

$$V^*_I(e) = \Sigma\, i_j e_j, \quad V^*_I(b) = \Sigma\, i_j b_j.$$

Proof. Let $\varphi_{(i_1\dots i_n)}:(Z/p)^n \to (Z/p)^n$ be given by $\varphi_{(i_1\dots i_n)}T_j=(T_j)^{i_j}$. Then V_I is the composite $V_{(1,\dots,1)}(\varphi_{i_1\dots i_n})$. By the Künneth theorem

$$\varphi^*_{(i_1\dots i_n)}(e_j)=i_j(e_j)$$

$$\varphi^*_{(i_1\dots i_n}(b_j)=i_j(b_j)$$

so 2.3 follows from 2.2.

3.

The total Chern class of the representation λ in (1.2) is $1+b=C(\lambda)$. Moreover, if R_1 and R_2 are two representations of G then

$$C(R_1\oplus R_2)=C(R_1)C(R_2). \tag{3.1}$$

Thus, we have from (1.1), (2.3)

$$C(R(Z/p)^n)=\prod_{(i_1\dots i_n)\in(Z/p)^n}(1+i_1b_1+\dots+i_nb_n). \tag{3.2}$$

To calculate (3.2) explicitly we introduce

Definition 3.3. The Chern polynomial $CP(R)$ of a representation R of dimension n is the element

$$CP(R)=\Sigma\, c_i(R)x^{n-i}$$

in the ring of polynomials in x with coefficients in $H^*(G,Z/p)$.

We see directly that

$$CP(R_1\oplus R_2)=CP(R_1)CP(R_2)$$

so we have

$$\begin{aligned} CP(R(Z/p)^n)&=\prod_{(i_1\dots i_n)\in(Z/p)^n}(x+i_1b_1+\dots+i_nb_n)\\ &=x\left(\sum_{(i_1\dots i_n)\neq(0,\dots,0)}c_i(R(Z/p)^n)x^{p^n-i-1}\right)\\ &=CP(R(Z/p)^n-1)\circ CP(1)\end{aligned} \tag{3.4}$$

where 1 is the trivial representation.

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4.

The general linear group $GL_n(Z/p)$ acts on $(Z/p)^n$ as its group of automorphisms, and hence by composition on the representation ring, ($\alpha(S(G))$ is the composite

$$(Z/p)^n \xrightarrow{\alpha} (Z/p)^n \xrightarrow{S} U(m)$$

where $G=(Z/p)^n$, $\alpha \in GL_n(Z/p)$, and S is a representation) and $\alpha S_I = S_{\alpha(I)}$. Clearly, $R(Z/p)^n - 1$ and 1 form a basis for the set of $GL_n(Z/p)$ invariant representations of $(Z/p)^n$.

The action of $GL_n(Z/p)$ also induces an action on $H^*((Z/p)^n, Z/p)$ (if $\alpha(T_j) = T_1^{i_{j,1}}, \ldots, T_n^{i_{j,n}}$, then $\alpha^*(e_k) = \Sigma\, i_{s,k} e_s$ and similarly for b_k) and from invariance of $R(Z/p)^n$ we have

Lemma 4.1. *$c_i(R(Z/p)^n) \in P(b_1 \ldots b_n)^{GL_n(Z/p)}$ the invariant subring of $P(b_1 \ldots b_n)$.*

5.

Let $p(x_1 \ldots x_m)$ be a polynomial in m indeterminants with coefficients in the field Z/p. Since the pth power homomorphism is the identity on Z/p we have

$$(p(x_1 \ldots x_m))^p = p(x_1^p, \ldots, x_m^p). \tag{5.1}$$

Consequently, the determinant

$$\begin{vmatrix} x_1^{p^{i_1}} & x_2^{p^{i_1}} & \ldots & x_m^{p^{i_1}} \\ x_1^{p^{i_2}} & x_2^{p^{i_2}} & \ldots & x_m^{p^{i_2}} \\ — & — & — & — \\ x_1^{p^{i_m}} & x_2^{p^{i_m}} & \ldots & x_m^{p^{i_m}} \end{vmatrix} = D(x_1 \ldots, x_m, p^{i_1}, p^{i_2}, \ldots, p^{i_m}) \tag{5.2}$$

satisfies

$$D(\alpha(x_1), \ldots, \alpha(x_m), p^{i_1}, \ldots, p^{i_m}) = \det(\alpha) D(x_1, \ldots, x_m, p^{i_1}, \ldots, p^{i_m}) \tag{5.3}$$

where $\alpha \in GL_m(Z/p)$ and $\alpha(x_i) = x_i' = \Sigma\, d_{ij} x_j$.

We need the following particular cases, which, we should note are non-zero.

Definition 5.4. $D_{n,j} = D(x_1, \ldots, x_n, 1, p, p^2, \ldots, \hat{p}^j, \ldots, p^n)$.

In particular, $D_{n,n}$ is usually written L and $D_{n,0} = L^p$. L has an explicit factorisation first discovered by E. H. Moore in 1896.

Lemma 5.5. (Moore [5])

$$L = \prod_{(j_1 \ldots j_n)} (j_1 x_1 + \ldots + j_n x_n)$$

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where $(j_1 \ldots j_n)$ runs over all elements in $(Z/p)^n$ with first non-zero coefficient equal to one.

Proof. (Compare [1, p. 76]). L is invariant under the special linear group $SL_n(Z/p)$ which acts transitively on the non-zero elements of $(Z/p)^n$. Hence, since x_1 is a factor of L it follows that $\alpha(x_1)=j_1x_1+\ldots+j_nx_n$ is a factor as well. Hence the product above divides L (the factors are all relatively prime). But both sides have the same degree, hence they differ only up to a constant factor. On the other hand, $x_1^{p^{n-1}}x_2^{p^{n-2}}\ldots x_n$ the diagonal term occurs in both sides only once and each time with coefficient 1.

More generally x_1 is a factor of $D_{n,j}$ for all j so L is also a factor of $D_{n,j}$ by the argument above and we have

Lemma 5.6. *$D_{n,j}=Q_{n,j}L$ where $Q_{n,j}$ is a non-zero polynomial invariant under $GL_n(Z/p)$.*

6.

Dickson's main result in [1], proved by induction on n (for $n=2$ the method is by Galois theory) specifies the $GL_n(Z/p)$ invariants as

Theorem 6.1. (Dickson) $P(x_1 \ldots x_n)^{GL_n(Z/p)}$

$$=P(Q_{n,n-1}, Q_{n,n-2}, \ldots, Q_{n,0})$$

(Actually, he showed $P(x_1 \ldots x_n)^{SL_n(Z/p)}=P(Q_{n,n-1}, \ldots, Q_{n,1}, L)$, but this gives (6.1) directly.)

Here is our main result.

Theorem 6.2. *The Chern polynomial for the regular representation of $(Z/p)^n$ is*

$$CP(R(Z/p)^n)=x((-1)^n\sum_{i=0}^{n-1}(-1)^iQ_{n,i}(b_1 \ldots b_n)x^{p^i-1}).$$

The proof decomposes into three steps. First, we note that (6.2) is true up to some undetermined coefficients in Z/p. Next, we verify that

$$c_{p^n-1}=(-1)^nQ_{n,0}.$$

These two steps are purely formal algebra. Finally, using the Steenrod pth power operations and a mod(p) Wu formula for the Chern classes we obtain the remaining coefficients and finish the proof.

7.

Lemma 7.1. *In degrees less than p^n the only elements in $P(x_1 \ldots x_n)^{GL_n(Z/p)}$ are the $Q_{n,j}$.*

Proof. The class of least degree, according to (6.1), is $Q_{n,n-1}$ which has degree $p^n - p^{n-1}$ and since

$$2(p^n - p^{n-1}) \geqq p^n > p^n - 1 = \dim(Q_{n,0})$$

for all $p \geqq 2$ we have (7.1).

Consequently, the only invariants in the range of degrees involved in $CP(R(Z/p)^n)$ are the $Q_{n,i}$ and this gives step 1.

8.

In (3.4) we set $x=0$ in $CP(R(Z/p)^n - 1)$ we obtain

$$\begin{aligned} c_{p^n-1} &= \prod_{(i_1 \ldots i_n) \neq 0} (i_1 b_1 + \ldots + i_n b_n) \\ &= \prod_{k=1}^{n} (p-1)!^{p^{n-k}} \prod_{(0 \circ 0 1, j_{k+1}, \ldots j_n)} (b_k + j_{k+1} b_{k+1} + \ldots + j_n b_n)^{p-1} \\ &= (p-1)!^n L^{p-1} = (p-1)!^n Q_{n,0}. \end{aligned} \tag{8.1}$$

Now, by Wilson's theorem $(p-1)! \equiv -1 (\operatorname{mod} p)$ and we have

$$c_{p^n-1}(R(Z/p)^n - 1) = c_{p^n-1}(R(Z/p)^n) = (-1)^n Q_{n,0}. \tag{8.2}$$

This is step 2.

9.

In $H^*(CP^\infty, Z/p) = P(b)$ where b is the two dimensional generator, the action of the Steenrod pth power operation P^i is given by

$$P^i(b^j) = \binom{j}{i} b^{j+i(p-1)}. \tag{9.1}$$

In particular, $P^j(b^{p^i}) = 0$ unless $j = p^i$ in which case $P^{p^i}(b^{p^i}) = b^{p^{i+1}}$. Using the Cartan formula $P^i(x \cup y) = \sum_{j=0}^{i} P^j(x) \cup P^{i-j}(y)$ we have the action of P^i in $H^*((CP^\infty)^n, Z/p) = P(b_1, \ldots, b_n)$.

In particular, the above remarks and (5.3) show

$$P^j D_{n,k} = 0 \quad \text{if} \quad 0 < j < p^n \quad \text{and} \quad j \neq p^{k-1}$$

but $P^{p^{k-1}}D_{n,k}=D_{n,k-1}$. Thus, we have

Lemma 9.2. *$P^{p^k}(Q_{n,j}(b_1,\ldots,b_n))=Q_{n,j-1}(b_1,\ldots,b_n)$ if $k=j-1$ and is zero otherwise for $k<n$.*

Next, we require

Lemma 9.3. *The Chern classes c_n reduced* mod p *universally satisfy the Wu formula*

$$P^i(c_n)=\binom{n-1}{i}c_{n+(p-1)i}+D$$

where D is a decomposable polynomial in $c_1,\ldots,c_{n+(p-1)i-1}$.

Proof. There is a map $f:\Sigma^2 CP^\infty\to B_U$ which satisfies $f^*(c_i)=\sigma^2(b^{i-1})$ for all i, and, of course, $f^*(D)=0$ for any decomposable. For details see [**8**, Chapter 4]. Since P^i commutes with suspension and with f^*, (9.3) is a consequence of (9.1).

Thus, we have

$$P^{p^r-1}(c_{p^n-p^r})=\binom{p^n-p^r-1}{p^r-1}c_{p^n-p^r-1}=-1c_{p^n-p^r-1} \tag{9.4}$$

on comparing coefficients in the expansion

$$(x+1)^{p^n-p^r-1}=(x^{p^{n-1}}+1)^{p-1}\ldots(x^{p^r}+1)^{p-2}(x^{p^{r-1}}+1)^{p-1}\ldots(x+1)^{p-1}.$$

Now using (9.2) and (9.4) together with (8.2) the proof of (6.2) is direct.

10.

If G is a finite group of order n it has a natural embedding $p:G\to\mathscr{S}_n$ obtained by regarding left multiplication by $g\in G$ as a permutation of the elements of G. Also, there is the permutation representation of $\mathscr{S}_n$

$$P:\mathscr{S}_n\to U(n)$$

where $P(\alpha)(z_1\ldots z_n)=(z_{\alpha^{-1}(1)},\ldots,z_{\alpha^{-1}(n)})$, and we evidently have

Lemma 10.1. *The composite*

$$G\xrightarrow{p}\mathscr{S}_n\xrightarrow{P}U(n)$$

is the regular representation $R(G)$.

Corollary 10.2. *$H^*(\mathscr{S}_n, Z/p)$ contains a polynomial subring $\mathscr{P}$ on generators $c_{p^n-p^{n-1}}(P), \ldots, c_{p^n-1}(P)$, and $\mathscr{P}$ maps onto*

$$P(b_1, \ldots, b_n)^{GL_n(Z/p)}$$

under the induced map

$$p^*: H^*(\mathscr{S}_{p^n}, Z/p) \to H^*((Z/p)^n, Z/p).$$

11.

We need more precise information about the Chern classes of the permutation representation of $\mathscr{S}_{p^n}$. To this end consider the inclusion

$$(\mathscr{S}_{p^{n-1}})^p \xrightarrow{i_{n-1}} \mathscr{S}_{p^n}$$

where the p-tuple $(\alpha_1, \ldots, \alpha_p)$ acts by regarding α_i as a permutation of the ith block of p^{n-1} letters

$$((i-1)p^{n-1}+1, \ldots, ip^{n-1}).$$

Let $q_i(\mathscr{S}_{p^{n-1}})^p \to \mathscr{S}_{p^{n-1}}$ be projection onto the ith factor, then by inspection we have

Lemma 11.1. *The representation $P_{p^n}(i_{n-1})$*

$$(\mathscr{S}_{p^{n-1}})^p \to U(p^n) \qquad \textit{is}$$

$$\sum_{i=1}^{p} P_{p^{n-1}}(q_i).$$

Consequently, under the composite map

$$\varphi_{n,1}: [(Z/p)^{n-1}]^p \xrightarrow{p \times \cdots \times p} (\mathscr{S}_{p^{n-1}})^p \xrightarrow{i_{n-1}} \mathscr{S}_{p^n} \tag{11.2}$$

we see that

$$\varphi_{n,1}^{*}C(P)=C(R(Z/p)^{n-1})\otimes\ldots\otimes C(R(Z/p)^{n-1})$$

since the total Chern class is multiplicative for sums of bundles.

12.

We may iterate the construction of (11.2) obtaining maps

$$\varphi_{n,i}:[(Z/p)^{n-i}]^{p^i}\to\mathscr{S}_{p^n}. \tag{12.1}$$

Recall the theorem of [**4**], see also [**6**, Section 3].

Theorem 12.2. *The intersection of the kernels of the maps p^*, i_{n-1}^* is zero.*

Consequently, the images under the $\varphi_{n,i}^*$ of $C(P)$ completely characterise $C(P)$ and we have

$$C(P\circ\varphi_{n,i})=\bigotimes^{p^i}C(R(Z/p)^{n-i}). \tag{12.3}$$

13.

The constructions above and some obvious generalisations provide nearly complete information about the parts of $H^*(\mathscr{S}_{p^n}, Z/p)$ which map onto the pure polynomial pieces of the cohomology of the detecting groups in (12.1). However, there are many classes which map into classes involving the exterior algebras as well. These are much more complex and most of the work in [**4**] is involved in classifying them.

Some of them, though by no means all, can be obtained by a modification of the techniques above by lifting the stabilisation of the permutation representation in the diagram

$$\begin{tikzcd} & GL_{p^n}(Z/p) \arrow[d, "g"] \\ \mathscr{S}_{p^n} \arrow[ur, "\bar{P}"] \arrow[r, "P"'] & U(p^n) \end{tikzcd} \tag{13.1}$$

where $\bar{P}$ is the permutation representation in the modular group, and g is the Green's representation [**2**]. Then use Quillen's calculation [**7**] of $H^*(GL_n(Z/q), Z/p)$ together with a slight extension of the techniques in Section 9 and Section 10, for appropriate q.

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