Note on the Complete Jacobian Elliptic Integrals.

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The complete elliptic integrals K and E are functions of their modulus k which satisfy the equations

$$kk'^2 \frac{dK}{dk} = E - k'^2 K$$

$$k \frac{dE}{dk} = E - K$$

respectively.

If we write Π for $\Pi(K, a; k)$ where

$$\Pi(u, a; k) \equiv \int_0^u \frac{k^2 \, sn a \, cn a \, dn a \, sn^2 u}{1 - k^2 sn^2 a \, sn^2 u} du$$

is the elliptic integral of the third kind, then Π is a function of a and k, and $\frac{d\Pi}{dk}$ has a meaning only when a and k are connected by a functional relation. In this note the value of the derivate $\frac{d\Pi}{dk}$ is found in certain cases, on the assumption that such a relation does exist. On account of their simplicity, the results appear to be worth recording. They arose in discussing the geodesics on an ellipsoid of revolution.

Write $x = a \sin\theta \cos\phi$

$y = a \sin\theta \sin\phi$

$z = c \cos\theta$

to define a point on the ellipsoid $\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$, and suppose $c > a$ and $c^2 e^2 = c^2 - a^2$ (prolate spheroid).

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Consider the geodesic which touches the parallels of latitude $\theta = \pm\alpha$. The first integral of geodesics on a surface of revolution is known to be $r \sin\eta = \text{const}$, where η is the angle between the curve and the meridian. Expressing this fact for the above surface, we obtain for the equation of the geodesic in question

$$d\phi = \frac{c}{a}\frac{\sin\alpha}{\sin\theta}\cdot\frac{\sqrt{1-e^2\cos^2\theta}}{\sqrt{\cos^2\alpha-\cos^2\theta}}\,d\theta$$

and the difference in longitude between a turning point of the geodesic and the point where it crosses the equator is

$$\frac{c}{a}\int_\alpha^{\frac{\pi}{2}}\frac{\sin\alpha}{\sin\theta}\,\frac{\sqrt{1-e^2\cos^2\theta}}{\sqrt{\cos^2\alpha-\cos^2\theta}}\,d\theta = \frac{c}{a}\int_\alpha^{\frac{\pi}{2}} f(\theta,\alpha)\,d\theta = \frac{c}{a}\,.\,I, \text{ say.}$$

Now let us find $\dfrac{dI}{d\alpha}$. We have

$$\frac{\partial f}{\partial\alpha} = \cos\alpha\,\sin\theta\,\frac{\sqrt{1-e^2\cos\;\theta}}{(\cos^2\alpha-\cos^2\theta)^{3/2}}$$

and notice that it contains an infinity of order $\frac{3}{2}$ at the lower limit of integration. We can write, however, in this case (*cf.* Hardy: *Quarterly Journal*, Vol. 32, 1901)

$$\frac{dI}{d\alpha} = \lim_{\beta\to\alpha}\left[\int_\beta^{\frac{\pi}{2}}\frac{\partial f}{\partial\alpha}\,d\theta - f(\beta,\alpha)\right]$$

Now transform to the notation of the Jacobian elliptic functions by writing

$$\cos\theta = \cos\alpha\,.\,sn(u+K) = \cos\alpha\,\frac{cn\,u}{dn\,u}, \text{ where } k = e\cos\alpha;$$

we get

$$f(\theta,\alpha) = \frac{1}{\cos\alpha}\cdot\frac{dn\,u}{sn\,u}\cdot\frac{1}{\sqrt{1+\cot^2\alpha(1-e^2)sn^2 u}}$$

$$\int\frac{\partial f}{\partial\alpha}\,d\theta = \frac{1}{\cos\alpha}\int\frac{du}{sn^2\,u} = \frac{1}{\cos\alpha}\left[u - \frac{cn\,u\;dn\,u}{sn\,u} - E(u)\right]$$

and
$$I = \frac{1-e^2\cos^2\alpha}{\sin\alpha}\int_0^K\frac{du}{1+\cot^2\alpha(1-e^2)sn^2 u}$$

and without difficulty $\dfrac{dI}{d\alpha} = \dfrac{1}{\cos\alpha}\,.\,(K-E)$.

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Substituting for α in terms of k throughout this equation we arrive at the result

$$\frac{d}{dk}\,\frac{ek'^2}{\sqrt{e^2-k^2}}\int_0^K \frac{du}{1+k^2\dfrac{1-e^2}{e^2-k^2}sn^2u}=\frac{e\,(E-K)}{k\sqrt{e^2-k^2}} \quad \ldots\ldots\ldots\ldots(1)$$

in which we can remove the restriction that e should be less than the unity.

But we can write

$$I=\frac{ek'^2}{\sqrt{e^2-k^2}}\int_0^K\left[du-\frac{k^2\dfrac{1-e^2}{e^2-k^2}sn^2u}{1+k^2\dfrac{1-e^2}{e^2-k^2}sn^2u}du\right]$$

$$=\frac{ek'^2}{\sqrt{e^2-k^2}}\cdot K+i\sqrt{1-e^2}\int_0^K k^2\,\frac{\dfrac{ik'^2e\sqrt{1-e^2}}{(e^2-k^2)^{3/2}}}{1+k^2\dfrac{1-e^2}{e^2-k^2}sn^2u}du$$

$$=\frac{ek'^2}{\sqrt{e^2-k^2}}\cdot K+i\sqrt{1-e^2}\,\Pi(K,a\,;\,k)$$

where $sn^2a=-\dfrac{1-e^2}{e^2-k^2}$.

Taking I in this form and differentiating again with respect to k and equating the result to the right side of (1), we finally obtain

$$\frac{d\Pi}{dk}=sn\,a\;cn\,a\;dn\,a\,.\,\frac{k}{k'^2}\,.\,K. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

This is the value of $\dfrac{d\Pi}{dk}$ under the condition $sn^2a=-\dfrac{1-e^2}{e^2-k^2}$,

which is easily seen to be equivalent to

$$\frac{cn^2a}{dn^2a}=sn^2(a+K)=\frac{1}{e^2}=\text{const.}$$

or $\quad sn\,(a+K,\,k)=\text{const.}$

Now if we carry out the same work for the case of the oblate spheroid, using the transformation given by Forsyth (*Differential*

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Geometry, p. 139), we arrive at the result

$$\frac{d}{dk}\frac{e}{\sqrt{e^2-k^2}}\int_0^K \frac{dn^2 u}{1+k^2 \cdot \frac{1-e^2}{e^2-k^2} sn^2 u}\, du = \frac{e}{\sqrt{e^2-k^2}}\,\frac{E-k'^2 . K}{kk'^2} \quad \text{.........(3)}$$

corresponding to (1) above.

After dividing by e, subtract (1) from (3) and get

$$\frac{d}{dk}\frac{1}{\sqrt{e^2-k^2}}\int_0^K \frac{dn^2 u - k'^2}{1+k^2 \frac{1-e^2}{e^2-k^2}\, sn^2 u} . du = \frac{1}{\sqrt{e^2-k^2}} . \frac{k}{k'^2} . E$$

which easily reduces to

$$\frac{d}{dk}\int_0^K \frac{\sqrt{e^2-k^2} . \frac{k^2}{e^2}\frac{cn^2 u}{dn^2 u}}{1-\frac{k^2}{e^2}\frac{cn^2 u}{dn^2 u}}\, du = \frac{1}{\sqrt{e^2-k^2}} . \frac{k}{k'^2} E$$

or, since $sn\,(u+K) = \dfrac{cn\,u}{dn\,u}$

and $sn\,(2K-u) = sn\,u$

$$\frac{d}{dk}\int_0^K \frac{k^2\, \frac{\sqrt{e^2-k^2}}{e} . \frac{1}{e} . sn^2 u}{1-k^2 . \frac{1}{e^2} . sn^2 u}\, du = \frac{1}{\sqrt{e^2-k^2}} . \frac{k}{k'^2} . E. \quad \text{.........(4)}$$

Put $sn\,a = \dfrac{1}{e}$, and after multiplying by $cn\,a$ we get

$$\frac{d\Pi}{dk} = \frac{sn\,a\; cn\,a}{dn\,a} . \frac{k}{k'^2} . E. \quad \text{........................(5)}$$

This then is the value of $\dfrac{d\Pi}{dk}$ under the condition $sn\,a = \text{const.}$

We may notice in passing that the values of $\dfrac{dK}{dk}$ and $\dfrac{dE}{dk}$ may be derived as special cases of the equations (1), (3) or (4). For example, putting $e=1$ in (1) we get $kk'^2\dfrac{dK}{dk} = E - k'^2 K$, and $e = \infty$ gives $k\dfrac{dE}{dk} = E - K$.

The results (2) and (5) could have been obtained in a more straightforward manner as follows, though one could hardly have predicted that they would turn out to be as simple as they are.

The expression of $\Pi(u, a; k)$ in terms of the Jacobian Θ function is $\Pi(u, a; k) = \frac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)} + u \, . \, Z(a)$

and on writing $u = K$ it becomes

$$\Pi(K, a; k) = K \, . \, Z(a) = K \, . \, E(a) - a \, . \, E. \qquad \text{...........(6)}$$

And since $$E(a) = \int_0^a dn^2 u \, du,$$

and therefore $$\frac{dE(a)}{dk} = dn^2 a \, . \, \frac{da}{dk} + \int_0^a \frac{\partial}{\partial k}(dn^2 u)\, du,$$

we obtain on differentiating

$$\frac{d\Pi}{dk} = E(a) \, . \, \frac{dK}{dk} - a\frac{dE}{dk} + (Kdn^2 a - E)\frac{da}{dk} + K\int_0^a \frac{\partial}{\partial k}(dn^2 u)\, du. \quad (7)$$

Now, from the equation $$u = \int_0^{sn\, u} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}$$

it is easy to prove that, while u remains constant

$$\frac{\partial sn\, u}{\partial k} = -\frac{cn\, u\, dn\, u}{kk'^2}\left[E(u) - k'^2 \, . \, u - k^2\frac{sn\, u \, . \, cn\, u}{dn\, u}\right] \quad \text{......(8)}$$

and hence that

$$\frac{\partial dn^2 u}{\partial k} = \frac{2k}{k'^2}\left[sn\, u\; cn\, u\; dn\, u\{E(u) - k'^2 \, . \, u\} - sn^2 u\, dn^2 u\right], \quad \text{...(9)}$$

and after a little reduction we obtain

$$\int_0^a \frac{\partial}{\partial k}(dn^2 u) du = \frac{k}{k'^2}\left[sn\, a\; cn\, a\; dn\, a - cn^2 a \, . \, E(a) - k'^2 \, . \, a \, . \, sn^2 a\right] \quad (10)$$

Substituting in (7)

$$\frac{d\Pi}{dk} = (Kdn^2 a - E)\left[\frac{da}{dk} - \frac{1}{kk'^2}\{E(a) - k'^2 . a\}\right] + sn\, a\; cn\, a\; dn\, a \, . \, \frac{k}{k'^2} \, . K$$

.........(11)

This equation gives the value of $\frac{d\Pi}{dk}$ when a and k are connected by any relation of the form $f(a, k) = \text{const.}$, provided we introduce the appropriate value of $\frac{da}{dk}$.

For example, if we make the assumptions

$$sn\, a = \text{const.}, \quad sn(a+K) = \text{const.},$$

and evaluate the right side of (11) in each case, we arrive at the results (5) and (2) respectively.

Again, since we are led to define

$$\Pi(u, a; k) = \int_0^u \frac{k^2\, sn\, a\, cn\, a\, dn\, a\, sn^2 u}{1 - k^2 sn^2 a\, sn^2 u}\, du$$

from a consideration of the integral $\int \frac{\alpha + \beta\, sn^2 u}{1 + \gamma\, sn^2 u}\, du$, it is natural to discuss the value of $\frac{d\Pi}{dk}$ when $\gamma\,[= -k^2 sn^2 a]$ remains constant; that is, when $dn\, a$ is constant.

Making use of (9) to differentiate $dn^2 a = \text{const.}$, we get

$$kk'^2 \frac{da}{dk} = E(a) - k'^2 . a - \frac{sn\, a\, dn\, a}{cn\, a}$$

and substituting in (11), we have

$$\left(\frac{d\Pi}{dk}\right)_{dn\, a = \text{const.}} = \frac{sn\, a\, dn\, a}{cn\, a} \cdot \frac{dK}{dk}. \quad \ldots\ldots\ldots(12)$$

Finally, when $dn(a+K, k) = \text{const.}$, we obtain

$$kk'^2 \frac{da}{dk} = E(a) - k'^2 . a + \frac{cn\, a\, dn\, a}{sn\, a},$$

and in this case

$$\left(\frac{d\Pi}{dk}\right)_{dn(a+K) = \text{const.}} = -\frac{cn\, a\, dn\, a}{sn\, a} \frac{1}{k'^2} \cdot \frac{dE}{dk}. \quad \ldots\ldots\ldots(13)$$

Collecting the results (11), (5), (2), (12), and (13), we may summarise the foregoing work in the following table:—

Relation between a and k.	Value of $\frac{d\Pi}{dk}$.
$a = \text{const.}$	$\frac{(E - dn^2 a \, . \, K)}{kk'^2} (E(a) - k'^2 \, . \, a) + sn\,a\,cn\,a\,dn\,a \, \frac{k}{k'^2} \, . \, K$
$sn\,(a \, . \, k) = \text{const.}$	$\frac{sn\,a\,cn\,a}{dn\,a} \, . \, \frac{k}{k'^2} \, . \, E$
$sn\,(a + K, \, k) = \text{const.}$	$sn\,a\,cn\,a\,dn\,a \, . \, \frac{k}{k'^2} \, . \, K$
$dn\,(a \, . \, k) = \text{const.}$	$\frac{sn\,a\,dn\,a}{cn\,a} \, . \, \frac{dK}{dk}$
$dn\,(a + K, \, k) = \text{const.}$	$- \frac{cn\,a\,dn\,a}{sn\,a} \, . \, \frac{1}{k'^2} \, . \, \frac{dE}{dk}$