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VARIATION REDUCING PROPERTIES OF DECREASING REARRANGEMENTS

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Introduction. One well-established characteristic of the operation of decreasing rearrangement is its variation reducing property. A systematic study of this property has been made in considerable detail by G.F.D. Duff in [**5**] and [**6**]. He proved some inequalities related to the operation of rearrangement in decreasing order showing that the total variation of a sequence or an absolutely continuous function is in general diminished by such rearrangement. He also showed that the L^p norm of the difference sequence (or the derivative function) is diminished by this rearrangement operation unless the given sequence (or absolutely continuous function) is already monotonic (or equal to a monotonic function almost everywhere). One of his inequalities [**5**, Theorem 4.1, p. 1168] was later generalized (except for the case of equality) for almost everywhere differentiable functions by J.V. Ryff in [**10**, p. 455].

In this paper, we establish some spectral inequalities (i.e. expressions of the form $f \prec g$ or $f \ll g$ where $\prec$ and $\ll$ denote the Hardy-Littlewood-Pólya spectral order relations) showing that the variation reducing properties of decreasing rearrangements can also be expressed in the sense of the weak spectral order $\ll$. With these spectral inequalities, we obtain some results of Duff and Ryff as particular cases. Moreover, we give conditions for equality in Ryff's generalization of Duff's inequality which was not discussed by Ryff in [**10**, p. 455].

1. Preliminaries. Let (X, Λ, μ) be a finite measure space, i.e., X is a non-empty point set provided with a countably additive non-negative measure μ on a σ-algebra Λ of subsets of X such that $\mu(X) < \infty$. Whenever X is clear from the context, we shall often write $\int \cdot \, d\mu$ for integration over X. By $M(X, \mu)$ we denote the set of all extended real valued measurable functions on X. Two functions $f \in M(X, \mu)$ and $g \in M(X', \mu')$, where $\mu'(X') = \mu(X)$, are said to be *equimeasurable* (written $f \sim g$) whenever

(1) $\quad \mu(\{x : f(x) > t\}) = \mu'(\{x : g(x) > t\})$

for all real t. If $f \sim g$, it is not hard to see that

(2) $\quad \Phi(f) \sim \Phi(g)$

whenever $\Phi : \mathbf{R} \to \mathbf{R}$ is a Borel measurable function. Moreover, if (X', Λ', μ')

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is any other measure space with $\mu'(X') = \mu(X)$, then one can easily verify that

(3) $f \circ \sigma \sim f$

whenever $\sigma : X \to X'$ is a measure-preserving map, i.e., $\sigma^{-1}(E) \in \Lambda$ and $\mu(\sigma^{-1}(E)) = \mu'(E)$ for all $E \in \Lambda'$.

If $f \in M(X, \mu)$, it is well-known that there exists a unique right continuous non-increasing function δ_f on the interval $[0, \mu(X)]$, called the *decreasing rearrangement* of f, such that δ_f and f are equimeasurable. In fact,

(4) $\delta_f(s) = \inf \{t \in \mathbf{R} : \mu(\{x : f(x) > t\}) \leqq s\}$

for all $s \in [0, \mu(X)]$.

In what follows, we denote the Lebesgue measure on the real line $\mathbf{R}$ by m.

If $f, g \in M(X, \mu)$ and $f^+, g^+ \in L^1(X, \mu)$ where $\mu(X) = a < \infty$, then we write $f \ll g$ whenever

$$\int_0^t \delta_f dm \leqq \int_0^t \delta_g dm, \quad t \in [0, a]$$

and $f \prec g$ whenever $f \ll g$ and

$$\int_0^a \delta_f dm = \int_0^a \delta_g dm.$$

In the sequel, expressions of the form $f \prec g$ (respectively $f \ll g$) are called *strong* (respectively *weak*) *spectral inequalities.*

In establishing the spectral inequalities to be given below, we need the following results proved earlier in [**2**].

THEOREM 1.1 (Chong [**2**, Theorems 2.3, 2.8, 3.1 and Corollaries 2.4 and 3.2]). *Suppose (X, Λ, μ) is a finite measure space. Suppose $f, g \in M(X, \mu)$ with integrable positive parts. Then $f \ll g$ if and only if $\int \Phi(f) d\mu \leqq \int \Phi(g) d\mu$ for all non-decreasing convex functions $\Phi : \mathbf{R} \to \mathbf{R}$ or, equivalently, $\Phi(f) \ll \Phi(g)$ for all non-decreasing convex functions $\Phi : \mathbf{R} \to \mathbf{R}$ such that $\Phi^+(g) \in L^1(X, \mu)$.*

If $f \ll g$ and if $\Phi : \mathbf{R} \to \mathbf{R}$ is strictly convex increasing such that $\Phi(g) \in L^1(X, \mu)$, then $\int \Phi(f) d\mu = \int \Phi(g) d\mu$ if and only if $f \sim g$.

Moreover, if $\Phi : \mathbf{R} \to \mathbf{R}$ is strictly increasing convex and if $f, g \in L^1(X, \mu)$ are such that $f \ll g$, then the strong spectral inequality $f \prec g$ holds whenever the integrals $\int \Phi(f) d\mu$ and $\int \Phi(g) d\mu$ are finite and equal.

THEOREM 1.2 (Hardy, Littlewood and Pólya [**7**, Theorem 10, p. 152] and Chong [**3**, Theorem 2.5 and Corollary 2.6]). *Suppose $f, g \in L^1(X, \mu)$, where $\mu(X) < \infty$. Then $f \prec g$ if and only if $\int \Phi(f) d\mu \leqq \int \Phi(g) d\mu$ for all convex functions $\Phi : \mathbf{R} \to \mathbf{R}$ or, equivalently, $\Phi(f) \ll \Phi(g)$ for all convex functions $\Phi : \mathbf{R} \to \mathbf{R}$ such that $\Phi^+(g) \in L^1(X, \mu)$.*

If $f \prec g$ and if $\Phi : \mathbf{R} \to \mathbf{R}$ is stricly convex such that $\Phi(g) \in L^1(X, \mu)$, then the equality $\int \Phi(f) d\mu = \int \Phi(g) d\mu$ holds if and only if $f \sim g$.

2. Variation reducing properties of decreasing rearrangements. In this section, we prove some new spectral inequalities showing that the variation reducing properties of decreasing rearrangements can also be expressed in the sense of the weak spectral order $\ll$.

We assume in the sequel without further comments that for any given n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$, the n-tuple $\mathbf{a}^* = (a_1^*, a_2^*, \ldots, a_n^*)$ will always denote the decreasing rearrangement of $\mathbf{a}$; here we have regarded $\mathbf{a}$ as a measurable function defined on a discrete measure space with n atoms of equal measures. We also denote by $\Delta\mathbf{a}$ the vector $(\Delta a_1, \Delta a_2, \ldots, \Delta a_{n-1})$ in $\mathbf{R}^{n-1}$, where $\Delta a_k = a_{k+1} - a_k$, $k = 1, 2, \ldots, n - 1$.

THEOREM 2.1. *If* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *is any n-tuple in* $\mathbf{R}^n$, *then*

$$|\Delta\mathbf{a}^*| \ll |\Delta\mathbf{a}|$$

where strong spectral inequality holds if and only if $|\Delta\mathbf{a}^*| \sim |\Delta\mathbf{a}|$ *or, equivalently, the sequence* $\{a_1, a_2, \ldots, a_n\}$ *is monotonic.*

Proof. By definition, we have $|\Delta\mathbf{a}| = (|a_2 - a_1|, |a_3 - a_2|, \ldots, |a_n - a_{n-1}|)$.

After interchanging the summands in each component of $|\Delta\mathbf{a}|$ if necessary, we have

$$|\Delta\mathbf{a}| \sim (|a_{i_1} - a_1^*|, |a_{i_2} - a_2^*|, \ldots, |a_{i_{n-1}} - a_{n-1}^*|)$$

for some permutation $(i_1, i_2, \ldots, i_{n-1})$ of $n - 1$ integers from the sequence $\{1, 2, \ldots, n\}$.

Now, if $a_1^* \notin \{a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}}\}$, then $(a_{i_1}, \ldots, a_{i_{n-1}}) \sim (a_2^*, \ldots, a_n^*)$ and so, by [**3**, Theorem 3.3] (which is a generalization for general L^1 functions of a spectral inequality of Lorentz and Shimogaki [**8**, Proposition 1, p. 34] via Luxemburg's Theorem [**9**, Theorem 9.5]), we have

$$\begin{aligned}(|a_2^* - a_1^*|, |a_3^* - a_2^*|, \ldots, |a_n^* - a_{n-1}^*|)\\ \ll (|a_{i_1} - a_1^*|, |a_{i_2} - a_2^*|, \ldots, |a_{i_{n-1}} - a_{n-1}^*|).\end{aligned}$$

Hence $|\Delta\mathbf{a}^*| \ll |\Delta\mathbf{a}|$.

If $a_1^* \in \{a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}}\}$, say $a_1^* = a_{i_j}$ for some $1 \leqq j \leqq n - 1$ and if

$$\{a_1, a_2, \ldots, a_n\} - \{a_{i_1}, a_{i_2}, \ldots, a_{i_{n-1}}\} = a_k$$

for some $1 < k \leqq n$, then

$$\begin{aligned}|\Delta\mathbf{a}| &\sim (|a_{i_1} - a_1^*|, \ldots, |a_{i_j} - a_j^*|, \ldots, |a_{i_{n-1}} - a_{n-1}^*|)\\ &\geqq (|a_{i_1} - a_1^*|, \ldots, |a_{i_{j-1}} - a_{j-1}^*|, |a_k - a_j^*|,\\ &\qquad |a_{i_{j+1}} - a_{j+1}^*|, \ldots, |a_{i_{n-1}} - a_{n-1}^*|)\\ &\gg (|a_2^* - a_1^*|, |a_3^* - a_2^*|, \ldots, |a_n^* - a_{n-1}^*|),\end{aligned}$$

again by [**3**, Theorem 3.3].

Finally, if the sequence $\{a_1, \ldots, a_n\}$ is monotonic, then clearly $|\Delta\mathbf{a}^*| \sim |\Delta\mathbf{a}|$ and, *a fortiori*, $|\Delta\mathbf{a}^*| \prec |\Delta\mathbf{a}|$. If the sequence $\{a_1, \ldots, a_n\}$ is not monotonic,

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then at least one of the components of the vector $\Delta\mathbf{a}$ is the sum of two or more non-zero components of the vector $\Delta\mathbf{a}^*$ (since there must be three non-monotonic consecutive components of $\mathbf{a}$) and so

$$\sum_{i=1}^{n-1} |\Delta a_i^*| < \sum_{i=1}^{n-1} |\Delta a_i|,$$

i.e. the spectral inequality $|\Delta\mathbf{a}^*| \ll |\Delta\mathbf{a}|$ is (strictly) weak.

COROLLARY 2.2. *If* $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$, *then*

$$\Phi(|\Delta\mathbf{a}^*|) \ll \Phi(|\Delta\mathbf{a}|)$$

and, in particular,

$$\sum_{i=1}^{n-1} \Phi(|\Delta a_i^*|) \leqq \sum_{i=1}^{n-1} \Phi(|\Delta a_i|)$$

for all increasing convex functions $\Phi : \mathbf{R}^+ \to \mathbf{R}$.

If Φ *is strictly increasing and convex, then the strong spectral inequality* $\Phi(|\Delta\mathbf{a}^*|) \prec \Phi(|\Delta\mathbf{a}|)$ *holds or, equivalently,*

$$\sum_{i=1}^{n-1} \Phi(|\Delta a_i^*|) = \sum_{i=1}^{n-1} \Phi(|\Delta a_i|)$$

if and only if the sequence $\{a_1, a_2, \ldots, a_n\}$ *is monotonic.*

Proof. This follows easily from Theorem 2.1 by virtue of Theorem 1.1.

COROLLARY 2.3. *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$ *where* $a_i > 0$, $i = 1, 2, \ldots, n$. *If* $b_{i,i+1} = a_i/a_{i+1}$ *or* a_{i+1}/a_i *whichever is greater or equal to* 1, $i = 1, 2, \ldots, n - 1$, *then*

$$(b_{12}, b_{23}, \ldots, b_{n-1,n}) \ll \left(\frac{a_1^*}{a_2^*}, \frac{a_2^*}{a_3^*}, \ldots, \frac{a_{n-1}^*}{a_n^*}\right)$$

where strong spectral inequality holds if and only if the seuqence $\{a_1, \ldots, a_n\}$ *is monotonic.*

Proof. The result follows easily on applying Theorem 2.1 to the vector $\log \mathbf{a} = (\log a_1, \ldots, \log a_n)$ and then exponentiating, i.e. using Theorem 1.1.

As a direct consequence of Corollary 2.2, we have the following theorem of G.F.D. Duff [**5**, Theorem (2.1), p. 1156].

THEOREM 2.4 (Duff). *If* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{R}^n$, *then*

$$\sum_{k=1}^{n-1} |\Delta a_k^*|^p \leqq \sum_{k=1}^{n-1} |\Delta a_k|^p, \quad p \geqq 1$$

where equality holds if and only if the sequence $\{a_1, \ldots, a_n\}$ *is monotonic.*

In [**6**, Theorem 3, p. 423], Duff obtained an improved version of Theorem 2.4. We observe that the proof given by him (except for the case of equality)

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is immediately applicable to extending his result to more general concave or convex functions, provided we use the following lemma. Our method gives necessary (and sufficient) conditions for equality to hold in his theorem.

LEMMA 2.2. *If* $a_i \geqq 0$, $i = 1, 2, \ldots, n$, *then*

$$\Phi(a_1) + \ldots + \Phi(a_n) \leqq \Phi(a_1 + a_2 + \ldots + a_n) + (n-1)\Phi(0)$$

for all convex functions $\Phi : \mathbf{R}^+ \to \mathbf{R}$ *and the inequality is reversed if* $\Phi : \mathbf{R}^+ \to \mathbf{R}$ *is concave.*

If Φ *is strictly convex or strictly concave, then equality holds if and only if all except possibly one of the* a_i*'s*, $i = 1, 2, \ldots, n$, *are zero.*

Proof. It is easy to see that the strong spectral inequality

$$(a_1, a_2, \ldots, a_n) \prec (a_1 + a_2 + \ldots + a_n, 0, \ldots, 0)$$

holds. With this spectral inequality, the result then follows easily from Theorem 1.2.

THEOREM 2.6. *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *be any n-tuple in* $\mathbf{R}^n$. *If* N_k *denotes the number of intervals of* $\{a_k\}$ *that contain the open interval* (a_k^*, a_{k+1}^*), $k = 1, 2, \ldots, n-1$, *and if* $\Phi : \mathbf{R}^+ \to \mathbf{R}$ *is a convex function satisfying* $\Phi(0) = 0$, *then*

$$\sum_{k=1}^{n-1} N_k \Phi(|\Delta a_k^*|) \leqq \sum_{k=1}^{n-1} \Phi(|\Delta a_k|)$$

where equality holds if Φ *is the identity map of* $\mathbf{R}^+$. *The inequality is reversed if* $\Phi : \mathbf{R}^+ \to \mathbf{R}$ *is concave and satisfying* $\Phi(0) = 0$.

If Φ *is strictly convex or strictly concave, then equality holds if and only if the sequence* $\{a_1, a_2, \ldots, a_n\}$ *is monotonic.*

Proof. Using Lemma 2.5, the proof of the first part of the theorem is essentially the same as that given in [**6**, Theorem 3, p. 423].

The sufficiency of the condition for equality is clear. To prove the necessity, let $\Phi : \mathbf{R}^+ \to \mathbf{R}$ be strictly convex. If the sequence $\{a_1, a_2, \ldots, a_n\}$ is not monotonic, then there must be at least three non-monotonic consecutive components of $\mathbf{a}$, and so at least one of the components of the vector $\Delta\mathbf{a}$ is the sum of two or more non-zero components of the vector $\Delta\mathbf{a}^*$. Using the fact that $\bigcup_{i=1}^{n-1}[a_i, a_{i+1}] = [a_n^*, a_1^*]$, it is then not hard to see that

$$\sum_{k=1}^{n-1} N_k \Phi(|\Delta a_k^*|) < \sum_{k=1}^{n-1} \Phi(|\Delta a_k|).$$

Remark. Theorem 2.6 implies, in particular, that

$$\sum_{k=1}^{n-1} \Phi(|\Delta a_k^*|) \leqq \sum_{k=1}^{n-1} \Phi(|\Delta a_k|)$$

for all increasing convex functions $\Phi : \mathbf{R}^+ \to \mathbf{R}$, and so it gives an alternative proof for Theorem 2.1 by virtue of Theorem 1.1.

In [**10**, p. 455], Ryff obtained a continuous analogue of Theorem 2.4 for almost everywhere differentiable functions defined on the unit interval [0, 1]. We observe that the proof given by him is readily applicable to extending the spectral inequality given in Theorem 2.1 to almost everywhere differentiable functions and to obtain conditions for strong spectral inequality. His proof implies, in particular, the following interesting result.

THEOREM 2.7. *If a function f is differentiable almost everywhere on a finite interval* $[0, a]$, *then there exists a measure preserving transformation* $\sigma : [0, a] \to [0, a]$ *such that*

$$|\delta_f'| \circ \sigma \leqq |f'| \quad m\text{-}a.e.$$

Proof. In [**10**], Ryff proved that there exist a measure preserving map σ of the interval $[0, a]$ into itself and a set $D \subset [0, a]$ satisfying $m(D) = a$, $|f'(t)| < \infty$, $|\delta_f'| \circ \sigma(t) < \infty$, $f(t) = \delta_f \circ \sigma(t)$ for all $t \in D$, and $\sigma'(t) \geqq 1$, $(\delta_f' \circ \sigma(t))\sigma'(t) = f'(t)$ whenever $\delta_f' \circ \sigma(t) \neq 0$, $t \in D$.

Thus, on the subset of D where $\delta_f' \circ \sigma \neq 0$, the preceding paragraph implies

$$|\delta_f'| \circ \sigma \leqq |f'|$$

which is also trivially satisfied on the subset of D where $\delta_f' \circ \sigma = 0$. Since $m(D) = a$, the result follows.

THEOREM 2.8. *If f is almost everywhere differentiable on a finite interval* $[0, a]$, *then*

$$\int_0^a \Phi(|\delta_f'|)dm \leqq \int_0^a \Phi(|f'|)dm$$

for all non-decreasing (not necessarily convex) *functions* $\Phi : \mathbf{R}^+ \to \mathbf{R}$ *and, in particular,*

$$\Phi(|\delta_f'|) \ll \Phi(|f'|)$$

for all non-decreasing functions $\Phi : \mathbf{R}^+ \to \mathbf{R}$ *such that* $\Phi^+(|f'|)$ *is integrable.*

If Φ is strictly increasing such that $\Phi(|f'|)$ is integrable, then the strong spectral inequality $\Phi(|\delta_f'|) \prec \Phi(|f'|)$ *or the equality*

$$\int_0^a \Phi(|\delta_f'|)dm = \int_0^a \Phi(|f'|)dm$$

holds if and only if $\Phi(|\delta_f'|) \sim \Phi(|f'|)$ *or* $|\delta_f'| \sim |f'|$.

If f is absolutely continuous, then $|\delta_f'| \prec |f'|$ *or* $|\delta_f'| \sim |f'|$ *if and only if f is monotonic.*

Proof. By (2), (3) and Theorem 2.7, we clearly have $\Phi(|\delta_f'|) \sim \Phi(|\delta_f'| \circ \sigma) \leqq$

$\Phi(|f'|)$ and so $\Phi(|\delta_f'|) \ll \Phi(|f'|)$ whenever Φ is an increasing function such that $\Phi^+(|f'|)$ is integrable.

If $\Phi : \mathbf{R}^+ \rightarrow \mathbf{R}$ is strictly increasing such that $\Phi(|f'|)$ is integrable, then $\int \Phi(|\delta_f'|)dm = \int \Phi(|f'|)dm$ implies $\int \Phi(|\delta_f' \circ \sigma|)dm = \int \Phi(|f'|)dm$ and so $\Phi(|\delta_f'| \circ \sigma) = \Phi(|f'|)$ m-a.e., which, in turn, implies that $|\delta_f'| \circ \sigma = |f'|$ m-a.e. Hence $|\delta_f'| \sim |f'|$ since $\sigma : [0, a] \rightarrow [0, a]$ is measure preserving.

Clearly, the condition is sufficient.

The last assertion follows directly from the preceding paragraph and Duff's Theorem [**5**, Theorem (4.1), p. 1168].

Remarks. (i) Theorem 2.8 extends Duff-Ryff Theorem [**10**, p. 455].

(ii) If f is only required to be differentiable almost everywhere on $[0, a]$, then the equality

$$\int_0^a |\delta_f'|dm = \int_0^a |f'|dm$$

does not necessarily imply that f is monotonic. Take, for example,

$$f = \chi_{[0,a/3)} + 3\chi_{[a/3,2a/3)} + 2\chi_{[2a/3,a)}$$

where χ denotes characteristic function. Then, clearly, $f' = 0 = \delta_f'$ m-a.e., but f is not monotonic.

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