(C, γ, μ)-HOMOGENEITY OF PROJECTIVE PLANES AND POLARITIES

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In **(1)** Baer introduced the concept of (C, γ)-transitivity and (C, γ)-homogeneity. A projective plane (see **(5)** for the requisite definitions and axioms) is (C, γ)-transitive if, given an ordered pair (P_1, P_2) of points collinear with C but distinct from C and not on γ, there is a collineation which maps P_1 into P_2 and leaves fixed every point on γ as well as every line through C. A projective plane is (C, γ)-homogeneous for a non-incident point-line-pair if it is (C, γ) transitive and there is a correlation which maps every line through C into its intersection with γ and every point on γ into its join with C.

The concept of (C, γ)-homogeneity was extended in **(4)** to what was there called (C, γ, μ)-homogeneity.

A projective plane is (C, γ, μ)*-homogeneous if*

(1) *it is* (C, γ)*-transitive,*

(2) *there is a correlation* τ *whose square is a central collineation with centre* C *and axis* γ (*i.e.,* τ^2 *fixes every point on* γ *and every line through* C).

The correlation τ induces a mapping μ of the lines through C onto the points of γ. Clearly τ and $\sigma\tau$ (where σ is a central collineation with centre C and axis γ) induce the same mapping μ.

Note. *C could be incident with γ.* There are examples of projective planes which are (C, γ, μ)-homogeneous for both incident and non-incident point-line-pairs **(4)**.

It is of some interest to know whether one can always choose the correlation τ in such a way that τ^2 is the identity. In what follows, it will be seen that this is always possible if C lies on γ. If C does not lie on γ this is still an open question.

Theorem 1. *Let* $\mathfrak{E}$ *be a projective plane which is* (C, γ, μ)*-homogeneous,* $C \in \gamma$. *Then* $\mathfrak{E}$ *has a polarity* (*correlation of order* 2) *which interchanges* C *with* γ *and induces the mapping* μ.

Proof. Set up a ternary ring (we use here Pickert's version of the Hall ternary ring **(5)** in the plane with the fundamental quadrangle O, U, V, E). Choose $C = V$, O not on γ, $U = (OV)^\mu$, and for later convenience we choose $E = OW \cap (W)^\mu$, where W is a point on UV distinct from U and V. Points have co-ordinates (x, y) with $x, y \in \mathfrak{T}$ if they are not on UV; and (m), $m \in \mathfrak{T}$,

Received January 14, 1964.

if they are on UV but distinct from V. The points on the line OE but not on UV satisfy the equation $y = x$; $O = (0, 0)$, $E = (1, 1)$, $U = (0)$. The line joining (m) and $(0, b)$ has the co-ordinates $[c]$. The ternary operation $\mathbf{T}$ maps $\mathfrak{T} \times \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ and is defined by $y = \mathbf{T}(m, x, b)$ if and only if the point (x, y) lies on the line $[m, b]$. Addition $(\mathfrak{T}, +)$ is defined by $a + b = \mathbf{T}(1, a, b)$, $a, b \in \mathfrak{T}$; multiplication $(\cdot)$ is defined by $a \cdot b = \mathbf{T}(a, b, 0)$.

$\mathfrak{E}$ is (V, UV, μ)-homogeneous and therefore (V, UV)-transitive. It is well known (**5**, p. 100) that (V, UV)-transitivity is equivalent to the first splitting law, $\mathbf{T}(m, x, b) = mx + b$, together with the associativity of addition.

We now consider an analytic representation of $\mathfrak{E}$. Since a line through V is mapped into a point on UV, and furthermore $(UV)^\mu = V$, there is a mapping of $\mathfrak{T}$ onto $\mathfrak{T}$ which may (without danger of confusion) also be called μ, such that:

$$[c]^\mu = (c^\mu).$$

Because $(OV)^\tau = U, 0^\mu = 0$ and because $(EV)^\tau = W, 1^\mu = 1$. Since a point on UV is mapped onto a line through V, there is a mapping ν of $\mathfrak{T}$ onto $\mathfrak{T}$ such that:

$$(m)^\tau = [m^\nu].$$

Because $U^\tau = OV, 0^\nu = 0$ and because $W^\tau = EV, 1^\nu = 1$. Since a point of OV is mapped onto a line through U, there is a mapping π of $\mathfrak{T}$ onto $\mathfrak{T}$ such that:

$$(0, b)^\tau = [0, b^\pi].$$

By definition,

$$[m, b] = (m) \cup (0, b).$$

Thus

$$[m, b]^\tau = (m)^\tau \cap (0, b)^\tau = [m^\nu] \cap [0, b^\pi] = (m^\nu, b^\pi);$$

therefore $[0, b]^\tau = (0, b^\pi)$ because $0^\nu = 0$. Because $(x, y) = [x] \cap [0, y]$, we get

$$(x, y)^\tau = [x]^\tau \cup [0, y]^\tau = (x^\mu) \cup (0, y^\pi) = [x^\mu, y^\pi].$$

Incidence is preserved by a correlation; thus

$$y = mx + b \rightleftharpoons b^\pi = x^\mu m^\nu + y^\pi$$

and we get the incidence equation

$$b^\pi = x^\mu m^\nu + (mx + b)^\pi$$

for all $m, x, b \in \mathfrak{T}$. Let

$$x = 1: b^\pi = m^\nu + (m + b)^\pi;$$
$$m = 1: b^\pi = x^\mu + (x + b)^\pi.$$

This clearly implies that $\mu = \nu$.

Let $x = 1$, $b = 0$: $0^\pi = m^\nu + m^\pi$. Thus $m^\pi - 0^\pi = -m^\nu$ for all $m \in \mathfrak{T}$ since $\mathfrak{T}$ is a group under addition. Consider the incidence equation

$$\begin{aligned} b^\pi &= x^\nu m^\nu + (mx + b)^\pi, \\ b^\pi - 0^\pi &= x^\nu m^\nu + (mx + b)^\pi - 0^\pi, \\ -b^\nu &= x^\nu m^\nu - (mx + b)^\nu. \end{aligned}$$

Setting $b = 0$ gives $x^\nu m^\nu = (mx)^\nu$. Setting $m = 1$ gives $-x^\nu - b^\nu = -(x + b)^\nu$, i.e.,

$$(x + b)^\nu = b^\nu + x^\nu.$$

Thus ν is an anti-isomorphism with respect to both addition and multiplication.

The mapping ρ defined below is the required polarity:

$$\begin{aligned} (x, y)^\rho &= [x^\nu, -y^\nu], & [m, b]^\rho &= (m^\nu, -b^\nu), \\ (m)^\rho &= [m^\nu], & [x]^\rho &= (x^\nu), \\ V^\rho &= UV, & (UV)^\rho &= V. \end{aligned}$$

We need only check two things: first that ρ preserves incidence, and second that ρ induces the same mapping μ of lines through V onto points on UV as did τ. The second is easily verified. To verify that ρ preserves incidence we need only show that

$$y = mx + b \rightleftarrows -b^\nu = x^\nu m^\nu - y^\nu,$$

and this follows immediately from the fact that ν is an anti-isomorphism with respect to both addition and multiplication.

The existence of this anti-isomorphism has as an immediate consequence that $\mathfrak{E}$ is (U, OV, μ')-homogeneous for a suitable mapping μ' if and only if $\mathfrak{E}$ is (U, OV)-transitive. This is because the polarity ρ interchanges U with OV.

In **(4)** a Lenz-Barlotti classification of projective planes according to the amount of (C, γ, μ)-homogeneity was given. This can be thought of as a refinement of the original classification of Lenz and Barlotti. They classified projective planes according to the amount of (C, γ)-transitivity.

Because of Theorem 1 and the above remarks we have the following:

COROLLARY 1. *A plane of class* III-2 *belongs to either class* A-β, C-β, *or* A-α.

COROLLARY 2. *A plane of class* II-2 *belongs to one of the classes* A-α, A-β *or* B-β.

Proof. Here the notation of **(2)** and **(3)** is used. A plane belongs to class III-2 if there is a point R and a line r not incident with R such that $\mathfrak{E}$ is (C, γ)-transitive for the point-line-pairs of the set

$$\{(R, r)\} \cup \{(P, PR); PIr\}$$

and for no others.

A plane belongs to class II-2 if it is (C, γ)-transitive for the two point-line-pairs (C_1, γ_1), (C_2, γ_2) whereby $C_i \, I \, \gamma_1$, $C_1 \, I \, \gamma_j$, $i, j = 1, 2$. A plane belongs to class A-α according to **(4)** if there is no point-line-pair (C, γ) such that $\mathfrak{E}$ is (C, γ, μ)-homogeneous. Clearly a plane of class II-2 or III-2 could belong to the class A-α.

A plane belongs to class A-β according to **(4)** if there is exactly one point-line-pair (C, γ) such that C is not on γ, for there is a mapping μ such that $\mathfrak{E}$ is (C, γ, μ)-homogeneous. If the plane were also of class II-2, then clearly $C = C_2, \gamma = \gamma_2$ because the (C, γ, μ)-homogeneity implies the (C, γ)-transitivity. If the plane were also of class III-2 then $C = R$ and $\gamma = r$.

A plane belongs to class C-β if there is a point R and a line r such that those point-line-pairs (C, γ) for which there is a μ with $\mathfrak{E}$ (C, γ, μ)-homogeneous is the set $\{(R, r)\} \cup \{(P, PR); P\,I\,r\}$.

Assume that $\mathfrak{E}$ belongs to class III-2 and that there is a $P\,I\,r$ for which there exists a μ such that $\mathfrak{E}$ is (P, PR, μ)-homogeneous. Clearly the group of central collineations of $\mathfrak{E}$ is transitive on the points of R. Therefore to every QIr there is a collineation σ such that $P^\sigma = Q, R^\sigma = R, r^\sigma = r$. Because of Theorem 1, there is a polarity ρ which interchanges P and PR and induces μ. The correlation $\sigma^{-1}\rho\sigma$ is also a polarity and $Q^{\sigma^{-1}\rho\sigma} = P^{\sigma\sigma^{-1}\rho\sigma} = (PR)^\sigma = QR$. Consider $R^{\sigma^{-1}\rho\sigma}$; every automorphism of $\mathfrak{E}$ maps the pair (R, r) onto itself since this is the only point-line-pair for which $\mathfrak{E}$ is (C, γ)-transitive. Hence, since $\sigma^{-1}\rho\sigma$ is a correlation, $R^{\sigma^{-1}\rho\sigma} = r$. Furthermore, $\sigma^{-1}\rho\sigma$ induces the mapping $\sigma^{-1}\mu\sigma$ of the lines through Q onto the points of QR. The (P, PR)-transitivity implies the $(P^\sigma, (PR)^\sigma) = (Q, QR)$-transitivity. Therefore $\mathfrak{E}$ is $(Q, QR, \sigma^{-1}\mu\sigma)$-homogeneous. Because of previous remarks, there is also a mapping μ' of the lines through R onto the points of r such that $\mathfrak{E}$ is (R, r, μ')-homogeneous. This shows that the plane belongs to the class C-β. There are no other possibilities for a plane of class III-2.

In **(4)** it was shown that the classical Moulton plane belongs to class C-β.

Corollary 2 is proved in the same way as Corollary 1, so the proof will be omitted.

References

1. R. Baer, *Homogeneity of projective planes*, Amer. J. Math., *64* (1942), 137–52.

2. A. Barlotti, *Le possibili configurazioni del sistema delle coppie punto-retta* (A, a) *per cui un piano grafico risulta* (A, a) *transitivo*, Boll. U.M.I., *12* (1957), 212–26.

3. H. Lenz, *Kleiner Desarguesscher Satz und Dualität in projektiven Ebenen*, Jber. Dtsch. Math. Ver., *57* (1954), 20–31.

4. W. J. Jónsson, *Transitivität und Homogenität projektiver Ebenen*, Math. Z., *80* (1963), 269–92.

5. G. Pickert, *Projektive Ebene* (Berlin, 1955).

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