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Singular Integrals With Rough Kernels

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Abstract. In this paper we establish the L^p boundedness of a class of singular integrals with rough kernels associated to polynomial mappings.

1 Introduction

Let $n \geq 2$ and $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let $\mathbf{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Consider the Calderón-Zygmund singular integral operator

$$(T_\Omega f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y) \frac{\Omega(y)}{|y|^n}\, dy, \tag{1}$$

where Ω is a homogeneous function of degree zero and satisfies $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(y)\, d\sigma(y) = 0. \tag{2}$$

Since the publication of the fundamental papers of Calderón and Zygmund, the operators T_Ω have been studied by many authors. Calderón and Zygmund showed that $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$ is essentially the weakest possible size condition on Ω for the L^p boundedness of T_Ω to hold ([1]). Subsequently, it was proved by Connet ([2]) and Ricci-Weiss ([9]) independently that T_Ω is bounded on L^p for every Ω in the Hardy space $H^1(\mathbf{S}^{n-1})$ (which contains $L\log^+ L(\mathbf{S}^{n-1})$ as a proper subspace) and $p \in (1,\infty)$.

In a more recent paper, Grafakos and Stefanov introduced the following condition:

$$\sup_{\xi \in \mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} |\Omega(y)| \left(\log \frac{1}{|\langle \xi, y \rangle|} \right)^{1+\alpha} d\sigma(y) < \infty, \tag{3}$$

and showed that it implies the L^p boundedness of T_Ω for p in a range dependent on the positive exponent α. For $\alpha > 0$ let $F_\alpha(\mathbf{S}^{n-1})$ denote the space of all integrable functions Ω on $\mathbf{S}^{n-1}$ which satisfy (3).

Theorem 1 ([7]) *Let $\Omega \in F_\alpha(\mathbf{S}^{n-1})$ and satisfy (2). Then T_Ω extends to a bounded operator from $L^p(\mathbb{R}^n)$ into itself for $p \in (\frac{2+\alpha}{1+\alpha}, 2+\alpha)$.*

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The range for p was later improved to $(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ in [4]. It should also be noted that Grafakos and Stefanov showed that

$$\bigcap_{\alpha>0} F_\alpha(\mathbf{S}^{n-1}) \not\subset H^1(\mathbf{S}^{n-1}) \not\subset \bigcup_{\alpha>0} F_\alpha(\mathbf{S}^{n-1}).$$

For details, see [7].

The main purpose of this paper is to investigate the L^p boundedness of singular integrals along subvarieties with kernels satisfying conditions similar to (3). More specifically, let $\mathcal{P} = (P_1, \dots, P_d)$, where P_j is a real-valued polynomial in $\mathbb{R}^n$ for $j = 1, \dots, d$. Define the operator $T_{\Omega,\mathcal{P}}$ in $\mathbb{R}^d$ by

$$(4) \qquad (T_{\Omega,\mathcal{P}} f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f\big(x - \mathcal{P}(y)\big) \frac{\Omega(y)}{|y|^n}\, dy,$$

where $x \in \mathbb{R}^d$. Clearly, when $d = n$ and $\mathcal{P}(y) = y$, one obtains $T_{\Omega,\mathcal{P}} = T_\Omega$. For general polynomial mappings $\mathcal{P}$, the L^p boundedness was first established for $\Omega \in C^1(\mathbf{S}^{n-1})$ as the model case for singular Radon transforms ([10]), and more recently for $\Omega \in H^1(\mathbf{S}^{n-1})$ (see [6]).

In order to state our main results, we let $\mathcal{A}(n, m)$ denote the set of polynomials on $\mathbb{R}^n$ which have real coefficients and degrees not exceeding m, and let $V(n, m)$ denote the collection of polynomials in $\mathcal{A}(n, m)$ which are homogeneous of degree m.

For $P(y) = \sum_{|\beta|\le m} a_\beta y^\beta$ we set $\|P\| = (\sum_{|\beta|\le m} |a_\beta|^2)^{1/2}$.

Definition Let $n \ge 2$, $m \in \mathbb{N}$ and $\alpha > 0$. An integrable function Ω on $\mathbf{S}^{n-1}$ is said to be in the space $F(n, m, \alpha)$ if

$$(5) \qquad \sup_{P\in V(n,m), \|P\|=1} \int_{\mathbf{S}^{n-1}} |\Omega(y)| \left(\log \frac{1}{|P(y)|}\right)^{1+\alpha} d\sigma(y) < \infty.$$

Clearly $F(n, 1, \alpha) = F_\alpha(\mathbf{S}^{n-1})$. We have the following:

Theorem 2 *Let $n \ge 2$, $m, d \in \mathbb{N}$ and $\mathcal{P} = (P_1, \dots, P_d) \in \big(\mathcal{A}(n, m)\big)^d$. Let Ω satisfy (2) and $\Omega \in \bigcap_{s=1}^m F(n, s, \alpha)$ for some $\alpha > 0$. Then the operator $T_{\Omega,\mathcal{P}}$ is bounded on $L^p(\mathbb{R}^d)$ for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$. Moreover, the bound for the operator norm $\|T_{\Omega,\mathcal{P}}\|_{p,p}$ is independent of the coefficients of the polynomials $\{P_j\}$.*

For $n = 2$ we shall show that (see Lemma 3.2)

$$\bigcap_{m=1}^{\infty} F(2, m, \alpha) = F_\alpha(\mathbf{S}^1),$$

which leads to the following:

Corollary 3 *Let $d \in \mathbb{N}$ and $\mathcal{P} = (P_1, \dots, P_d)$ where $P_j \colon \mathbb{R}^2 \to \mathbb{R}$ is a polynomial for $1 \le j \le d$. If $\Omega \in F_\alpha(\mathbf{S}^1)$ for some $\alpha > 0$ and satisfies (2), then, for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, there exists a $C_p > 0$ such that*

$$\|T_{\Omega,\mathcal{P}} f\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}$$

for $f \in L^p(\mathbb{R}^d)$. The constant C_p may depend on $\deg(\mathcal{P}) = \max_{1\le j\le m} \deg(P_j)$, but it is independent of the coefficients of the polynomials $P_1, \dots, P_d$.

2 Some Lemmas

Lemma 2.1 *Let $d, m \in \mathbb{N}$, $\alpha > 0$, and $L\colon \mathbb{R}^d \to \mathbb{R}^m$ be a linear transformation. Let $\{a_k\}_{k\in\mathbb{Z}}$ be a sequence of positive numbers satisfying $\inf_{k\in\mathbb{Z}}(a_{k+1}/a_k) = a > 1$, $\{\sigma_k\}_{k\in\mathbb{Z}}$ be a sequence of uniformly bounded measures on $\mathbb{R}^d$ and set $Tf = \sum_{k\in\mathbb{Z}} \sigma_k * f$, initially for $f \in \mathcal{S}(\mathbb{R}^d)$. Suppose that*

(i) $|\hat{\sigma}_k(\xi)| \le C \min\{a_{k+1}|L\xi|, [\log^+(a_k|L\xi|)]^{-(1+\alpha)}\}$ *holds for $\xi \in \mathbb{R}^d$ and $k \in \mathbb{Z}$;*
(ii) $\big\| (\sum_{k\in\mathbb{Z}} |\sigma_k * g_k|^2)^{1/2} \big\|_q \le A_q \big\| (\sum_{k\in\mathbb{Z}} |\sigma_k * g_k|^2)^{1/2} \big\|_q$ *holds for arbitrary functions $\{g_k\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^d$ and $1 < q < \infty$.*

Then T extends to a bounded operator from $L^p(\mathbb{R}^d)$ into itself for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2 + 2\alpha)$. Moreover, the bound on $\|T\|_{p,p}$ is independent of L.

Proof We shall combine the method of Duoandikoetxea and Rubio de Francia ([3]) with ideas from [4, 6, 7]. By an argument in [6], we may assume that $m \le d$ and $L\xi = (\xi_1, \dots, \xi_m) = \xi'$ for $\xi = (\xi_1, \dots, \xi_d) = (\xi', \xi'') \in \mathbb{R}^d$. Choose C^∞ functions $\{\psi_j\}_{j\in\mathbb{Z}}$ on $\mathbb{R}$ such that $\operatorname{supp}(\psi_j) \subseteq [a_{j+1}^{-1}, a_{j-1}^{-1}]$, $|\psi_j^{(s)}(t)| \le Ct^{-s}$, and

$$\sum_{j\in\mathbb{Z}} [\psi_j(t)]^2 = 1$$

for $t > 0$, $s \ge 0$. Define the operator S_j by

$$\widehat{S_j f}(\xi) = \psi_j(|\xi'|)\hat{f}(\xi)$$

for $j \in \mathbb{Z}$ and set

$$T_j f = \sum_{k\in\mathbb{Z}} S_{j+k}(\sigma_k * S_{j+k} f).$$

Thus we have

$$Tf = \sum_{j\in\mathbb{Z}} T_j f. \tag{6}$$

It follows from Littlewood-Paley theory and (ii) that

$$\|T_j f\|_q \le C_q \|f\|_q \tag{7}$$

holds for $1 < q < \infty$, $f \in L^q(\mathbb{R}^d)$ and $j \in \mathbb{Z}$ with C_q independent of j. Let $\Gamma_j = \{\xi \in \mathbb{R}^d : a_{j+1}^{-1} \le |\xi'| < a_{j-1}^{-1}\}$ and $\chi_j = \chi_{\Gamma_j}$. By Plancherel's Theorem,

$$\|T_j f\|_2^2 \le C \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \Big[\sum_{k\in\mathbb{Z}} |\hat{\sigma}_k(\xi)|^2 \chi_{j+k}(\xi) \Big]\, d\xi. \tag{8}$$

For $j > 1$ and $k \in \mathbb{Z}$,

$$|\hat{\sigma}_k(\xi)|^2 \chi_{j+k}(\xi) \le C \Big(\frac{a_{k+1}}{a_{j+k-1}} \Big)^2 \le C a^{-2j+4}. \tag{9}$$

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On the other hand, when $j < -1$,

$$|\hat{\sigma}_k(\xi)|^2 \chi_{j+k}(\xi) \le C\Big[\log\Big(\frac{a_{k+1}}{a_{j+k-1}}\Big)\Big]^{-2(1+\alpha)} \le C|j|^{-2(1+\alpha)} \tag{10}$$

holds for $k \in \mathbb{Z}$. By (7)–(10) and the finite overlapping property of $\{\Gamma_{j+k} : k \in \mathbb{Z}\}$, we obtain

$$\|T_j f\|_2 \le C(1+|j|)^{-(1+\alpha)}\|f\|_2. \tag{11}$$

By interpolating between (7) and (11), for every $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, there is a $\theta_p > 1$ such that

$$\|T_j f\|_p \le C(1+|j|)^{-\theta_p}\|f\|_p \tag{12}$$

holds for $j \in \mathbb{Z}$. The lemma now follows from (6) and (12).

Lemma 2.2 *Let $\alpha > 0$, $m, d \in \mathbb{N}$ and $\{\sigma_{s,k} : 0 \le s \le m$ and $k \in \mathbb{Z}\}$ be a family of uniformly bounded Borel measures on $\mathbb{R}^d$ with $\sigma_{0,k} = 0$ for every $k \in \mathbb{Z}$. Let $\{\eta_s : 1 \le s \le m\} \subset \mathbb{R}^+ \setminus \{1\}$, $\{l_s : 1 \le s \le m\} \subset \mathbb{N}$, and $L_s \colon \mathbb{R}^d \to \mathbb{R}^{l_s}$ be linear transformations for $1 \le s \le m$. Suppose that*

(i) $|\hat{\sigma}_{s,k}(\xi)| \le C[\log^+(\eta_s^k|L_s\xi|)]^{-(1+\alpha)}$ *for $\xi \in \mathbb{R}^d$, $k \in \mathbb{Z}$ and $1 \le s \le m$;*
(ii) $|\hat{\sigma}_{s,k}(\xi) - \hat{\sigma}_{s-1,k}(\xi)| \le C(\eta_s^k|L_s\xi|)$ *for $\xi \in \mathbb{R}^d$, $k \in \mathbb{Z}$ and $1 \le s \le m$;*
(iii) *For every $q \in (1, \infty)$ there exists an $A_q > 0$ such that*

$$\Big\|\sup_{k\in\mathbb{Z}}(|\sigma_{s,k}| * |f|)\Big\|_q \le A_q\|f\|_q \tag{13}$$

for all $f \in L^q(\mathbb{R}^d)$ and $1 \le s \le m$.

Then for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, there exists a $C_p > 0$ such that

$$\Big\|\sum_{k\in\mathbb{Z}}\sigma_{m,k} * f\Big\|_p \le C_p\|f\|_p \tag{14}$$

holds for all $f \in L^p(\mathbb{R}^d)$. Moreover, the constant C_p is independent of the linear transformations $\{L_s : 1 \le s \le m\}$.

One may use the arguments in Section 5 of [5] and Lemma 2.1 to obtain a proof of Lemma 2.2. Details are omitted.

3 Proofs of Main Results

Proof of Theorem 2 Let $n \ge 2$, $m, d \in \mathbb{N}$ and $\mathcal{P} = (P_1, \dots, P_d)$, where

$$P_j(y) = \sum_{|\beta|\le m} a_{j\beta} y^\beta$$

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for $j = 1, \dots, d$. Let Ω satisfy (2) and $\Omega \in \bigcap_{s=1}^{m} F(n, s, \alpha)$ for some $\alpha > 0$. For $0 \le s \le m$ and $k \in \mathbb{Z}$ we define the measure $\sigma_{s,k}$ on $\mathbb{R}^d$ by

$$\int_{\mathbb{R}^d} f \, d\sigma_{s,k} = \int_{2^{k-1} \le |y| < 2^k} f\Big(\sum_{|\beta| \le s} a_{1\beta} y^\beta, \dots, \sum_{|\beta| \le s} a_{d\beta} y^\beta\Big) \frac{\Omega(y)}{|y|^n} \, dy. \tag{15}$$

It follows from (2) that $\sigma_{0,k} = 0$ for all $k \in \mathbb{Z}$ and

$$T_{\Omega,\mathcal{P}} f = \sum_{k \in \mathbb{Z}} \sigma_{m,k} * f. \tag{16}$$

By Theorem 7.4 in [6], (13) holds for all $f \in L^q(\mathbb{R}^d)$ and $1 \le s \le m$. Let l_s denote the number of multi-indices $\beta = (\beta_1, \dots, \beta_n)$ satisfying $|\beta| = s$ and define the linear transformation $L_s \colon \mathbb{R}^d \to \mathbb{R}^{l_s}$ by

$$L_s \xi = \big((L_s \xi)_\beta\big)_{|\beta| = s} = \Big(\sum_{j=1}^{d} a_{j\beta} \xi_j\Big)_{|\beta| = s}. \tag{17}$$

It follows from (15) and (17) that

$$\begin{aligned} |\hat{\sigma}_{s,k}(\xi) - \hat{\sigma}_{s-1,k}(\xi)| &\le \int_{2^{k-1} \le |y| < 2^k} \Big| \exp\Big[i\Big(\sum_{j=1}^{d} \sum_{|\beta| = s} a_{j\beta} \xi_j y^\beta\Big)\Big] - 1 \Big| \frac{|\Omega(y)|}{|y|^n} \, dy \\ &\le C(2^{sk} |L_s \xi|) \end{aligned}$$

for $1 \le s \le m$ and $k \in \mathbb{Z}$. Write

$$\hat{\sigma}_{s,k}(\xi) = \int_{\mathbf{S}^{n-1}} I_{s,k}(\xi, y) \Omega(y) \, d\sigma(y),$$

where

$$I_{s,k}(\xi, y) = \int_{1/2}^{1} \exp\big[i\big(2^{sk} |L_s \xi| Q_{s\xi}(y) t^s + \text{lower powers in } t\big)\big] t^{-1} \, dt$$

with

$$Q_{s\xi}(y) = |L_s \xi|^{-1} \sum_{|\beta| = s} (L_s \xi)_\beta y^\beta.$$

Then by van der Corput's lemma,

$$|I_{s,k}(\xi, y)| \le C\big[2^{sk} |L_s \xi| \, |Q_{s\xi}(y)|\big]^{-1/s}. \tag{18}$$

By combining (18) with the trivial inequality $|I_{s,k}(\xi, y)| \le 1$ we obtain that

$$|I_{s,k}(\xi, y)| \le C\big[\log^+(2^{sk} |L_s \xi|)\big]^{-(1+\alpha)} \Big(s + \alpha + \log \frac{1}{|Q_{s\xi}(y)|}\Big)^{1+\alpha}. \tag{19}$$

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Since $Q_{s\xi} \in V(n,s)$, $\|Q_{s\xi}\| = 1$, and $\Omega \in F(n,s,\alpha)$ for $1 \le s \le m$, by (5) and (19) we obtain

$$|\hat{\sigma}_{s,k}(\xi)| \le C\left[\log^+(2^{sk}|L_s\xi|)\right]^{-(1+\alpha)}$$

for $1 \le s \le m$, $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^d$. It follows from Lemma 2.2 and (16) that $T_{\Omega,\mathcal{P}}$ is bounded on $L^p(\mathbb{R}^d)$ for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ with a bound on $\|T_{\Omega,\mathcal{P}}\|_{p,p}$ independent of the coefficients of the P_j's. The proof of Theorem 2 is now complete.

We now show that $F_\alpha(\mathbf{S}^1) \subseteq F(2,m,\alpha)$ for $m \in \mathbb{N}$.

Lemma 3.1 *Let $m \in \mathbb{N}$, $a_0, a_1, \dots, a_m \in \mathbb{C}$ and $g(z) = a_0 + a_1 z + \cdots + a_m z^m$ for $z \in \mathbb{C}$. If $z_1, \dots, z_l$ are the roots of $g(z)$ which lie in $\{z \in \mathbb{C} : |z| \le 2\}$, then*

$$|g(z)| \ge 6^{-m}\big(\sup_{|z|=1}|g(z)|\big)\prod_{s=1}^{l}|z - z_s|$$

holds for $|z| \le 1$.

Proof Without loss of generality we may assume that $a_m = 1$. Let $z_{l+1}, \dots, z_m$ denote the roots of $g(z)$ which lie in $\{z \in \mathbb{C} : |z| > 2\}$. By

$$g(z) = \prod_{s=1}^{m}(z - z_s),$$

we have

$$|a_j| \le \sum_{1 \le k_1 < \cdots < k_{m-j} \le m} |z_{k_1} \cdots z_{k_{m-j}}| \le \frac{(2^{m-j}\, m!)|z_{l+1}| \cdots |z_m|}{j!\,(m-j)!}$$

for $j = 0, 1, \dots, m$, which implies that

$$\prod_{s=l+1}^{m}|z_s| \ge 3^{-m}\Big(\sum_{j=0}^{m}|a_j|\Big).$$

Thus, for $|z| \le 1$,

$$|g(z)| \ge \Big(\prod_{s=1}^{l}|z - z_s|\Big)\Big(\prod_{s=l+1}^{m}\frac{|z_s|}{2}\Big) \ge 6^{-m}\Big(\sum_{j=0}^{m}|a_j|\Big)\Big(\prod_{s=1}^{l}|z - z_s|\Big)$$

$$\ge 6^{-m}\big(\sup_{|z|=1}|g(z)|\big)\Big(\prod_{s=1}^{l}|z - z_s|\Big).$$

Lemma 3.1 is proved.

Corollary 3 follows from Theorem 2 and the following lemma:

Lemma 3.2 $\bigcap_{m=1}^{\infty} F(2,m,\alpha) = F_\alpha(\mathbf{S}^1)$.

Proof It suffices to show that $F_\alpha(\mathbf{S}^1) \subseteq F(2, m, \alpha)$ for $m \in \mathbb{N}$. Let $\Omega \in F_\alpha(\mathbf{S}^1)$. Then it follows from [7] that

$$\sup_{\zeta \in \mathbf{S}^1} \int_{\mathbf{S}^1} |\Omega(y)| \Big(\log^+ \frac{1}{|y - \zeta|} \Big)^{1+\alpha} d\sigma(y) = C_\Omega < \infty. \tag{20}$$

For a fixed $m \in \mathbb{N}$, there exists a $\lambda_m > 0$ such that

$$\sup_{y \in \mathbf{S}^1} |P(y)| \geq \lambda_m \|P\|$$

holds for every $P \in V(2, m)$.

Let

$$P(y) = P(y_1, y_2) = \sum_{j+k=m} a_{jk} y_1^j y_2^k \in V(2, m)$$

and $\|P\| = 1$. Define $g = g_P$ on $\mathbb{C}$ by

$$g(z) = 2^{-m} \sum_{j+k=m} (-i)^k a_{jk} (z^2 + 1)^j (z^2 - 1)^k.$$

Then $|P(y_1, y_2)| = |g(y_1 + y_2 i)|$ for $(y_1, y_2) \in \mathbf{S}^1$. Let $z_1, \dots, z_l$ denote the roots of $g(z)$ in $\{0 < |z| \leq 2\}$. By Lemma 3.1, for $y = (y_1, y_2) \in \mathbf{S}^1$,

$$\begin{aligned} |P(y)| &\geq 6^{-2m} \lambda_m \prod_{s=1}^{l} |(y_1 + y_2 i) - z_s| \\ &\geq (12)^{-2m} \lambda_m \prod_{s=1}^{l} \Big| (y_1 + y_2 i) - \frac{z_s}{|z_s|} \Big|. \end{aligned}$$

Thus, by (20),

$$\int_{\mathbf{S}^1} |\Omega(y)| \Big(\log \frac{1}{|P(y)|} \Big)^{1+\alpha} d\sigma(y) \leq C(\|\Omega\|_{L^1(\mathbf{S}^1)} + C_\Omega),$$

which implies that $F_\alpha(\mathbf{S}^1) \subseteq F(2, m, \alpha)$ for $m \in \mathbb{N}$. Lemma 3.2 is proved.

There is no analogue of Lemma 3.1 when $n \geq 3$. We shall illustrate this with the following example for $n = 3$:

Example For $y = (y_1, y_2, y_3) \in \mathbf{S}^2$, let

$$\Omega(y) = \frac{y_3 \chi_{[\sqrt{2}/2, \sqrt{3}/2]}(|y_3|)}{|y_3| \, |y_1^2 + y_2^2 - y_3^2| \log(100 |y_1^2 + y_2^2 - y_3^2|^{-1}) \{\log[\log(100 |y_1^2 + y_2^2 - y_3^2|^{-1})]\}^2}.$$

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Clearly

$$\int_{\mathbf{S}^2} \Omega(y)\, d\sigma(y) = 0.$$

For $\alpha > 0$, $\varphi \in [\frac{\pi}{6}, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{6}] = E$, and $\xi \in \mathbf{S}^2$, let

$$J_\alpha(\varphi, \xi) = \int_0^{2\pi} \Big(\log \frac{1}{|\xi_1 \sin\varphi \cos\theta + \xi_2 \sin\varphi \sin\theta + \xi_3 \cos\varphi|} \Big)^{\alpha+1} d\theta.$$

Then there exists a $C > 0$ such that

$$|J_\alpha(\varphi, \xi)| \le C$$

for $\alpha > 0$, $\xi \in \mathbf{S}^2$ and $\varphi \in E$. Thus

$$\int_{\mathbf{S}^2} |\Omega(y)| \Big(\log \frac{1}{|\langle \xi, y\rangle|} \Big)^{1+\alpha} d\sigma(y) \le \int_E \frac{(\sin\varphi) J_\alpha(\varphi,\xi)\, d\varphi}{|\cos 2\varphi| (\log|\frac{100}{\cos 2\varphi}|)[\log(\log|\frac{100}{\cos 2\varphi}|)]^2} < \infty$$

for $\alpha > 0$ and $\xi \in \mathbf{S}^2$. Thus $\Omega \in F_\alpha(\mathbf{S}^2)$ for every $\alpha > 0$.

On the other hand, if we take $P(y) = (y_1^2 + y_2^2 - y_3^2)/\sqrt{3} \in V(3,2)$, then

$$\int_{\mathbf{S}^2} |\Omega(y)| \Big(\log \frac{1}{|P(y)|} \Big)^{1+\alpha} d\sigma(y) = \infty$$

for $\alpha > 0$, which implies that $\Omega \notin F(3, 2, \alpha)$ for any $\alpha > 0$.

4 Additional Results

Let Ω and $\mathcal{P}$ be given as in Section 1. Define the maximal truncated singular integral operator $T^*_{\Omega,\mathcal{P}}$ by

$$(T^*_{\Omega,\mathcal{P}} f)(x) = \sup_{\varepsilon>0} \Big| \int_{|y|>\varepsilon} f\big(x - \mathcal{P}(y)\big) \frac{\Omega(y)}{|y|^n}\, dy \Big|. \tag{21}$$

We have the following results:

Theorem 4 *Let $n \ge 2$, $m, d \in \mathbb{N}$ and $\mathcal{P} = (P_1, \dots, P_d) \in \big(\mathcal{A}(n,m)\big)^d$. Let Ω satisfy (2) and $\Omega \in \bigcap_{s=1}^m F(n, s, \alpha)$ for some $\alpha > 1/2$. Then the operator $T^*_{\Omega,\mathcal{P}}$ is bounded on $L^p(\mathbb{R}^d)$ for $p \in (\frac{1+2\alpha}{2\alpha}, 1 + 2\alpha)$. Moreover, the bound for the operator norm $\|T^*_{\Omega,\mathcal{P}}\|_{p,p}$ is independent of the coefficients of the polynomials $\{P_j\}$.*

Corollary 5 *Let $d \in \mathbb{N}$, $\mathcal{P} = (P_1, \dots, P_d)$ where $P_j \colon \mathbb{R}^2 \to \mathbb{R}$ is a polynomial for $1 \le j \le d$. If $\Omega \in F_\alpha(\mathbf{S}^1)$ for some $\alpha > 1/2$ and satisfies (2), then, for $p \in (\frac{1+2\alpha}{2\alpha}, 1 + 2\alpha)$, there exists a $C_p > 0$ such that*

$$\|T^*_{\Omega,\mathcal{P}} f\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}$$

for $f \in L^p(\mathbb{R}^d)$. The constant C_p may depend on $\deg(\mathcal{P}) = \max_{1\le j\le m} \deg(P_j)$, but it is independent of the coefficients of the polynomials $P_1, \dots, P_d$.

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One may construct a proof for Theorem 4 by using the arguments in Section 3, [4] and [6] (see also [3] and [7]). We omit the details.

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